World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:7, No:5, 2013

On Properties of Generalized Bi-Γ-Ideals of Γ-Semirings

Teerayut Chomchuen and Aiyared Iampan

Abstract—The notion of Γ-semirings was introduced by Murali Krishna Rao as a generalization of the notion of Γ-rings as well as of semirings. We have known that the notion of Γ-semirings is a generalization of the notion of semirings. In this paper, extending Kaushik, Moin and Khan's work, we generalize the notion of generalized bi-Γ-ideals of Γ-semirings and investigate some related properties of generalized bi-Γ-ideals.

Keywords—Γ-semiring, bi-Γ-ideal, generalized bi-Γ-ideal.

I. Introduction and Preliminaries

THE notion of Γ-semirings was introduced and studied in 1995 by Murali Krishna Rao [10] as a generalization of the notion of Γ-rings as well as of semiring, and the notion of generalized bi-ideals was first introduced for rings in 1970 by Szász [12], [13] and then for semigroups by Lajos [8]. Many types of ideals on the algebraic structures were characterized by several authors such as: In 2000, Dutta and Sardar [3] studied the characterization of semiprime ideals and irreducible ideals of Γ-semirings. In 2004, Sardar and Dasgupta [11] introduced the notions of primitive Γ-semirings and primitive ideals of Γ-semirings. In 2008, Kaushik, Moin and Khan [7] introduced and studied bi-Γ-ideals in Γ-semirings, Pianskool, Sangwirotjanapat and Tipyota [9] introduced and studied valuation Γ-semirings and valuation Γ-ideals of a Γ-semiring, and Chinram [1] gave some properties of quasi-ideals in Γ-semirings. In 2009, Jagatap and Pawar [6] introduced the concept of minimal quasi-ideal in Γ-semirings. Some properties of minimal quasi-ideals in Γ-semirings are provided. In 2010, Ghosh and Samanta [5] studied the relation between the fuzzy left (respectively, right) ideals of Γ-semirings and that of operator semiring. In 2011, Dutta, Sardar and Goswami [4] introduced different types of operations on fuzzy ideals of Γ-semirings and proved subsequently that these operations give rise to different structures such as complete lattice, modular lattice on some restricted class of fuzzy ideals of Γ-semirings. In 2012, Bektaş, Bayrak and Ersoy [2] introduced and studied the characterization of soft Γ-semirings and soft sub-Γ-semiring.

The concept of ideals for many types of Γ-semirings is the really interested and important thing in Γ-semirings. Therefore, we will introduce and study generalized bi-Γ-ideals of Γ-semirings in the same way as of bi-Γ-ideals of Γ-semirings which was studied by Kaushik, Moin and Khan [7].

T. Chomchuen and A. Iampan* are with the Department of Mathematics, School of Science, University of Phayao, Phayao 56000, THAILAND.

This research is supported by the Group for Young Algebraists in University of Phayao (GYA), Thailand.

*Corresponding author. e-mail: aiyared.ia@up.ac.th.

To present the main results we first recall the definition of a Γ-semiring which is important here and discuss some elementary definitions that we use later.

Definition I.1. [10] Let M and Γ be two additive commutative semigroups. Then M is called a Γ-*semiring* if there exists a mapping $\cdot : M \times \Gamma \times M \to M$ (the image $\cdot(a, \alpha, b)$ to be denoted by $a\alpha b$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$) satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(2) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(3) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Let M be a Γ-semiring, A and B nonempty subsets of M, and Λ a nonempty subset of Γ. Then we define

$$A + B := \{a + b \mid a \in A \text{ and } b \in B\}$$

and

$$A\Lambda B := \left\{ \sum_{i=1}^{n} a_i\lambda_i b_i \mid n \in \mathbb{Z}^+, a_i \in A, b_i \in B \text{ and } \lambda_i \in \Lambda \text{ for all } i \right\}.$$

If $A = \{a\}$, then we also write $\{a\} + B$ as $a + B$, and $\{a\}\Lambda B$ as $a\Lambda B$, and similarly if $B = \{b\}$ or $\Lambda = \{\lambda\}$.

Example I.2. [6] Let $\mathbb{Q}$ be set of rational numbers. Let $(S, +)$ be the commutative semigroup of all 2×3 matrices over $\mathbb{Q}$ and $(\Gamma, +)$ commutative semigroup of all 3×2 matrices over $\mathbb{Q}$. Define $W\alpha Y$ usual matrix product of W, α and Y for all $W, Y \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semiring but not a semiring.

Example I.3. [6] Let $\mathbb{N}$ be the set of natural numbers and $\Gamma = \{1, 2, 3\}$. Then $(\mathbb{N}, \max)$ and $(\Gamma, \max)$ are commutative semigroups. Define the mapping $\mathbb{N} \times \Gamma \times \mathbb{N} \to \mathbb{N}$, by $a\alpha b = \min\{a, \alpha, b\}$ for all $a, b \in \mathbb{N}$ and $\alpha \in \Gamma$. Then $\mathbb{N}$ is a Γ-semiring.

Example I.4. [6] Let $\mathbb{Q}$ be set of rational numbers and $\Gamma = \mathbb{N}$ the set of natural numbers. Then $(\mathbb{Q}, +)$ and $(\mathbb{N}, +)$ are commutative semigroups. Define the mapping $\mathbb{Q} \times \Gamma \times \mathbb{Q} \to \mathbb{Q}$, by $a\alpha b$ usual product of a, α, b; for all $a, b \in \mathbb{Q}$ and $\alpha \in \Gamma$. Then $\mathbb{Q}$ is a Γ-semiring.

Example I.5. [2] For consider the additively abelian groups $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $\Gamma = \{2, 4, 6\}$. Let $\cdot : \mathbb{Z}_8 \times \Gamma \times \mathbb{Z}_8 \to \mathbb{Z}_8$, $(y, \alpha, s) = y\alpha s$. Then $\mathbb{Z}_8$ is a Γ-semiring.

Definition I.6. A nonempty subset A of a Γ-semiring M is called

(1) a *sub-*Γ*-semiring* of M if $(A,+)$ is a subsemigroup of $(M,+)$ and $a\gamma b \in A$ for all $a,b \in A$ and $\gamma \in \Gamma$.
(2) a Γ*-ideal* of M if $(A,+)$ is a subsemigroup of $(M,+)$, and $x\gamma a \in A$ and $a\gamma x \in A$ for all $a \in A, x \in M$ and $\gamma \in \Gamma$.
(3) a *quasi-*Γ*-ideal* of M if A is a sub-Γ-semiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(4) a *bi-*Γ*-ideal* of M if A is a sub-Γ-semiring of M and $A\Gamma M\Gamma A \subseteq A$.
(5) a *generalized bi-*Γ*-ideal* of M if $A\Gamma M\Gamma A \subseteq A$.

Remark I.7. Let M be a Γ-semiring. We have the following:

(1) Every quasi-Γ-ideal of M is a bi-Γ-ideal.
(2) Every bi-Γ-ideal of M is a generalized bi-Γ-ideal.

Definition I.8. A Γ-semiring M is called a *GB-simple* Γ*-semiring* if M is the unique generalized bi-Γ-ideal of M.

II. Main Results

Before the characterizations of generalized bi-Γ-ideals of Γ-semirings for the main results, we give some auxiliary results which are necessary in what follows. By Lemma I.7 (2) and [7], we have the following lemma.

Lemma II.1. *Let M be a Γ-semiring and $a \in M$. Then $a\Gamma M$ and $M\Gamma a$ are generalized bi-Γ-ideals of M.*

Lemma II.2. *Let M be a Γ-semiring and $\{B_i \mid i \in I\}$ a nonempty family of generalized bi-Γ-ideals of M with $\bigcap_{i\in I} B_i \neq \emptyset$. Then $\bigcap_{i\in I} B_i$ is a generalized bi-Γ-ideal of M.*

Proof: For all $i \in I$, we have

$$\left(\bigcap_{i\in I} B_i\right)\Gamma M\Gamma\left(\bigcap_{i\in I} B_i\right) \subseteq B_i\Gamma M\Gamma B_i \subseteq B_i.$$

Thus

$$\left(\bigcap_{i\in I} B_i\right)\Gamma M\Gamma\left(\bigcap_{i\in I} B_i\right) \subseteq \bigcap_{i\in I} B_i.$$

Hence $\bigcap_{i\in I} B_i$ is a generalized bi-Γ-ideal of M. ■

Lemma II.3. *Let M be a Γ-semiring and $\emptyset \neq A \subseteq M$. Then*

$$A \cup A\Gamma M\Gamma A \tag{1}$$

is the smallest generalized bi-Γ-ideal of M containing A.

Proof: Let $B = A \cup A\Gamma M\Gamma A$. Then $A \subseteq B$. Therefore

$$\begin{aligned}
& B\Gamma M\Gamma B \\
= \; & (A \cup A\Gamma M\Gamma A)\Gamma M\Gamma(A \cup A\Gamma M\Gamma A) \\
\subseteq \; & [A(\Gamma M\Gamma)(A \cup A\Gamma M\Gamma A)] \cup \\
& [A\Gamma M\Gamma A(\Gamma M\Gamma)(A \cup A\Gamma M\Gamma A)] \\
\subseteq \; & [A(\Gamma M\Gamma)A \cup A(\Gamma M\Gamma)A\Gamma M\Gamma A] \cup \\
& [A\Gamma M\Gamma A(\Gamma M\Gamma)A \cup A\Gamma M\Gamma A(\Gamma M\Gamma)A\Gamma M\Gamma A] \\
\subseteq \; & [A\Gamma M\Gamma A \cup A\Gamma M\Gamma A] \cup [A\Gamma M\Gamma A \cup A\Gamma M\Gamma A] \\
= \; & A\Gamma M\Gamma A \\
\subseteq \; & A \cup A\Gamma M\Gamma A \\
= \; & B.
\end{aligned}$$

Thus $B = A \cup A\Gamma M\Gamma A$ is a generalized bi-Γ-ideal of M. We shall show that B is the smallest generalized bi-Γ-ideal of M containing A. Let C be a generalized bi-Γ-ideal of M containing A. Then

$$A\Gamma M\Gamma A \subseteq C\Gamma M\Gamma C \subseteq C.$$

Thus

$$B = A \cup A\Gamma M\Gamma A \subseteq C.$$

Hence B is the smallest generalized bi-Γ-ideal of M containing A. ■

By Lemma II.3, let (A) be the smallest generalized bi-Γ-ideal of M containing A. Therefore

$$(A) = A \cup A\Gamma M\Gamma A. \tag{2}$$

It is also very common to denote the smallest generalized bi-Γ-ideal of M containing $\{a\}$ as (a).

Lemma II.4. *Let T be a sub-Γ-semiring of a Γ-semiring M, $a \in M$ and $(a\Gamma T\Gamma a) \cap T \neq \emptyset$. Then $(a\Gamma T\Gamma a) \cap T$ is a generalized bi-Γ-ideal of T.*

Proof: Consider

$$\begin{aligned}
& (a\Gamma T\Gamma a \cap T)\Gamma T\Gamma(a\Gamma T\Gamma a \cap T) \\
\subseteq \; & [(a\Gamma T\Gamma a)\Gamma T \cap T\Gamma T]\Gamma(a\Gamma T\Gamma a \cap T) \\
\subseteq \; & [(a\Gamma T\Gamma a)\Gamma T \cap T]\Gamma(a\Gamma T\Gamma a \cap T) \\
\subseteq \; & [[(a\Gamma T\Gamma a\Gamma T)\Gamma(a\Gamma T\Gamma a)] \cap [T\Gamma(a\Gamma T\Gamma a \cap T)]] \\
\subseteq \; & [(a\Gamma T\Gamma a) \cap (T\Gamma a\Gamma T\Gamma a)] \cap T \\
\subseteq \; & (a\Gamma T\Gamma a) \cap T.
\end{aligned}$$

Hence $(a\Gamma T\Gamma a) \cap T$ is a generalized bi-Γ-ideal of T. ■

Lemma II.5. *Let M be a Γ-semiring and $a \in M$. Then $a\Gamma M\Gamma a$ is a generalized bi-Γ-ideal of M.*

Proof: Consider

$$(a\Gamma M\Gamma a)\Gamma M\Gamma(a\Gamma M\Gamma a) = a\Gamma(M\Gamma a\Gamma M\Gamma a\Gamma M)\Gamma a \subseteq a\Gamma M\Gamma a$$

Hence $a\Gamma M\Gamma a$ is a generalized bi-Γ-ideal of M. ■

Proposition II.6. *Let M be a Γ-semiring and T a sub-Γ-semiring of M. Then every subset of T containing $M\Gamma T$ is a sub-Γ-semiring of M.*

Proof: Let A be a subset of T such that $M\Gamma T \subseteq A$. Then

$$A\Gamma A \subseteq M\Gamma T \subseteq A.$$

Hence A is a sub-Γ-semiring of M. ■

Proposition II.7. *Let M be a Γ-semiring and T a Γ-ideal of M. Then every subset of T containing $M\Gamma T \cup T\Gamma M$ is a Γ-ideal of M.*

Proof: Let B be a subset of T such that $M\Gamma T \cup T\Gamma M \subseteq B$. Then

$$M\Gamma B \subseteq M\Gamma T \subseteq M\Gamma T \cup T\Gamma M \subseteq B$$

and

$$B\Gamma M \subseteq T\Gamma M \subseteq T\Gamma M \cup M\Gamma T \subseteq B.$$

Hence B is a Γ-ideal of M. ■

Proposition II.8. *Let M be a Γ-semiring and T a quasi-Γ-ideal of M. Then every subset of T containing $T\Gamma M \cap M\Gamma T$ is a quasi-Γ-ideal of M.*

Proof: Let C be a subset of T such that $T\Gamma M \cap M\Gamma T \subseteq C$. Then

$$C\Gamma C \subseteq T\Gamma M \cap M\Gamma T \subseteq C$$

and

$$C\Gamma M \cap M\Gamma C \subseteq T\Gamma M \cap M\Gamma T \subseteq C.$$

Hence C is a quasi-Γ-ideal of M. ■

Proposition II.9. *Let M be a Γ-semiring and T a bi-Γ-ideal of M. Then every subset of T containing $T\Gamma M\Gamma T$ and all of its images is a bi-Γ-ideal of M.*

Proof: Let D be a subset of T such that $T\Gamma M\Gamma T \subseteq D$ and $D\Gamma D \subseteq D$. Then

$$D\Gamma M\Gamma D \subseteq T\Gamma M\Gamma T \subseteq D.$$

Hence D is a bi-Γ-ideal of M. ■

Proposition II.10. *Let M be a Γ-semiring and T a generalized bi-Γ-ideal of M. Then every subset of T containing $T\Gamma M\Gamma T$ is a generalized bi-Γ-ideal of M.*

Proof: Let E be a subset of T such that $T\Gamma M\Gamma T \subseteq E$. Then

$$E\Gamma M\Gamma E \subseteq T\Gamma M\Gamma T \subseteq E.$$

Hence E is a generalized bi-Γ-ideal of M. ■

Theorem II.11. *Let M be a Γ-semiring. Then the following statements are equivalent.*

(1) *M is a GB-simple Γ-semiring.*
(2) *$a\Gamma M\Gamma a = M$ for all $a \in M$.*
(3) *$(a) = M$ for all $a \in M$.*

Proof: (1) $\Rightarrow$ (2) Assume that M is a GB-simple Γ-semiring and $a \in M$. By Lemma II.5, we have $a\Gamma M\Gamma a$ is a generalized bi-Γ-ideal of M. Since M is a GB-simple Γ-semiring, we have $a\Gamma M\Gamma a = M$.
(2) $\Rightarrow$ (3) Assume that $a\Gamma M\Gamma a = M$ for all $a \in M$ and let $a \in M$. Then, by (2), we have

$$(a) = \{a\} \cup a\Gamma M\Gamma a = \{a\} \cup M = M.$$

(3) $\Rightarrow$ (1) Assume that $(a) = M$ for all $a \in M$, and let A be a generalized bi-Γ-ideal of M and $a \in A$. Then $(a) \subseteq A$. By assumption, we have

$$M = (a) \subseteq A \subseteq M.$$

Thus $M = A$. Therefore M is a GB-simple Γ-semiring. ■

Lemma II.12. *Let B be a generalized bi-Γ-ideal of a Γ-semiring M and T a sub-Γ-semiring of M. If T is a GB-simple Γ-semiring such that $T \cap B \neq \emptyset$, then $T \subseteq B$.*

Proof: Assume that T is a GB-simple Γ-semiring such that $T \cap B \neq \emptyset$ and let $a \in T \cap B$. By Lemma II.3, we have $\{a\} \cup a\Gamma T\Gamma a$ is a generalized bi-Γ-ideal of T. Since T is a GB-simple Γ-semiring, we have $\{a\} \cup a\Gamma T\Gamma a = T$. Thus

$$T = \{a\} \cup a\Gamma T\Gamma a \subseteq B \cup B\Gamma M\Gamma B \subseteq B \cup B \subseteq B.$$

Hence $T \subseteq B$. ■

Theorem II.13. *Let M be a Γ-semiring, B a generalized bi-Γ-ideal of M and $\emptyset \neq A \subseteq M$. Then $B\Gamma A$ and $A\Gamma B$ are generalized bi-Γ-ideals of M.*

Proof: Since B is a generalized bi-Γ-ideal of M, we have

$$(B\Gamma A)\Gamma M\Gamma(B\Gamma A) = (B\Gamma(A\Gamma M)\Gamma B)\Gamma A \subseteq (B\Gamma M\Gamma B)\Gamma A \subseteq B\Gamma A$$

and

$$(A\Gamma B)\Gamma M\Gamma(A\Gamma B) = A\Gamma(B\Gamma(M\Gamma A)\Gamma B) \subseteq A\Gamma(B\Gamma M\Gamma B) \subseteq A\Gamma B.$$

Therefore $B\Gamma A$ and $A\Gamma B$ are generalized bi-Γ-ideals of M. ■

Theorem II.14. *Let M be a Γ-semiring and B a bi-Γ-ideal of M. Then B is a minimal generalized bi-Γ-ideal of M if and only if B is a GB-simple Γ-semiring.*

Proof: Assume that B is a minimal generalized bi-Γ-ideal of M. By assumption, B is a Γ-semiring. Let C be a generalized bi-Γ-ideal of B. Then

$$C\Gamma B\Gamma C \subseteq C \subseteq B. \tag{3}$$

Since B is a generalized bi-Γ-ideal of M and by Theorem II.13, we have $C\Gamma B\Gamma C$ is a generalized bi-Γ-ideal of M. Since B is a minimal generalized bi-Γ-ideal of M, we get $C\Gamma B\Gamma C = B$. Thus, by (3), we have $B = C$. Hence B is a GB-simple Γ-semiring.

Conversely, assume that B is a GB-simple Γ-semiring. Let C be a generalized bi-Γ-ideal of M such that $C \subseteq B$. Then

$$C\Gamma B\Gamma C \subseteq C\Gamma M\Gamma C \subseteq C.$$

Thus C is a generalized bi-Γ-ideal of B. Since B is a GB-simple Γ-semiring, we have $B = C$. Hence B is a minimal generalized bi-Γ-ideal of M. ■

Theorem II.15. *Let M be a Γ-semiring having a proper generalized bi-Γ-ideal. Then every proper generalized bi-Γ-ideal of M is minimal if and only if the intersection of any two distinct proper generalized bi-Γ-ideals is empty.*

Proof: Assume that every proper generalized bi-Γ-ideal of M is minimal and let B_1 and B_2 be two distinct proper generalized bi-Γ-ideals of M. By assumption, we have B_1 and B_2 are minimal. We shall show that $B_1 \cap B_2 = \emptyset$. Suppose that $B_1 \cap B_2 \neq \emptyset$. By Lemma II.2, we have $B_1 \cap B_2$ is a proper generalized bi-Γ-ideal of M. Since $B_1 \cap B_2 \subseteq B_1$ and $B_1 \cap B_2 \subseteq B_2$, we get $B_1 \cap B_2 = B_1$ and $B_1 \cap B_2 = B_2$. Thus $B_1 = B_2$ which is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

Conversely, assume that the intersection of any two distinct proper generalized bi-Γ-ideals is empty. Let B be a proper generalized bi-Γ-ideal of M and C a generalized bi-Γ-ideals of M such that $C \subseteq B$. Suppose that $C \neq B$. Then C is a proper generalized bi-Γ-ideal of M. Since $C \subset B$ and by assumption, we have $C = C \cap B = \emptyset$ which is a contradiction. Therefore $C = B$, so B is minimal. ■

ACKNOWLEDGMENT

The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

REFERENCES

[1] R. Chinram, *A note on quasi-ideals in* Γ*-semirings*, International Mathematical Forum **26** (2008), 1253–1259.
[2] Ö. Bektaş, N. Bayrak, and A. Ersoy, *Soft* Γ*-semirings*, arXiv:1202.1496 [math.RA], 7 Feb 2012.
[3] T. K. Dutta and S. K. Sardar, *Semiprime ideals and irreducible ideals of* Γ*-semirings*, Novi Sad Journal of Mathematics **30** (2000), 97–108.
[4] T. K. Dutta, S. K. Sardar, and S. Goswami, *Operations on fuzzy ideals of* Γ*-semirings*, arXiv:1101.4791 [math.RA], 25 Jan 2011.
[5] J. Ghosh and T. K. Samanta, *Fuzzy ideals in* Γ*-semiring*, arXiv:1010.2469 [math.GM], 12 Oct 2010.
[6] R. D. Jagatap and Y. S. Pawar, *Quasi-ideals and minimal quasi-ideals in* Γ*-semirings*, Novi Sad Journal of Mathematics **39** (2009), 79–87.
[7] J. P. Kaushik, M. A. Ansari, and M. R. Khan, *On bi-*Γ*-ideal in* Γ*-semirings*, International Journal of Contemporary Mathematical Sciences **3** (2008), 1255–1260.
[8] S. Lajos, *Notes on generalized bi-ideals in semigroups*, Soochow Journal of Mathematics **10** (1984), 55–59.
[9] S. Pianskool, S. Sangwirotjanapat, and S. Tipyota, *Valuation* Γ*-semirings and valuation* Γ*-ideals*, Thai Journal of Mathematics **Special Issue (Annual Meeting in Mathematics)** (2008), 93–102.
[10] M. Murali Krishna Rao, Γ*-semiring I*, Southeast Asian Bulletin of Mathematics **19** (1995), 49–54.
[11] S. K. Sardar and U. Dasgupta, *On primitive* Γ*-semirings*, Novi Sad Journal of Mathematics **34** (2004), 1–12.
[12] F. A. Szász, *Generalized biideals of rings I*, Mathematische Nachrichten **47** (1970), 355–360.
[13] F. A. Szász, *Generalized biideals of rings II*, Mathematische Nachrichten **47** (1970), 361–364.

Teerayut Chomchuen was born in Nakhon Sawan, Thailand, in 1990. He received his M.S. from University of Phayao, Thailand, under the independent study advisor of Asst. Prof. Dr. Aiyared Iampan.

Aiyared Iampan was born in Nakhon Sawan, Thailand, in 1979. He received his M.S. and Ph.D. from Naresuan University, Thailand, under the thesis advisor of Assoc. Prof. Dr. Manoj Siripitukdet.