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Mechanical Quadrature Methods for Solving First Kind Boundary Integral Equations of Stationary Stokes Problem

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Abstract—By means of Sidi-Israeli's quadrature rules, mechanical quadrature methods (MQMs) for solving the first kind boundary integral equations (BIEs) of steady state Stokes problem are presented. The convergence of numerical solutions by MQMs is proved based on Anselone's collective compact and asymptotical compact theory, and the asymptotic expansions with the odd powers of the errors are provided, which implies that the accuracy of the approximations by MQMs possesses high accuracy order $O(h^3)$. Finally, the numerical examples show the efficiency of our methods.

Keywords—Stokes problem; boundary integral equation; mechanical quadrature methods; asymptotic expansions.

I. Introduction

Consider the following plane Stokes equation

$$\begin{cases} -\nu\Delta u + \text{grad}p = 0, & \text{in } \Omega\cup\Omega', \\ \text{div } u = 0, & \text{in } \Omega\cup\Omega', \\ u = u_0, & \text{on } \Gamma, \end{cases} \quad (1)$$

where ν is viscosity, Ω is a bounded domain with the boundary Γ. Compared with the other methods including the finite difference method [16] and finite element method [14, 15] for solving eq. (1), the boundary element method [7, 10, 11] which turn eq. (1) into a boundary integral equation is of advantage in the following aspects: (a) The dimensions are decreased; (b) The trouble in numerical process for diverse equation is avoided; (c) Outward problem is easy to deal. Up to now there have been various methods to make (1) a boundary integral equations. In this paper, the reformulated problem becomes the first kind boundary integral equation (BIE) with a logarithmic function. First, we solved density function $t = t_1,\ t_2$ and constant $c_1,\ c_2$ satisfy integral equations

$$u_{ok}(x) = \Sigma_{i=1}^2 \int_\Gamma U_{ik}(x-y)t_i(y)ds_y + c_k, \quad x\in\Gamma, \quad k=1,2, \quad (2)$$

and constraint conditions

$$\int_\Gamma t_i(y)ds_y = 0, \quad i=1,2. \quad (3)$$

Secondly, by the values of $t_i,\ c_i (i=1,2)$, we can computing integrals

$$\begin{cases} u_k(x) = \Sigma_{i=1}^2 \int_\Gamma U_{ik}(x-y)t_i(y)ds_y + c_k, \quad x\in\Re^2\backslash\Gamma, \\ p(x) = \Sigma_{i=1}^2 \int_\Gamma P_i(x-y)t_i(y)ds_y, \ k=1,2, \ x\in\Re^2\backslash\Gamma, \end{cases} \quad (4)$$

where

$$\begin{cases} U_{ik}(x) = \frac{1}{4\pi\nu}[\delta_{ik}\ln(1/|x-y|) + \frac{(x_i-y_i)(x_k-y_k)}{|x-y|^2}], \\ P_i = (x_i-y_i)/(2\pi|x-y|^2), i,k=1,2 \end{cases} \quad (5)$$

are fundamental solutions[1]. Notice that the solution of equations (2) and (3) are not unique, because the unit normal vector $n = (n_1, n_2)$ satisfies

$$\begin{cases} \sum_{i=1}^2 \int_\Gamma U_{ik}(x-y)n_i(y)ds_y = \int_\Omega \text{div}U_k dy = 0, \\ \int_\Gamma n_i(y)ds_y = \int_\Gamma e_i\cdot n ds_y = \int_\Omega \text{div}e_i dy = 0, i=1,2, \end{cases}$$

where e_i is the unit vector of y_i axis. Then, in order to obtain a unique solution, we add a constraint condition

$$\langle t, n\rangle = \sum_{i=1}^2 \int_\Gamma t_i(y)n_i(y)ds_y = 0 \quad (6)$$

Kress R.[6] has already proved the solution of equation (1) is unique, when (2), (3) and (6) are satisfied.

Due to the difficulties in theory , all the numerical methods except Galerkin method [17] were unable to discuss the convergence of numerical methods for computing (2), (3) and (6). Since the discrete matrix is full, we have to calculate a double integral for each entry in it by Galerkin method, which increase the computation cost [8]. Obviously, the entries of discrete matrices of the MQMs are explicit in computation, without any singular integral [11]. However, the analysis of the MQMs is more difficult than that of Galerkin and collocation methods [9, 17, 18], because it is no longer within the framework of projection theory. In this paper, we make use of Sidi's quadrature rules [5] to compute weakly singular and singular integral. Using Anselone' asymptotically compact theory theorem [13], the existence, the uniqueness, and the convergence and the error estimation with $O(h^3)$ of the discrete equations are shown. Some numerical examples are provided to illustrate the features of the methods discussed in this paper.

This paper is organized as follows: in Section II, we present the MQMs, and prove the convergence of MQMs. in Section III, we provide the asymptotic expansion of errors.

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Two numerical examples are provided to verify the theoretical results in Section IV, and some useful conclusions are listed in Section V.

II. MECHANICAL QUADRATURE METHODS

Assume that Γ is a smooth closed curve described by the parameter mapping: $x(s) = (x_1(s),\ x_2(s)) : [0, 2\pi] \rightarrow \Gamma$ with $|x^{'}(s)| = [|x_1(s)|^2 + |x_2(s)|^2]^{1/2} > 0,\ x_i(s) \in \tilde{C}^m[0, 2\pi],\ i = 1,\ 2$. Let $\tilde{C}^m[0, 2\pi]$ denote the set of m times differentiable periodic functions with periodic 2π. Define the boundary integral operators on $\tilde{C}^m[0, 2\pi]$

$$(A_0 v)(s) = -\frac{1}{4\pi\nu}\int_0^{2\pi} \ln|2e^{-1/2}\sin\frac{s-\tau}{2}| \cdot v(\tau) \cdot |x^{'}(\tau)| d\tau, \tag{7}$$

and

$$\begin{aligned}(Av)(s) &= -\frac{1}{4\pi\nu}\int_0^{2\pi} \ln|x(s) - x(\tau)| \cdot v(\tau) \cdot |x^{'}(\tau)| d\tau \\ &= (A_0 v)(s) + (A_1 v)(s),\end{aligned} \tag{8}$$

where

$$(A_1 v)(s) = -\frac{1}{4\pi\nu}\int_0^{2\pi} \ln\frac{|x(s) - x(\tau)|}{|2e^{-1/2}\sin\frac{s-\tau}{2}|} \cdot v(\tau) \cdot |x^{'}(\tau)| d\tau, \tag{9}$$

$$(K_i v)(s) = \frac{1}{4\pi\nu}\int_0^{2\pi} k_i(s,\tau) v(\tau) d\tau, \quad i = 0,\ 1,\ 2 \tag{10}$$

with

$$\begin{cases} k_0(s,\tau) = \frac{(x_1(s)-x_1(\tau))(x_2(s)-x_2(\tau))}{(x_1(s)-x_1(\tau))^2+(x_2(s)-x_2(\tau))^2}|x^{'}(\tau)|, \\ k_i(s,\tau) = \frac{(x_i(s)-x_i(\tau))^2}{(x_1(s)-x_1(\tau))^2+(x_2(s)-x_2(\tau))^2}|x^{'}(\tau)|, \quad i = 1,\ 2 \end{cases} \tag{11}$$

Evidently, both the kernels of operators A_1 and K_i ($i = 0,\ 1,\ 2$) are smooth kernel functions. However, the kernel function of A_0 has logarithmic singularity. Assume that $u \in \tilde{C}[0, 2\pi]$, we have

$$l(u, \cdot)v = l(u, v) = \int_0^{2\pi} u(\tau) v(\tau) \cdot |x(\tau)| d\tau, \tag{12}$$

simply denoted by $l(u, \cdot)$. Let $A_1 + K_1 = K_3$ and $A_1 + K_2 = K_4$. Combing with lagrange multiplier, the integral equations (2), (3) and (6) can describe be block construct pattern

$$\begin{bmatrix} A_0 + K_3 & K_0 & 1 & 0 & n_1 \\ K_0 & A_0 + K_4 & 0 & 1 & n_2 \\ l(1,\cdot) & 0 & 0 & 0 & 0 \\ 0 & l(1,\cdot) & 0 & 0 & 0 \\ l(n_1,\cdot) & l(n_2,\cdot) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ c_1 \\ c_2 \\ \mu \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13}$$

where $g_i(s) = u_{0i}(x_1(s), x_2(s)),\ t_i(s) = t_i(x_1(s), x_2(s))$ with i=1,2 and c_1, c_2 and μ are real number. The eq. (13) is an operator equation from $V = \tilde{C}^m[0, 2\pi] \times \tilde{C}^m[0, 2\pi] \times \Re^3$ to V. Let $\tilde{A} = diag(A_0, A_0, 1, 1, 1)$ be an diagonal operator. Hence, eq.(13) is equivalent to

$$\tilde{D}w = (\tilde{A} + \tilde{B}) = g \tag{14}$$

with $w = (t_1, t_2, c_1, c_2, \mu)^T$, $g = (g_1, g_2, 0, 0, 0)^T$. Applying $\tilde{A}^{-1}$ to both side of eq. (14), then we have

$$(I + \tilde{A}^{-1}\tilde{B})w = \tilde{A}^{-1}g = f, \tag{15}$$

where

$$\tilde{A}^{-1}\tilde{B} = \begin{bmatrix} A_0^{-1}K_3 & A_0^{-1}K_0 & A_0^{-1} & 0 & A_0^{-1}n_1 \\ A_0^{-1}K_0 & A_0^{-1}K_4 & 0 & A_0^{-1} & A_0^{-1}n_2 \\ l(1,\cdot) & 0 & -1 & 0 & 0 \\ 0 & l(1,\cdot) & 0 & -1 & 0 \\ l(n_1,\cdot) & l(n_2,\cdot) & 0 & 0 & -1 \end{bmatrix}. \tag{16}$$

Take the mesh width $h = 2\pi/m,\ \tau_i = ih\ (i = 0, \cdots, m-1)$, we know that kernel $k_j(s,\tau)\ (j = 0,\ 3,\ 4)$ of operator K_j is smooth periodic function. By the trapezoidal and quadrature rule [2,3], we construct a high accuracy approximate Nyström operator. Let $v \in \tilde{C}[0, 2\pi]$, we have

$$(K_{jm} v)(s) = h\sum_{i=0}^{m-1} k_j(s, \tau_i) v(\tau_i), \quad j = 0, 3, 4. \tag{17}$$

Since the kernel of A_0 is of logarithmic singularity, we use Side-Israeli's quadrature rules [12] to construct the Nyström approximate operator

$$\begin{aligned}(A_{0m} v)(s) = -\frac{h}{4\pi\nu}|\sum_{i=0,\tau_i\neq s}^{m-1} \ln|2e^{-1/2}\sin\frac{s-\tau_i}{2}| \\ \cdot v(\tau_i) \cdot |x^{'}(\tau_i)| + \ln(\frac{h}{2\pi}e^{-1/2}) \cdot v(s) \cdot |x^{'}(s)||.\end{aligned} \tag{18}$$

Let $s = \tau_j\ (j = 0, \cdots, m-1)$ in eqs. (17) and (18), the approximate equation of eq. (15) is a system of linear equations with $2m + 3$ unknowns, which is

$$\bar{D}_m w^m = (\bar{A}_m + \bar{B}_m) w^m = g^m, \tag{19}$$

where $w^m = (t_1^m(\tau_0), \cdots, t_1^m(\tau_{m-1}), t_2^m(\tau_0), \cdots, t_2^m(\tau_{m-1}), c_1^m, c_2^m, \mu^m)^T$ and $g^m = (g_1^m(\tau_0), \cdots, g_2^m(\tau_0), \cdots, 0, 0, 0)^T$ are column vectors of $2m + 3$ dimensions and $\bar{D}_m \in C^{(2m+3),(2m+3)}$. Once w^m is solved by (19), by the quadrature rule we obtain

$$\begin{cases} \bar{U}_k(x) = h\sum_{i=1}^2\sum_{j=0}^{m-1} U_{ik}(x - y(\tau_i)) \cdot t_i^m(\tau_j) \cdot |x^{'}(\tau_j)|, \\ p(x) = h\sum_{i=1}^2\sum_{j=0}^{m-1} P_i(x - y(\tau_i)) \cdot t_i^m(\tau_j) \cdot |x^{'}(\tau_i)|, \end{cases} \tag{20}$$

with $U_k(x) = \bar{U}_k(x) + c_k^m$ and $y(\tau_j) = (x_1(\tau_i), x_2(\tau_j))$, $x \in \Re^2\backslash\Gamma$, by using (4), we can obtain the solution of eq. (1).

Remarks 1: The entries of $\bar{D}_m$ of the MQMs are explicit in computation according to formula (17) and (18), without any singular integral. So this method reduce the computation cost.

III. ESTIMATION AND ASYMPTOTIC EXPANSIONS OF ERRORS

Eq. (19) is equivalent to

$$(I + \bar{A}_m^{-1}\bar{B}_m) w^m = \bar{A}_m^{-1} g^m = f^m, \tag{21}$$

where $\bar{A}_m^{-1}\bar{B}_m$ is equivalent to

$$\begin{bmatrix} A_{0m}^{-1}K_{3m} & A_{0m}^{-1}K_{0m} & A_{0m}^{-1} & 0 & A_{0m}^{-1}n_1 \\ A_{0m}^{-1}K_{0m} & A_{0m}^{-1}K_{4m} & 0 & A_{0m}^{-1} & A_{0m}^{-1}n_2 \\ l_m(1,\cdot) & 0 & -1 & 0 & 0 \\ 0 & l_m(1,\cdot) & 0 & -1 & 0 \\ l_m(n_1,\cdot) & l_m(n_2,\cdot) & 0 & 0 & -1 \end{bmatrix}, \quad (22)$$

with $n_i = (n_i(\tau_0), \cdots, n_i(\tau_{m-1}))^T,\ i = 1, 2.\ A_{0m},\ K_{jm} \in C^{m,m}$. Define

$$l_m(u, \nu) = h\sum_{i=0}^{m-1} u(\tau_i)v(\tau_i)|x^{'}(\tau_i)| \quad (23)$$

discrete inner product which is denoted by $l_m(u, \nu)$ for convenience. In order to discuss the convergence of numerical solution of eq. (21), we introduce two mappings. One is R_m: $\Re^m \longrightarrow S_m$ satisfying

$$R_m Z = \sum_{i=0}^{m-1} Z_i e_i(t), \quad \forall\ Z \in \Re^m \quad (24)$$

where S_m is a continue piecewise linear function subspace and $e_i(t)$ are the basis functions of S_m, which satisfy $e_i(\tau_j) = \delta_{ij},\ i, j = 0, \cdots, m-1$. Denoted by $\tilde{R}_m : \Re^{2m+3} \to S_m \times S_m \times \Re^3$ its prolongation operator, which is

$$\tilde{R}_m(u, \nu, c_1, c_2, c_3)^T = (R_m u, R_m \nu, c_1, c_2, c_3)^T,$$

with $u, \nu \in \Re^m,\ c_i \in \Re,\ i = 1, 2, 3$.

Another mapping is $I_m : C[0, 2\pi] \to \Re^m$ satisfying

$$I_m v = \nu = (v(\tau_0), \cdots, v(\tau_{m-1}))^T. \quad (25)$$

Similarly, denoted by $\tilde{I}_m : \Re^{2m+3} \to V$ its prolongation operator, which is $\tilde{I}_m(u, v, c_1, c_2, c_3) = (I_m u, I_m v, c_1, c_2, c_3)$ with $V = C[0, 2\pi] \times C[0, 2\pi] \times \Re^3$. Consider the following operator equation

$$(I + \hat{A}_m^{-1}\hat{B}_m)w_m = \tilde{R}_m f^m, \quad (26)$$

where $\hat{A}_m^{-1}\hat{B}_m$ has an analogy construction with $\bar{A}_m^{-1}\bar{B}_m$, just by replacing A_{0m}^{-1} and $K_{km}, (k = 0, 3, 4)$ with $R_m A_{0m}^{-1}$ and $I_m K_{km}$ in (22), respectively.

Obviously, if w^m is a solution of (21), then $\tilde{R}_m w^m$ is a solution of (26) and vice versa. So the convergence of approximate solution can be ascribed to prove $\hat{A}_m^{-1}\hat{B}_m$ is collectively compact convergent to $\tilde{A}^{-1}\tilde{B}$. Now we first recall the following lemma from [4].

Lemma 1[4] *Suppose that A_0 is an integral operator of eq. (7), K_0 is also an integral operator with the smooth kernel function, if kernel function $k_0(s, \tau) \in \tilde{C}^3([0, 2\pi]^2)$, then*

$$R_m A_{0m}^{-1} I_m K_{0m} \overset{c.c}{\to} A_0^{-1}K_0$$

where $\overset{c.c}{\to}$ denotes the collectively compact convergence.

Remarks 2: There are some difficulties in proving this lemma. The main work is to estimate the upper and lower bound of the eigenvalues of A_{0m}. Since A_{0m} is a symmetric circulant matrix, by means of circulant matrix theory we obtained $\lambda_j > \frac{1}{2j},\ j = 1, \cdots, m-1$ of A_{0m}. Therefore, the inverse of A_{0m} is existence and $||A_{0m}^{-1}|| = O(m)$[4]. Finally, by the above estimation, and Sidi's quadrature rule and collectively compact operator theory, the lemma is proved.

Theorem 2 *Assume Γ is a smooth curve. Then the operator sequence $\hat{A}_m^{-1}\hat{B}_m$ is collectively compact convergent to $\tilde{A}^{-1}\tilde{B}$ in V. That is, we have*

$$\hat{A}_m^{-1}\hat{B}_m \overset{c.c}{\to} \tilde{A}^{-1}\tilde{B}.$$

Proof: First, we prove $\hat{A}_m^{-1}\hat{B}_m$ is of collectively compact operator sequence. Choose an arbitrary sequence $Z_m \subset V$, $Z_m = (Z_{1m}, Z_{2m}, c_{1m}, c_{2m}, c_{3m})^T$. Then there exists a convergent subsequence $\hat{A}_m^{-1}\hat{B}_m Z_m$. In fact, consider the first complement

$$\begin{aligned} &R_m A_{0m}^{-1} I_m K_{0m} Z_{1m} + R_m A_{0m}^{-1} I_m K_{3m} Z_{2m} \\ &+ c_{1m} R_m A_{0m}^{-1} I_m I + c_{3m} R_m A_{0m}^{-1} I_m n_1, \end{aligned} \quad (27)$$

of $\hat{A}_m^{-1}\hat{B}_m Z_m$. From Lemma 1 we have

$$R_m A_{0m}^{-1} I_m K_{im} \overset{c.c}{\to} A_0^{-1}K_i, \quad i = 0,\ 3,\ 4 \quad (28)$$

and

$$R_m A_{0m}^{-1} I_m n_i \overset{c.c}{\to} A_0^{-1}n_i, \quad i = 1,\ 2, \quad (29)$$

since $\{c_{im}, i = 1, 2, 3\}$ is a bounded sequence of $\Re$, there exist a infinite subsequence $\{m_1\}$ of inferior index sequence $\{m\}$, with regard to (27) which is a convergence sequence. Following the above arguments, we can also find an infinite subsequence $\{m_5\} \subset \{m_4\} \subset \cdots \{m_1\} \subset \{m\}$ such that $\hat{A}_m^{-1}\hat{B}_m$ is a convergent sequence in V. Obviously, this implies the pointwise convergence, i.e.,

$$\hat{A}_m^{-1}\hat{B}_m \overset{p}{\to} \tilde{A}^{-1}\tilde{B}.$$

We completed the proof of Theorem 2.

■

Corollary 3 *When m is sufficiently large, there exists a unique solution w_m of (26) such that*

$$\|w_m - w\| \le \|(I+\tilde{L})^{-1}\| \frac{\|(\hat{L}_m - \tilde{L})f\| + \|(\hat{L}_m - \tilde{L})\hat{L}_m w\|}{1 - \|(I - \hat{L}_m)^{-1}(\hat{L}_m - \tilde{L})\tilde{L}_m\|}, \quad (30)$$

where $\|\cdot\|$ is the norm of V space. $\tilde{L} = \tilde{A}^{-1}\tilde{B},\ \hat{L}_M = \hat{A}_m^{-1}\hat{B}_m$.

Remarks 3: According to the collectively compact convergent theory, we can immediacy deduce Corollary 3.

Theorem 4 *Suppose that $\Gamma \in C^{2m+3},\ u_0 \in \tilde{C}^{2m+2}(\Gamma) \times \tilde{C}^{2m+2}(\Gamma)$. Then there exists a vector function $\varphi_i \in V$ independent of h such that the following asymptotic expansions hold*

$$w_m(s) - w(s) = \sum_{k=1}^{m-1} h^{2k+1}\varphi_k(s) + O(h^{2m+1}), \quad s \in \tau_j. \quad (31)$$

where $w(s)$ and $w_m(s)$ be solutions of equations (15) and (26), respectively.

Proof: Based on the above assumptions, n_1, n_2, t_1, t_2 are in $\tilde{C}^{2m+3}[0, 2\pi]$. Let $\varepsilon = w_m - w$, we have

$$((I + \hat{L}_m)\varepsilon)(s) = ((\hat{L}_m - \tilde{L})w)(s), \quad \forall s \in \tau_i. \tag{32}$$

By the proved result of asymptotic expansions of errors of the first kind boundary integral equation (BIE), for $\forall v \in \tilde{C}^{2m+2}[0, 2\pi]$, we obtained

$$((R_m A_{0m}^{-1} I_m K_{im} - A_0^{-1} K_i)v)(s) = \sum_{k=1}^{m-1} h^{2k+1}\psi_{ik}(s) + O(h^{2m+2}), \quad s \in \tau_j,\ i = 0,\ 3,\ 4, \tag{33}$$

where $\psi_{ik} \in \tilde{C}^{2m+2-k}[0, 2\pi]$ are independent of h. Similarly, we also have

$$((R_m A_{0m}^{-1} I_m - A_0^{-1})n_i)(s) = \sum_{k=1}^{m-1} h^{2k+1}\xi_{ik}(s) + O(h^{2m+2}), \quad s \in \tau_j,\ i = 1,\ 2, \tag{34}$$

where $\xi_{ik} \in \tilde{C}^{2m+2-k}[0, 2\pi]$ are independent of h. Besides, according to the estimation of the convergence for the periodic functions obtained by trapezoidal rule, we know

$$l(1, n_i) - l_m(1, n_i) = O(h^{2m+2}),\ i = 1,\ 2, \tag{35}$$

$$l(n_i, t_i) - l_m(n_i, t_i) = O(h^{2m+2}),\ i = 1,\ 2. \tag{36}$$

From $\tilde{L}$ and $\hat{L}_m$, by (33)−(36), we know there exists a vector function $\psi_k \in (C^{2m+2-k}[0, 2\pi])^2 \times \Re^2$ which is independent of h and satisfies

$$((I + \hat{L}^m)\varepsilon)(s) = \sum_{k=1}^{m-1} h^{2k+1}\Psi_k(s) + O(h^{2m+2}), \quad s \in \tau_j. \tag{37}$$

According to Theorem 2, we know

$$\|\varepsilon\| = O(h^3). \tag{38}$$

Define φ_3 and φ_{3m} be solutions of equations

$$(I + \tilde{L})\varphi_3 = \Psi_3, \tag{39}$$

and

$$(I + \hat{L}_m)\varphi_{3m} = \tilde{R}_m\Psi_3, \tag{40}$$

respectively. We obtain the following result by estimating (40) again

$$(w_m - w - h^3\varphi_3)(s) = O(h^5), \quad s \in \tau_j \tag{41}$$

By mathematical induction, we can obtain result of Theorem 4. Hence, the proof of (31) is completed. ■

Remarks 4: From Theorem 4 and (31), we can easily obtain the following error ratio

$$log_2|\frac{w_m(s) - w(s)}{w_{2m}(s) - w(s)}| \approx 3. \tag{42}$$

Remarks 5: When the boundary Γ are polygons, we can obtain the same convergence and error ratio results similar to the case of closed smooth boundary.

TABLE I
e BY MQMS AND GALERKIN [1]

$r \setminus e \setminus h$	1/7	1/14	1/64 GM [1]
4	9.81E-4	1.09E-4	1.2E-3
8	4.60E-4	5.50E-5	1.0E-4
16	2.22E-4	2.75E-5	5.0E-5

TABLE II
THE ERRORS OF u AT THE POINT $(0.15,\ 1.15)$

Λ^k	Λ^3	Λ^4	Λ^5	Λ^6	Λ^7
e_k^u	4.062E-3	1.199E-4	1.407E-5	1.757E-6	2.195E-7
r_k^u	−	$2^{5.081}$	$2^{3.092}$	$2^{3.003}$	$2^{3.000}$

TABLE III
THE ERRORS OF u AT THE POINT $(0.35,\ 1.35)$

Λ^k	Λ^3	Λ^4	Λ^5	Λ^6	Λ^7
e_k^u	7.835E-4	1.670E-4	2.078E-5	2.593E-6	3.239E-7
r_k^u	−	$2^{2.231}$	$2^{3.006}$	$2^{3.002}$	$2^{3.001}$

TABLE IV
THE ERRORS OF p AT THE POINT $(0.15,\ 1.15)$

Λ^k	Λ^3	Λ^4	Λ^5	Λ^6	Λ^7
e_k^p	3.766E-3	1.453E-4	1.581E-5	1.973E-6	2.465E-7
r_k^p	−	$2^{4.696}$	$2^{3.200}$	$2^{3.003}$	$2^{3.001}$

IV. NUMERICAL EXAMPLES

Example 1. Consider (1), where an infinitely long circle cylinder Ω with radius $\tilde{r}$ rotates round a central axes at an invariably angular velocity ϑ, and $u_0 = (-y, x)$, $(x, y) \in \Gamma$, and exact solution $u = \nu\tilde{r}^2/r$ with $\nu = 1$ and $\tilde{r} = 1$. In Table I, we list the errors e in the second and thirth column by MQMs, and the errors e in the fourth column by Galerkin method [1].

Obviously, from Table I, we can see numerically $\frac{e_n|_{n=7}}{e_n|_{n=14}} \approx 8$, to agree with (42) very well.

Example 2 (see [11]). Consider (1), where $\Omega = (0, 1) \times (1, 2)$ with the edge $\Gamma = \cup_{m=1}^{4}\Gamma_m$, where $\Gamma_1 = \{(x_1, 1) :\ 0 \le x_1 \le 1\}$, $\Gamma_2 = \{(1, x_2 + 1) :\ 0 \le x_2 \le 1\}$, $\Gamma_3 = \{(x_1, 2) :\ 0 \le x_1 \le 1\}$, and $\Gamma_4 = \{(0, x_2 + 1) :\ 0 \le x_2 \le 1\}$. The Dirichlet condition $u_0 = (0, x_1(x_1 - 1)$, $(x_1, x_2) \in \Gamma$. The exact solution of (1) is $u = (0, x_1(x_1 - 1))$ and $p = 2\nu x_2$,where $\nu = 1$.

Let $e_k^u = |u_{exact} - u_{\Lambda^k}|$, $e_k^p = |p_{exact} - p_{\Lambda_k}|$, $r_k^u = e_k^u/e_{k-1}^u$, and $r_k^p = e_k^p/e_{k-1}^p$, where $k = 3, \cdots, 7$. Let Λ^k denote $(2^k,\ 2^k,\ 2^k,\ 2^k)$, where $(2^k,\ 2^k,\ 2^k,\ 2^k)$ $(k = 3, \cdots, 7)$ represents the piecewised boundary node number set of the boundary $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$. The errors and error ratio of u and p at the interior points $(0.15,\ 1.15)$, and $(0.35,\ 1.35)$ using $n\ (= 4 \times 2^k,\ k = 3, \cdots, 7)$ nodes by MQMs are listed in Table II-V respectively. From the numerical results we can see that $\log_2 r_k^u \approx 3$ and $\log_2 r_k^p \approx 3$, which mean that the convergence rates of u and p are $O\ (h^3)$ for MQMs.

TABLE V
THE ERRORS OF p AT THE POINT $(0.35,\ 1.35)$

Λ^k	Λ^3	Λ^4	Λ^5	Λ^6	Λ^7
e_k^p	6.973E-4	1.723E-4	2.134E-5	2.662E-6	3.326E-7
r_k^p	–	$2^{2.017}$	$2^{3.013}$	$2^{3.003}$	$2^{3.001}$

V. CONCLUSIONS

In this article, we construct MQMs for solving Stokes equations BIE by using sidi's quadrature rules to compute weakly singular integrals and prove that the methods is convergent. The calculation of the discrete matrix costs very little and the most of work can be saved.

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