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Pseudo-almost periodic solutions of a class of delayed chaotic neural networks

Farouk Cherif

Abstract—This paper is concerned with the existence and uniqueness of pseudo-almost periodic solutions to the chaotic delayed neural networks

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + C\int_{t-\sigma}^{t} f(x(\rho))d\rho + J(t).$$

Under some suitable assumptions on A, B, C, D, J and f, the existence and uniqueness of a pseudo-almost periodic solution to equation above is obtained. The results of this paper are new and they complement previously known results.

Keywords—Chaotic neural network,Hamiltonian systems, Pseudo almost periodic.

I. Introduction

THE human brain is made up of a large number of cells called neurons and their interconnections. An artificial neural network is an information processing system that has certain characteristics in common with biological neural networks. Since the pioneering work on Hopfield neural networks in [13], the investigation of the dynamics of neural networks has been the subject of much recent activity due to their promising potential applications such as signal processing, pattern recognition, optimization and associative memories. Some important results have been reported (see, for example, [6], [19], [20], [21], [22], and the references therein).

In particular, The chaotic neural network shows the more complex spatial-temporal chaotic dynamics compared to the coupled map lattice system in which each lattice site of coupled map lattice systems is only coupled to its nearest ones. Besides the neurons in the chaotic neural networks are connected to each other in whole spatial, the delay feedback of the networks is more complicated than the coupled map lattice system. It is believed that the investigation of the dynamics characters of chaotic neural networks is helpful to an understanding of the memory rules of the brain. For example In [2] Adachi and Aihara have studied in detail the non-periodic associative dynamics of the chaotic neural network. The network can retrieve the stored patterns, but they appear non-periodically since the network is in chaos.

It is well-known, that there exist time delays in the information processing of neurons due to various reasons. For example, time delays can be caused by the finite switching speed of amplifier circuits in neural networks or deliberately introduced to achieve tasks of dealing with motion-related problems, such as moving image processing. Time delays in the neural networks make the dynamic behaviors become more complicated, and may destabilize the stable equilibria and admit almost periodic oscillation, bifurcation and chaos. Therefore, considerable attention has been paid on the study of delay systems in control theory and a large body of work has been reported in the literature (see, for example, [5], [15], [23] and the references therein).

In this paper, motivated by the above discussions, we are concerned with the following delayed chaotic neural networks:

$$\begin{aligned}\dot{x}(t) &= -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) \\ &+C\int_{t-\sigma}^{t} f(x(\rho))d\rho + J(t)\end{aligned} \tag{1}$$

This model have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of periodic and almost periodic solutions of the chaotic neural networks in the literature (see, for example, [7], [8], [9], [11], [12], [14], [16], [17], [18], [24] and the references therein). But, to the best of our knowledge, the pseudo almost periodic oscillatory behavior for the chaotic neural network is never considered. However, in practice, many natural phenomena correspond to the pseudo almost periodic solution of a functional differential equation since the space of pseudo almost periodic functions contains the space of almost periodic functions and the space periodic functions. Hence, the main purpose of this paper is to study the existence and uniqueness of the pseudo almost periodic solution of system (1). To the best of our knowledge, this is the first paper considering the pseudo almost-periodic of chaotic neural network.

The rest of this paper is organized in the following way. In Section 2, we will recall the basic properties of the pseudo almost periodic functions. In section 3 we will introduce some necessary notations, definitions and preliminaries which will be used later. In Section 4, several sufficient conditions are derived for the existence and attractively of pseudo almost periodic solutions of the equation (1) in the suitable convex set of $PAP(\mathbb{R}, \mathbb{R}^n)$.

II. Almost periodic and pseudo almost periodic functions

Throughout this paper, we will use the following concepts and notations. For a vector $v = (v_1, v_2, ..., v_n)^T \in \mathbb{R}^n$, let $\|v\| = \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$ denote the euclidean vector norm, and for a matrix $M \in \mathbb{R}^{n\times n}$, let $\|M\|_2$ indicate the of M induced by the spectrum norm,i.e., $\|M\|_2 = \left(\lambda_{\max}\left(M^T M\right)\right)^{\frac{1}{2}}$, where $\lambda_{\max}(\cdot)$ represents the largest eigenvalue of a matrix. $BC(\mathbb{R}, \mathbb{R}^n)$

F. CHERIF is with the Department of Mathematics., EPAM., Avenue du Marechal Tito. Sousse., TUNISIA e-mail: faroukcheriff@yahoo.fr

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denotes the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{R},\mathbb{R}^n),\|.\|_\infty)$ is a Banach space where $\|.\|_\infty$ denotes the sup norm

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|. \tag{2}$$

Let $f \in BC(\mathbb{R},\mathbb{R}^n)$. We say that f is *almost periodic (Bohr a.p.) or uniformly almost periodic (u.a.p),* when the following property is satisfied: for all $\varepsilon > 0$

$$\exists l_\epsilon > 0, \forall \alpha \in \mathbb{R}, \exists \delta \in [\alpha, \alpha + l_\varepsilon[\,,\ \|f(\cdot+\delta) - f(\cdot)\|_\infty \le \varepsilon.$$

A subset D of $\mathbb{R}$ is called relatively dense in $\mathbb{R}$ when:

$$\exists l > 0, \forall \alpha \in \mathbb{R}, D \cap [\alpha, \alpha + l[\neq \varnothing.$$

And so, introducing the sets $E(f,\varepsilon) := \{r \in \mathbb{R}, \|f(\cdot+r) - f(\cdot)\|_\infty < \varepsilon\}$, we can formulate the definition of the Bohr almost periodicity of $f \in C(\mathbb{R},\mathbb{R}^n)$ in the following manner: for each $\varepsilon > 0$, the set $E(f,\varepsilon)$ is relatively dense in $\mathbb{R}$. An element of $E(f,\varepsilon)$ is called an $\varepsilon-$period of f. Consequently, a Bohr almost periodic function is a continuous function which possesses very much almost periods. We denote by $AP(\mathbb{R},\mathbb{R}^n)$ the set of the Bohr a.p. functions from $\mathbb{R}$ to $\mathbb{R}^n$. It is well-known that the set $AP(\mathbb{R},\mathbb{R}^n)$ is a Banach space with the supremum norm. We refer the reader to ([1], [4], and [10]) for the basic theory of almost periodic functions and their applications.

Besides, the concept of pseudo almost periodicity (pap) was introduced by Zhang (see for example [25] and [26]) in the early nineties It is a natural generalization of the classical almost periodicity. Define the class of functions $PAP_0(\mathbb{R},\mathbb{R}^n)$ as follows:

$$\left\{ f \in BC(\mathbb{R},\mathbb{R}^n) / \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|f(t)\|\, dt = 0 \right\}.$$

A function $f \in BC(\mathbb{R},X)$ is called pseudo almost periodic if it can be expressed as

$$f = h + \varphi,$$

where $h \in AP(\mathbb{R},\mathbb{R}^n)$ and $\varphi \in PAP_0(\mathbb{R},\mathbb{R}^n)$. The collection of such functions will be denoted by $PAP(\mathbb{R},\mathbb{R}^n)$.

The functions h and φ in above definition are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f.

The decomposition given in Definition above is unique.

Observe that $(PAP(\mathbb{R},\mathbb{R}^n),\|.\|_\infty)$ is a Banach space and

$$AP(\mathbb{R},\mathbb{R}^n) \subsetneq PAP(\mathbb{R},\mathbb{R}^n) \subset BC(\mathbb{R},\mathbb{R}^n)$$

since the function $\phi(t) = \cos^2 t + \cos^2\sqrt{3}t + \exp\left(-t^2\cos^2 t\right)$ is pseudo almost periodic function but not almost periodic.

III. Description system and preliminaries

The model of the delayed chaotic neural network considered in this paper is described by the following state equation (1)

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + C\int_{t-\sigma}^{t} f(x(\rho))d\rho + J(t)$$

where $(x_1(t),\cdots,x_n(t))^T \in \mathbb{R}^n$ is the state vector of the network at time t, n corresponds to the number of neurons;

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n \end{pmatrix}, \forall 1 \le i \le n, d_i > 0.$$

$A = (a_{ij})_{1\le i,j\le n}, B = (b_{ij})_{1\le i,j\le n}$ and $C = (c_{ij})_{1\le i,j\le n}$ are the interconnection weight matrices; $f(x(t)) = (f_1(x_1(t)),\cdots,f_n(x_n(t)))^T \in \mathbb{R}^n$ denotes the neuron activation at time $t, J(t) = (J_1(t),\cdots,J_n(t))^T \in \mathbb{R}^n$ is an external input vector function; τ and σ denote the time varying delay and the distributed delay, respectively.

Throughout this paper, we make the following assumptions:

(H_1) The activity function f is assumed to be global Lipschitz continuous, that is, : there exists $L_f > 0$ such that for all $u, v \in \mathbb{R}^n$

$$\|f(u) - f(v)\| < L_f \|u - v\|.$$

Furthermore, we suppose that $f(0) = 0$.

(H_2) $J(\cdot) \in PAP(\mathbb{R},\mathbb{R}^n)$ and τ and $\sigma > 0$.

(H_3) $r = \frac{L_f}{\alpha}\left(\|A\|_2 + \|B\|_2 + \sigma\|C\|_2\right) < 1$.

The class of delayed neural networks unifies several well-known neural networks such as Hopfield neural networks with or without delays and cellular neural networks with or without delays. If the activation function f such that f_i is sigmoid, then Eq. (1) describes the dynamics of Hopfield neural networks. Similarly, if the activation function satisfy $f_i(x_i) = \frac{|x_i+1|-|x_i+1|}{2}$ for all $1 \le i \le n$, then Eq. (1) describes the dynamics of cellular neural networks.

IV. Main results

In this section, we establish some results for the existence, uniqueness of pseudo almost periodic solution of (1).

Lemma 1: If $\varphi \in PAP(\mathbb{R},\mathbb{R}^n)$, then $\varphi(\cdot - h) \in PAP(\mathbb{R},\mathbb{R}^n)$.

Proof: By Definition, we can write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AP(\mathbb{R},\mathbb{R}^n)$ and $\varphi_2 \in AP_0(\mathbb{R},\mathbb{R}^n)$. Obviously,

$$\varphi(\cdot - h) = \varphi_1(\cdot - h) + \varphi_2(\cdot - h).$$

Observe that $\varphi_1(\cdot - h) \in AP(\mathbb{R},\mathbb{R}^n)$ and

$$0 \le \frac{1}{2T}\int_{-T}^{T} \|\varphi_2(t-h)\|\, dt = \frac{1}{2T}\int_{-T-h}^{T-h} \|\varphi_2(t\|\, dt$$
$$\le \frac{2T+2h}{2T}\frac{1}{2T+2h}\int_{-T-h}^{T-h} \|\varphi_2(t)\|\, dt,$$

which implies that $\varphi_2(\cdot\ -h) \in AP_0(\mathbb{R},\mathbb{R}^n)$. So $\varphi(\cdot - h) \in PAP(\mathbb{R},\mathbb{R}^n)$.

Lemma 2: Suppose that assumptions $(H_1),(H_2),(H_3)$ hold and $x(\cdot)) \in PAP(\mathbb{R},\mathbb{R}^n)$ then $\phi : t \longmapsto \int\limits_{t-\sigma}^{t} f(x(\rho))d\rho$ belongs to $PAP(\mathbb{R},\mathbb{R}^n)$.

Proof: By the composition theorem of pseudo-almost periodic functions [3], the functions $\psi : t \longmapsto f(x\,(t))$ belongs to $PAP(\mathbb{R},\mathbb{R}^n)$ whenever $x \in PAP(\mathbb{R},\mathbb{R}^n)$, then ψ can be expressed as

$$\psi = u + v,$$

where $u \in AP(\mathbb{R},\mathbb{R}^n)$ and $v \in PAP_0(\mathbb{R},\mathbb{R}^n)$. Consequently, $\phi\,(t) = \int\limits_{t-\sigma}^{t} u(\rho)d\rho + \int\limits_{t-\sigma}^{t} v(\rho)d\rho = \phi_1\,(t) + \phi_2\,(t)\,.$
Let us prove the almost periodicity of $t \longmapsto (\Gamma h)\,(t)$. For $\varepsilon > 0$, we consider, in view of the almost periodicity of f, a number L_ε such that in any interval $[\alpha, \alpha + L[$ one finds a number δ, with property that:

$$\sup_{t\in\mathbb{R}} \|u\,(\xi+\delta) - u(\xi)\| < \frac{\varepsilon}{\sigma}.$$

Afterwards, we can write:

$$\begin{aligned}
\|\phi_1\,(t+\delta) - \phi_1\,(t)\| &= \left\| \int\limits_{t+\delta-\sigma}^{t+\delta} u(\rho)d\rho - \int\limits_{t-\sigma}^{t} u(\rho)d\rho \right\| \\
&= \left\| \int\limits_{t-\sigma}^{t} u\,(\xi+\delta) - u(\xi)d\xi \right\| \\
&\leq \int\limits_{t-\sigma}^{t} \|u\,(\xi+\delta) - u(\xi)\|\,d\xi \\
&\leq \varepsilon,
\end{aligned}$$

which implies that $\phi_1(\cdot\ ) \in AP(\mathbb{R},\mathbb{R}^n)$. Now, we turn our attention to $\phi_2(\cdot\ )$. First, note that $s \longmapsto \phi_2\,(s)$ is a bounded continuous function. Thus, we have to prove that

$$\lim_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{T} \|\phi_2\,(s)\|\,ds = 0.$$

One has

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{T} \left\| \int\limits_{t-\sigma}^{t} v(\rho)d\rho \right\| dt \\
\leq\ &\lim_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{T} \left(\int\limits_{t-\sigma}^{t} \|v(\rho)\|\,d\rho \right) dt \\
=\ &\lim_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{T} \|v(t)\|\,dt \left(\int\limits_{t-\sigma}^{t} d\rho \right) dt \\
=\ &\lim_{T\to+\infty} \frac{\sigma}{2T} \int\limits_{-T}^{T} \|v(t)\|\,dt \\
=\ &0.
\end{aligned}$$

Theorem 1: Suppose that assumptions $(H_1),(H_3)$ and (H_3) hold. Define the nonlinear operator Γ by: for each $\varphi \in PAP(\mathbb{R},\mathbb{R}^n)$

$$\begin{aligned}
(\Gamma\varphi)(t) \ :\ = &\int\limits_{-\infty}^{t} e^{-(t-s)C}[Af(\varphi\,(s)) + Bf\,(\varphi\,(s-\tau)) \\
&+C \int\limits_{s-\sigma}^{s} f(x(\rho))d\rho + J\,(s)]ds,
\end{aligned}$$

Then Γ maps $PAP(\mathbb{R},\mathbb{R}^n)$ into itself.

Proof: First of all, let us check that Γ is well defined. Indeed, by lemma 1, for all if $\varphi \in PAP(\mathbb{R},\mathbb{R}^n)$ the function $T_h\,(\varphi) = \varphi(\cdot - h) \in PAP(\mathbb{R},\mathbb{R}^n)$. And hence, by the composition theorem of pseudo-almost periodic functions [3], the functions $s \longmapsto Af(\varphi\,(t))$ and $s \longmapsto Bf\,(\varphi\,(t-\tau))$ belong to $PAP(\mathbb{R},\mathbb{R}^n)$ whenever $\varphi \in PAP(\mathbb{R},\mathbb{R}^n)$. Thus,

$$\begin{aligned}
t \ \longmapsto\ &Af(\varphi\,(t)) + Bf\,(\varphi\,(t-\tau)) \\
&+C \int\limits_{t-\sigma}^{t} f(x(\rho))d\rho + J\,(t)
\end{aligned}$$

belongs to $PAP(\mathbb{R},\mathbb{R}^n)$. Consequently the integral (in Riemann's sense)

$$\begin{aligned}
(\Gamma\varphi)(t) \ :\ = &\int\limits_{R}^{t} e^{-(t-s)C}[Af(\varphi\,(s)) + Bf\,(\varphi\,(s-\tau)) \\
&+C \int\limits_{s-\sigma}^{s} f(x(\rho))d\rho + J\,(s)]ds,
\end{aligned}$$

has a sense. The integrand is estimated by

$$\begin{aligned}
\|\ \ &e^{-(t-s)D}(Af(\varphi\,(s)) + Bf\,(\varphi\,(s-\tau)) \\
&+C \int\limits_{s-\sigma}^{s} f(x(\rho))d\rho + J\,(s))\ \| \leq M \left\| e^{-(t-s)D} \right\|
\end{aligned}$$

where

$$\begin{aligned}
M \ =\ &\sup_{s\in\mathbb{R}} \|\ Af(\varphi\,(s)) + Bf\,(\varphi\,(s-\tau)) \\
&+C \int\limits_{s-\sigma}^{s} f(x(\rho))d\rho + J\,(s)\ \| < +\infty.
\end{aligned}$$

Thus, the integral

$$\begin{aligned}
&\int\limits_{-\infty}^{t} e^{-(t-s)C}[Af(\varphi\,(s)) + Bf\,(\varphi\,(s-\tau)) \\
&+C \int\limits_{s-\sigma}^{s} f(x(\rho))d\rho + J\,(s)]ds,
\end{aligned}$$

is absolutely convergent and

$$\|(\Gamma\varphi)(t)\| \leq M \int\limits_{-\infty}^{t} \left\| e^{-(t-s)D} \right\| ds = M \int\limits_{-\infty}^{t} e^{-(t-s)\alpha} ds = \frac{M}{c_j}.$$

where $\alpha = \min_{1\leq k\leq n} d_k$.
Since the function F belongs to $PAP(\mathbb{R},\mathbb{R}^n)$, then F can be expressed as

$$f = h + g,$$

where $h \in AP(\mathbb{R},\mathbb{R}^n)$ and $g \in PAP_0(\mathbb{R},\mathbb{R}^n)$. Consequently, $(\Gamma\varphi) = (\Gamma h) + (\Gamma g)$ where

$$\begin{aligned}(\Gamma h)(t) &= \int_{-\infty}^{t} e^{-(t-s)D}h(s)ds \\ \text{and } (\Gamma g)(t) &= \int_{-\infty}^{t} e^{-(t-s)D}g(s)ds.\end{aligned}$$

Let us prove the almost periodicity of $t \longmapsto (\Gamma h)(t)$. For $\varepsilon > 0$, we consider, in view of the almost periodicity of f, a number $L(\varepsilon\alpha)$ such that in any interval $[\mu, \mu + L[$ one finds a number δ, with property that:

$$\sup_{t\in\mathbb{R}} \|h(\xi+\delta) - h(\xi)\| < \varepsilon\alpha.$$

Afterwards, we can write for $s - \delta = \xi$

$$\begin{aligned}&(\Gamma h)(t+\delta) - (\Gamma h)(t) \\ &= \int_{-\infty}^{t+\tau} e^{-(t+\delta-s)D}h(s)ds - \int_{-\infty}^{t} e^{-(t-s)D}h(s)ds \\ &= \int_{-\infty}^{t} e^{-(t-\xi)D}\left[h(\xi+\delta) - h(\xi)\right]d\xi.\end{aligned}$$

and consequently:

$$\begin{aligned}&\|(\Gamma h)(t+\tau) - (\Gamma h)(t)\| \\ &\leq \int_{-\infty}^{t} \left\|e^{-(t-\xi)D}\right\| \|h(\xi+\tau) - h(\xi)\| d\xi \\ &\leq \varepsilon\alpha \int_{-\infty}^{t} \left\|e^{-(t-\xi)D}\right\| d\xi \leq \varepsilon,\end{aligned}$$

which implies the almost periodicity of (Γh). Now, we turn our attention to (Γg). First, note that $s \longmapsto (\Gamma g)(s)$ is a bounded continuous function. Thus, we have to prove that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|(\Gamma g)(s)\| ds = 0.$$

Clearly,

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|(\Gamma g)(s)\| ds \leq I + K$$

where

$$\begin{aligned}I &= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} dt \left(\int_{-T}^{t} \left\|e^{-(t-s)D}g(s)\right\| ds\right) \\ \text{and } K &= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} dt \left(\int_{-\infty}^{-T} \left\|e^{-(t-s)D}g(s)\right\| ds\right).\end{aligned}$$

It is clear that

$$\begin{aligned}I &= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} dt \left(\int_{-T}^{t} \left\|e^{-(t-s)D}g(s)\right\| ds\right) \\ &\leq \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} dt \left(\int_{-T}^{t} \left\|e^{-(t-s)D}\right\| \|g(s)\| ds\right) \\ &\leq \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|g(t)\| dt \left(\int_{0}^{t+T} e^{-\sigma\alpha}d\sigma\right) \\ &\leq \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|g(t)\| dt \left(\int_{0}^{+\infty} e^{-\sigma\alpha}d\sigma\right) \\ &\leq \lim_{T\to+\infty} \frac{1}{2T\alpha}\int_{-T}^{T} \|g(t)\| dt = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}J &= \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} dt \left(\int_{-\infty}^{-T} \left\|e^{-(t-s)D}g(s)\right\| ds\right) \\ &\leq \lim_{T\to+\infty} \frac{\sup_{t\in\mathbb{R}}\|g(t)\|}{2T}\int_{-T}^{T} dt \left(\int_{T+t}^{+\infty} e^{-\alpha s}ds\right) \\ &\leq \lim_{T\to+\infty} \frac{\sup_{t\in\mathbb{R}}\|g(t)\|}{2T}\int_{-T}^{T} dt \left(\int_{2T}^{+\infty} e^{-\alpha s}ds\right) \\ &\leq \lim_{T\to+\infty} \frac{\sup_{t\in\mathbb{R}}\|g(t)\|}{\alpha}e^{-2\alpha T} \\ &= 0.\end{aligned}$$

Consequently, the function (Γg) belongs to $PAP_0(\mathbb{R},\mathbb{R}^n)$.

Theorem 2: Suppose that assumptions $(H_1) - (H_3)$ hold. Then the delayed chaotic neural networks

$$\begin{aligned}\dot{x}(t) &= -Dx(t) + Af(\varphi(t)) + Bf(\varphi(t-\tau)) \\ &\quad + C\int_{t-\sigma}^{t} f(x(\rho))d\rho + J(t),\end{aligned}$$

has a unique pseudo almost periodic solution in the region

$$\begin{aligned}\mathcal{B} &= B(\varphi_0, r) \\ &= \left\{\varphi \in PAP(\mathbb{R},\mathbb{R}^n), \|\varphi - \varphi_0\| \leq \frac{r\|J\|_\infty}{\alpha(1-r)}\right\},\end{aligned}$$

where

$$\begin{aligned}\varphi_0(t) &= \int_{-\infty}^{t} e^{-(t-s)D} J(s)ds \\ \text{and } r &= \frac{L_f\left(\|A\|_2 + \|B\|_2 + \sigma\|C\|_2\right)}{\alpha}.\end{aligned}$$

Proof: Set

$$\begin{aligned}\mathcal{B} &= B(\varphi_0, r) \\ &= \left\{\varphi \in PAP(\mathbb{R},\mathbb{R}^n), \|\varphi-\varphi_0\| \le \frac{r\|J\|_\infty}{\alpha(1-r)}\right\},\end{aligned}$$

Clearly, $\mathcal{B}$ is a closed convex subset of $PAP(\mathbb{R},\mathbb{R}^n)$ and

$$\begin{aligned}\|\varphi_0(t)\| &= \left\|\int_{-\infty}^{t} e^{-(t-s)D} J(s)ds\right\| \\ &\le \int_{-\infty}^{t} \left\|e^{-(t-s)D}\right\| \|J(s)\| ds \\ &\le e^{-t\alpha}\|J\|_\infty \int_{-\infty}^{t} e^{s\alpha} ds \\ &= \frac{\|J\|_\infty}{\alpha}.\end{aligned}$$

Therefore, for any $\varphi \in \mathcal{B}$ by using the estimate just obtained, we see that

$$\begin{aligned}\|\varphi\| &\le \|\varphi-\varphi_0\| + \|\varphi_0\| \\ &\le \frac{r\|J\|_\infty}{\alpha(1-r)} + \frac{\|J\|_\infty}{\alpha} = \frac{\|J\|_\infty}{\alpha(1-r)}.\end{aligned}$$

Let us prove that the operator Γ is a self-mapping from $\mathcal{B}$ to $\mathcal{B}$. In fact, for any $\varphi \in \mathcal{B}$, we have

$$\begin{aligned}&\|(\Gamma\varphi)(t) - \varphi_0(t)\| \\ &= L_f \frac{\|A\|_2 + \|B\|_2 + \sigma\|C\|_2}{\alpha}\|\varphi\|_\infty \\ &\le L_f \frac{r\|J\|_\infty}{\alpha(1-r)}\end{aligned}$$

which implies that $(\Gamma\varphi) \in \mathcal{B}$. Next, we prove the mapping Γ is a contraction mapping of $\mathcal{B}$. Set

$$\begin{aligned}F(s,\varphi(s)) &= Af(\varphi(s)) + Bf(\varphi(s-\tau)) \\ &\quad + C\int_{s-\sigma}^{s} f(x(\rho))d\rho + J(s).\end{aligned}$$

In view of (H_2), for any $\varphi, \psi \in \mathcal{B}$, we have

$$\begin{aligned}&\|(\Gamma\varphi)(t) - (\Gamma\psi)(t)\| \\ &= \int_{-\infty}^{t} e^{-(t-s)D}\left[F(s,\varphi(s)) - F(s,\psi(s))\right] ds \\ &\le \int_{-\infty}^{t} \left\|e^{-(t-s)D}\right\| \|F(s,\varphi(s)) - F(s,\psi(s))\| ds \\ &\le \int_{-\infty}^{t} e^{-(t-s)\alpha} \|A\left[f(\varphi(s)) - f(\psi(s))\right]\| ds \\ &+ \int_{-\infty}^{t} e^{-(t-s)\alpha} \|B\left[f(\varphi(s-\tau(s))) - f(\psi(s-\tau(s)))\right]\| ds \\ &+ \int_{-\infty}^{t} \left\|C\int_{s-\sigma}^{s}\left[f(\varphi(\rho)) - f(\psi(\rho))\right] d\rho\right\| ds \\ &\le \frac{L_f}{\alpha}\left(\|A\|_2 + \|B\|_2 + \sigma\|C\|_2\right)\|\varphi-\psi\|_\infty \\ &\le r\|\varphi-\psi\|_\infty.\end{aligned}$$

Since, by (H_3) , $r < 1$, then Γ is a contraction mapping. Consequently, Γ possess a unique fixed point $\varphi_* \in \mathcal{B}$ that is $\Gamma(\varphi_*) = \varphi_*$. Hence, φ_* is the unique pseudo almost periodic solution of (1) in $\mathcal{B}$. Then Banach's fixed point theorem yields that Γ has a unique fixed point in $\mathcal{B} \subset PAP(\mathbb{R},\mathbb{R}^n)$. ■

V. Conclusion

In this paper, some novel sufficient conditions are presented ensuring the existence and uniqueness of the pseudo almost periodic solution for delayed chaotic neural networks. All criteria are found without assuming the networks have almost periodic or pseudo almost periodic activation functions. The only restriction for activation function is the lipschitz property. All criteria can be easily adapted to many classes of recurrent neural networks such as Hopfield neural networks, cellular neural networks with log-sigmoid, saturate linear or triangular basis, activation functions etc. Hence our results obtained extend and improve existing ones.

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