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# Positive solutions for three-point boundary value problems of third-order nonlinear singular differential equations in Banach space

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*Abstract*—In this paper, by constructing a special set and utilizing fixed point index theory, we study the existence of solution for singular differential equation in Banach space, which improved and generalize the result of related paper.

*Keywords*—Banach space, cone, fixed point index, singular differential equation.

## I. Introduction

THE singular differential equation arises in a variety of applied mathematics and physics, the theory of singular differential equation is emerging as an important area of investigation since it is much richer than the corresponding theory of concerning equation without singular. The beam of Sandwich

$$\begin{cases} x'''(t) - \lambda f(t, x(t)) = 0, & t \in (0,1), \\ x(0) = x'(\eta) = x''(1) = 0, \end{cases}$$

is a singular problem in special exogenic action. In recent years, some new results concerning the three-point boundary value problems of three-order nonlinear singular differential equations have been obtained by a variety of method (see$[1-4]$). In 1998, D.Anderson[5] got the existence of solution when $f(t,x) = f(x)$ and $f : [0,+\infty) \to [0,+\infty)$. In 2003, yao[6] got the existence of at least one positive solution when $f(t,x)$ is semipositone and superlinear. However the thesis above mentioned are all not consider the case of singularity of $f(t,x)$. In 2004, Yu [7] got the existence of multi-positive solution when $\lambda = 1$ and $f$ is super linearity and inferior linearity in real space. Motivated by the work of thesis [8], the present paper investigates the existence of positive solution for a class of three-point boundary value problems of three-order nonlinear singular differential equations in Banach Space. Compared with the paper above mentioned, this paper has different characters. Firstly, the result is more generally. Secondly, our approaches are method of fixed point theory and a new constructed cone, this is different with thesis above mentioned completely. Lastly, we obtained the result in abstract space. The organization of this paper is as follows, we shall introduce some definitions and lemmas in the rest of this section. The main result will be stated and proved in section 2.

Suppose $(E, \|.\|)$ is a Banach space, $I = [0,1]$, $J = (0,1)$, $P$ is a normal cone in $E$, let the normal constant be $N$, $P^*$ is a dual cone of $P$, the partial order induced by cone $P$ in $E$ is $\leq$: $x \leq y \Leftrightarrow y - x \in P$, we consider the following problem

$$\begin{cases} x'''(t) + a(t)f(x(t)) = \theta, & t \in (0,1), \\ x(0) = x'(\eta) = x''(1) = \theta, \frac{1}{2} < \eta < 1, \end{cases} \quad (1)$$

where $a(t) \in C([0,1],[0,+\infty))$, for any small subinterval $[\alpha,\beta] \subset I$, $a(t) \neq 0$, $\theta$ is zero element in $E$, $f(x)$ may be singular at $x = 0$.

We consider problem (1) in $C[I,E]$. For any $x \in C[I,E]$, let $\|x\|_c = \max_{t\in J} \|x(t)\|$, then $(C[I,E], \|.\|_c)$ is a Banach space. A map $x \in C[I,E] \bigcap C^1[(0,1],E] \bigcap C^2[(0,1],E] \bigcap C^3[J,E]$ is called a solution of (1.1) if it satisfies all equations of (1); if $x(t) > \theta, t \in (0,1)$, we call $x$ is positive solution.

We denote $\alpha$ the $Kuratowski$ noncompactness measure, $\alpha(.)$ and $\alpha_c(.)$ are the $Kuratowski$ noncompactness measure in $E$ and $C[I,E]$ respectively.

Let $G(t,s)$ be the Green function of the following equation:

$$\begin{cases} x'''(t) = 0, & t \in (0,1), \\ x(0) = x'(\eta) = x''(1) = 0, \end{cases}$$

then

$$G(t,s) = \begin{cases} ts - \frac{1}{2}t^2, & 0 \leq s \leq \eta, 0 \leq t \leq s, \\ \frac{1}{2}s^2, & 0 \leq s \leq \eta, 0 \leq s \leq t, \\ \eta t - \frac{1}{2}t^2, & \eta \leq s \leq 1, 0 \leq t \leq s, \\ \frac{1}{2}s^2 - ts + \eta t, & \eta \leq s \leq 1, 0 \leq s \leq t, \end{cases}$$

We first list some properties of the Green function.

(1) $G(t,s) \geq 0, \forall t,s \in [0,1]$.

(2) $\max_{t\in I} G(t,s) = J(s) = \begin{cases} \frac{1}{2}s^2, 0 \leq s \leq \eta, \\ \frac{1}{2}\eta^2, \eta \leq s \leq 1. \end{cases}$

(3) $\frac{1}{2} \geq G(t,s) \geq q(t)J(s) \geq q(t)G(\tau,s)$

where $q(t) = \begin{cases} \eta t, & 0 \leq t \leq \eta, \\ 2\eta t(1-t), \eta \leq t \leq 1, \end{cases}$ is nonnegative convex function in $I$.

We define the operator $T$:

$$Tv(t) = \int_0^1 G(t,s)a(s)v(s)ds, v \in C[I,R], \quad (2)$$

where $G(t,s)$ and $a(s)$ are same to above mentioned. For the convenience sake, we list some lemmas and conditions:

**lemma 1.1**[9] Suppose $T : C[I,R] \to C[I,R]$ is a completely continuous and positive operator, there exists $v_0 \in Q_1 = \{v \in C[I,R] \mid v(t) \geq 0, \forall t \in [0,1]\}$ with $v_0 \neq \theta$ such that $\lambda_1 T v_0 = v_0 (\lambda_1 > 0)$, for any $v \in Q_1 \setminus \{\theta\}$, there exists natural number $n = n(v)$ and real number

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$\alpha_0(v) > 0, \beta_0(v) > 0$ such that $\alpha_0 v \le T^n v \le \beta_0 v$. For $\forall v \in Q_1$, if $v \neq \mu v_0 (\mu \ge 0)$, then $\lambda_1 T v \nleq v, \lambda_1 T v \ngeq v$.

**Remark** The operator $T$ defined by (2) satisfied all conditions of lemma 1.1. $\lambda_1 = (r(T))^{-1}$ is the first eigenvalue of $T$, $T$ has no other positive eigenfunction except one corresponding to $\lambda_1$.

**Lemma 1.2**[10] Suppose $K$ is a cone in real Banach space, $R > r > 0, K_{r,R} = \{x \in K | r \le \| x \| \le R\}$. If $A : K_{r,R} \to K$ is a strictly set contraction, and satisfied one of the following two cases:

$$Ax \nleq x, \forall x \in K, \|x\| = r; Ax \ngeq x \forall x \in K, \|x\| = R,$$

or

$$Ax \ngeq x, \forall x \in K, \|x\| = r; Ax \nleq x, \forall x \in K, \|x\| = R,$$

then $A$ has at least one fixed point in $K_{r,R}$.

$(\mathbf{H_1})$ $f \in C[P, P]$, for any $r > 0, f$ is uniformly continuous on $P \bigcap B_r$, where $B_r = \{x \in E | \|x\| \le r\}$, there exists a constant $L_r$ such that $\alpha(f(D)) \le L_r \alpha(D), \forall D \subset P \bigcap B_r$, where $L_r : 0 \le L_r \le \frac{1}{2 \max a(t)}$.

$(\mathbf{H_2})$ For $x > \theta$, there exists $\varphi \in P^*$ such that $\varphi(x) > 0$, if $x \in P$ then $\liminf\limits_{\|x\| \to +\infty} \dfrac{\varphi(f(x))}{\varphi(x)} > \lambda_1$, where $\lambda_1$ is the first eigenvalue of the operator $T$.

$(\mathbf{H_3})$ If $x \in P$, then $\limsup\limits_{\|x\| \to 0} \dfrac{\|f(x)\|}{\|x\|} < \dfrac{\lambda_1}{N}$.

$(\mathbf{H_4})$ If $x \in P$, then $\limsup\limits_{\|x\| \to +\infty} \dfrac{\| f(x) \|}{\| x \|} < \dfrac{\lambda_1}{N}$.

$(\mathbf{H_5})$ For $x > \theta$, there exist $\varphi \in P^*$ such that $\varphi(x) > 0$, if $x \in P$ then $\liminf\limits_{\|x\| \to 0} \dfrac{\varphi(f(x))}{\varphi(x)} > \lambda_1$.

$(\mathbf{H_6})$ There exist $r_0 > 0$ such that $\sup\{\| f(x) \| | x \in P, \| x \| \le r_0\} < \dfrac{2r_0}{N} (\int_0^1 a(s) ds)^{-1}$.

$(\mathbf{H_7})$ For $x > \theta$, there exist $r_0 > 0$ and $\varphi \in P^*$ with $\|\varphi\| = 1$ such that $\varphi(x) > 0$, moreover if $q(t) r_0 \le \| x \| \le r_0$ then $\varphi(f(x)) > \alpha r_0$, where $\alpha = \{C \int_\tau^{1-\tau} J(s) a(s) ds\}^{-1}$, $C = \{\eta\tau, 2\eta\tau^2\}_{max}$.

## II. Conclusion

We consider the equivalent problem of (1)

$$Ax(t) = \int_0^1 G(t, s) a(s) f(x(s)) ds \tag{3}$$

Let $Q = \{x(t) \in C[I, E] | x(t) \ge \theta, t \in I\}$, then $Q$ is cone in $C[I, E]$. By the continuity of $G(t, s)$ and $f$, we can get $A : Q \to Q$ is continuous. Similar to the boundary value problem of ordinary differential equation in scalar space, we can get the problem (1.1) has solution in $C[I, E] \bigcap C^1[(0, 1], E] \bigcap C^2[(0, 1], E] \bigcap C^3[J, E]$ if and only if $(Ax)(t) = x(t)$ has fixed point, so we only need to show $A$ has at least one nontrivial fixed point.

In order to the overcome the difficulty caused by singularity, we construct a cone

$$K = \{x \in Q \mid x(t) \ge q(t) \| x \|, \forall t \in I\} \tag{4}$$

obviously $K$ is a cone in $E$ and $K \subset Q$.

Next we show $AK \subset K$, i.e. $A$ is a self-mapping in $K$. By the property of $G(t, s)$, we can get

$$\begin{aligned} Ax(t) &= \int_0^1 G(t, s) a(s) f(x(s)) ds \\ &\le \int_0^1 J(s) a(s) f(x(s)) ds, \forall t \in I, \end{aligned}$$

so

$$\| Ax(t) \| \le \int_0^1 J(s) a(s) f(x(s)) ds.$$

If $x \in K$, then

$$\begin{aligned} Ax(t) &= \int_0^1 G(t, s) a(s) f(x(s)) ds \\ &\ge q(t) \int_0^1 J(s) a(s) f(x(s)) ds \\ &\ge q(t) \| Ax(t) \|, \end{aligned}$$

so $AK \subset K$. Note that $0 < G(t, s) < \frac{1}{2}$, similar to the proof of lemma in thesis [11], for $\forall r > 0$, we can show $A : K_r \to K$ is a strictly set contraction, where $K_r = \{x \in K : \|x\|_c < r\}$. Base the work upon the preliminary, we give the following theorem:

**Theorem 2.1** Suppose conditions $(H_1) - (H_3)$ hold, or conditions $(H_1), (H_4), (H_5)$ hold, the problem (1) has at least one fixed point.

**Proof** We first suppose $(H_1) - (H_3)$ are satisfied. By $(H_3)$ it is easy to see there exist $r_1 : 0 < r_1 < 1$ and $\varepsilon : 0 < \varepsilon < \lambda_1$ such that

$$\|f(x)\| \le \frac{\lambda_1 - \varepsilon}{N} \|x\|, \forall x \in P, \|x\| \le r_1, \tag{5}$$

where $N$ is the regular constant of cone $P$, now we show

$$Ax \ngeq x, \forall x \in K, \|x\| = r_1. \tag{6}$$

In fact, if there exist $x_1 \in K$ with $\|x_1\|_c = r_1$ such that $Ax_1 \ge x_1$, then we have $\theta \le x_1(t) \le (Ax_1)(t), t \in I$. Let $v_1(t) = \| x_1(t) \|$, then $v_1(t) \in C[I, R]$, by the regularity of cone and ( 3 ), we can get

$$\begin{aligned} v_1(t) &= \| x_1(t) \| \\ &\le N \int_0^1 G(t, s) a(s) \| f(x_1(s)) \| ds \\ &\le (\lambda_1 - \varepsilon) \int_0^1 G(t, s) a(s) \| x_1(s) \| ds \\ &= (\lambda_1 - \varepsilon)(T v_1)(t), \end{aligned}$$

so

$$v_1(t) \le (\lambda_1 - \varepsilon)(T v_1)(t), t \in [0, 1]. \tag{7}$$

Next we show $v_1(t) \equiv 0, t \in [0, 1]$. If this is not true, then $v_1(t) \neq 0, t \in [0, 1]$, note that $v_1(t) \ge 0, t \in [0, 1]$, so $v_1(t) \le (\lambda_1 - \varepsilon)^n (T^n v_1)(t)$, correspondingly

$$\| T^n \| \ge \frac{1}{\|v_1(t)\|} \|(T^n v_1)(t)\| \ge \frac{1}{(\lambda_1 - \varepsilon)^n}, n = 1, 2, 3...,$$

According to Gelfand formula

$$r(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|} \ge \lim_{n \to \infty} \sqrt[n]{\frac{1}{(\lambda_1 - \varepsilon)^n}} = \frac{1}{\lambda_1 - \varepsilon} > \frac{1}{\lambda_1}.$$

This is in contradiction with $r(T) = \frac{1}{\lambda_1}$, so $v_1(t) \equiv 0, t \in [0,1]$, but this is in contradiction with $\| x_1 \| = r_1$, so (6) hold. Let

$$(T_\delta v)(t) = \int_\delta^{1-\delta} G(t,s)a(s)v(s)ds, v \in C[I,R],$$

where $\delta \in (0, \frac{1}{2})$, it is easy to see $T_\delta : C[I,R] \to C[I,R]$ is completely continuous and positive linear operator, it is satisfied all conditions of lemma 1.1, so $r(T_\delta) > 0, \lambda_\delta = (r(T_\delta))^{-1}$ is the first eigenvalue of $T_\delta$, $T_\delta$ has no other positive eigenfunction except one corresponding to $\lambda_\delta$. Choose $\delta_n \in (0, \frac{1}{2})(n = 1, 2, ...)$ such that $\delta_1 \geq \delta_2 \geq ... \geq \delta_n \geq ...$ with $\delta_n \to 0 (n \to +\infty)$. For $m > n, v \in Q_1$, we have $T\delta_n v(t) \leq T\delta_m v(t) \leq Tv(t), t \in I$, so $r(T_{\delta_n}) \leq r(T_{\delta_m}) \leq r(T)$. Let $\lambda_{\delta_n} = (r(T_{\delta_n}))^{-1}$, so $\lambda_{\delta_n} \geq \lambda_{\delta_n} \geq \lambda_1$, where $\lambda_1$ is the first eigenvalue defined by (2). Let $\lim\limits_{n \to +\infty} \lambda_{\delta_n} = \widetilde{\lambda_1}$, now we show $\widetilde{\lambda_1} = \lambda_1$.

We first show $\widetilde{\lambda_1}$ is eigenvalue of $T$. Let $v_{\delta_n}$ is a eigenfunction of operator $T_{\delta_n}$ corresponding to the first eigenvalue $\lambda_{\delta_n}$, and satisfied $\|v_{\delta_n}\| = 1$, i.e.

$$v_{\delta_n}(t) = \lambda_{\delta_n} \int_{\delta_n}^{1-\delta_n} G(t,s)a(s)v_{\delta_n}(s)ds, \qquad (8)$$

since $G(t,s)$ is uniformly continuous, we have $\{v_{\delta_n}\}$ is equicontinuous and uniformly bounded, by Arzela-Ascoli theorem, no loss of generality we assume $v_{\delta_n}(t) \to \widetilde{v_0}(t) (n \to +\infty)$, so $\widetilde{v_0} \in Q_1$ and $\|\widetilde{v_0}\| = 1$, by (8) we have

$$\widetilde{v_0}(t) = \widetilde{\lambda_1} \int_0^1 G(t,s)a(s)\widetilde{v_0}(s)ds = \widetilde{\lambda_1}(T\widetilde{v_0})(t),$$

so $\widetilde{\lambda_1}$ is eigenvalue of $T$, notice that $\widetilde{v_0} \in Q_1$ and Remark, we have $\widetilde{\lambda_1} = \lambda_1$.

By $(H_2)$, there exist $R_1 > 1$ and $\varepsilon(0 < \varepsilon < \lambda_1)$ such that

$$\varphi(f(x)) \geq (\lambda_1 + \varepsilon)\varphi(x), x \in P, \|x\| \geq R_1. \qquad (9)$$

According to previous discussion, there exist $\delta \in (0, \frac{1}{2})$ such that $\lambda_1 \leq \lambda_\delta = (r(T_\delta))^{-1} < \lambda_1 + \varepsilon$, take $R_2 = \max\{R_1, \frac{NR_1}{\eta\delta}, \frac{NR_1}{2\eta\delta^2}\}$, where $N$ is normal constant of cone $P$, now we show

$$Ax \nleq x, \forall x \in K, \|x\| = R_2. \qquad (10)$$

In fact, if there exist $x_2 \in K$ with $\|x_2\| = R_2$ such that $Ax_2 \leq x_2$, then $x_2(t) \geq q(t) \| x_2 \|, \forall t \in I$, $P$ is normal, so $\|x_2(t)\| \geq q(t) \| x_2 \| \geq R_1, t \in [\delta, 1-\delta]$. Let $v_2(t) = \varphi(x_2(t))$, by (9),

$$\begin{aligned} v_2(t) &= \varphi(x_2(t)) \\ &\geq \varphi((Ax_2))(t) \\ &= \varphi(\int_0^1 G(t,s)a(s)f(x_2(s))ds) \\ &\geq \int_\delta^{1-\delta} G(t,s)a(s)f(x_2(s))ds \\ &\geq (\lambda_1 + \varepsilon)\int_\delta^{1-\delta} G(t,s)a(s)\varphi(x_2(s))ds \\ &= (\lambda_1 + \varepsilon)(T_\delta)v_2(t) \end{aligned} \qquad (11)$$

so $v_2(t) \geq (\lambda_1+\varepsilon)(T_\delta)v_2(t), t \in I$, note that $v_2(t) \geq 0, t \in I$, by lemma 1.1, there exist $\mu \geq 0$ such that $v_2 = \mu v_\delta$, where $v_\delta$ is positive eigenfunction of operator $T_\delta$ corresponding to the first eigenvalue $\lambda_\delta$. If $\mu = 0$, then $v_2(t) \equiv 0$, so $x_2(t) \equiv 0$, this is in contradiction with $\| x_2 \| = R_2$; If $\mu > 0$, by (11), $\mu v_\delta = \varphi(x_2) \geq (\lambda_1 + \varepsilon)(T_\delta)\varphi(x_2) = \mu(\lambda_1 + \varepsilon)(T_\delta)v_\delta$, so $v_\delta \geq (\lambda_1 + \varepsilon)(T_\delta)v_\delta$, note that $\lambda_\delta < \lambda_1 + \varepsilon$, this is in contradiction with $v_\delta = \lambda_\delta T_\delta v_\delta$, so (10) is proved. To sum up, by (6),(10) and lemma 1.2, $A$ has a fixed point in $K_{r_1,R_2}$. Suppose $(H_1), (H_4), (H_5)$ hold, by $(H_4)$, there exist $R_3 > 1$ and $\varepsilon(0 < \varepsilon < \lambda_1)$ such that

$$\|f(x)\| \leq \frac{\lambda_1 - \varepsilon}{N} \| x \|, \forall x \in P, \|x\| \geq R_3,$$

by $(H_1)$, $\sup\{\| f(x) \| | x \in P, \| x \| \leq R_3\} = b < +\infty$, so

$$\| f(x) \| \leq \frac{\lambda_1 - \varepsilon}{N} \| x \| + b, \forall x \in P. \qquad (12)$$

Let $W = \{x \in K : Ax \geq x\}$, next we show $W$ is bounded. If $x \in W$, then $\theta \leq x(t) \leq (Ax)(t)$. Let $v(t) = \| x(t) \|$, by the normality of cone $P$ and (12) we can get

$$\begin{aligned} v(t) &= \| x(t) \| \\ &\leq N \int_0^1 G(t,s)a(s) \| f(x(s)) \| ds \\ &\leq (\lambda_1 - \varepsilon)(Tv)(t) + M, \end{aligned}$$

where $M = \max\limits_{t \in I} Nb \int_0^1 G(t,s)a(s)ds$, so $((I - (\lambda_1 - \varepsilon)T)v)(t) \leq M, t \in [0,1]$. $\lambda_1$ is the first eigenvalue of $T$, $r((\lambda_1 - \varepsilon)T) = \frac{\lambda_1 - \varepsilon}{\lambda_1} < 1$, so $(I - (\lambda_1 - \varepsilon)T)^{-1}$ exist and

$$\begin{aligned} (I - (\lambda_1 - \varepsilon)T)^{-1} = \; & I + (\lambda_1 - \varepsilon)T + ((\lambda_1 - \varepsilon)T)^2 + \\ & ... + ((\lambda_1 - \varepsilon)T)^n + ... \end{aligned}$$

since $T : Q_1 \to Q_1$ we have $(I - (\lambda_1 - \varepsilon)T)^{-1} : Q_1 \to Q_1$, so $v(t) \leq (I - (\lambda_1 - \varepsilon)T)^{-1}M, t \in [0,1]$, i.e. $W$ is bounded.

Take $R_4 > max\{R_3, \sup W\}$, we can get

$$Ax \ngeq x, \forall x \in K, \|x\| = R_4. \qquad (13)$$

By $(H_5)$, there exist $r_2$ and $\varepsilon(0 < \varepsilon < \lambda_1)$ such that

$$\varphi(f(x)) \geq (\lambda_1 + \varepsilon)\varphi(x), \forall x \in P, \|x\|_c \leq r_2, \qquad (14)$$

we show

$$Ax \nleq x, \forall x \in K, \|x\|_c \leq r_2. \qquad (15)$$

If it is false, there exist $x_2 \in K$ with $\| x_2 \|_c = r_2$ such that $Ax_2 \leq x_2$ , by (14), we can get

$$\begin{aligned} \varphi(x_2) &\geq \varphi(Ax_2) \\ &= \int_0^1 G(t,s)a(s)\varphi(f(x(s)))ds \\ &\geq (\lambda_1 + \varepsilon)T\varphi(x_2), \end{aligned} \qquad (16)$$

so $\varphi(x_2) \geq \lambda_1 T\varphi(x_2)(t), t \in I$. Note that $\varphi(x_2) \geq 0, t \in I$, by lemma 1.1 there exist $\mu \geq 0$ such that $\varphi(x_2) = \mu v_0$. If $\mu = 0$, then $\varphi(x_2)(t) = 0, t \in I$, so $x_2(t) \equiv 0$, this is in contradiction with $\| x_2 \|_c = r_2$; if $\mu > 0$, by (16) we can get $\mu v_0 = \varphi(x_2) \geq (\lambda_1 + \varepsilon)T\varphi(x_2) = \mu(\lambda_1 + \varepsilon)Tv_0$, so $v_0 \geq (\lambda_1 + \varepsilon)Tv_0$, this is in contradiction with $v_0 = \lambda_1 Tv_0$, so (15) is proved. By (13),(15) and lemma 1.2, $A$ has a fixed point in $K_{r_2,R_4}$, and the theorem is proved.

**Theorem 2.2** Suppose $P$ is a cone in $E$, and conditions $(H_1),(H_2),(H_5),(H_6)$ are satisfied, the problem (1) has at least two positive solutions.

**Proof:** Take cone $K$ in $E$, similar to theorem 2.1, we can show $A(K)\subset K$, by $(H_2),(H_5)$, take $R>r_0>r>0$ such that

$$Ax\nleq x,\forall x\in K,\|x\|_c\leq r. \tag{17}$$

$$Ax\nleq x,\forall x\in K,\|x\|_c\leq R. \tag{18}$$

On the other hand, we can show

$$Ax\ngeq x,\forall x\in K,\|x\|_c\leq r_0. \tag{19}$$

In fact, if there exist $x_0\in K$ with $\parallel x_0\parallel=r_0$ such that $Ax_0\geq x_0$, so we can get $\theta\leq x_0(t)\leq(Ax_0)(t),t\in I$, by the normality of cone and $(H_6)$,

$$\begin{aligned}\parallel x_0(t)\parallel &\leq N\int_0^1 G(t,s)a(s)\|f(x_0(s))\|ds\\ &\leq \tfrac{1}{2}N\int_0^1 a(s)\|f(x_0(s))\|ds\\ &< r_0,\forall t\in[0,1],\end{aligned}$$

so $\parallel x_0\parallel< r_0$, this is in contradiction with $\parallel x_0\parallel=r_0$, so (19) is proved.

Since $A$ is a strictly set contraction in $K_{r_0,R}=\{x\in K\mid r_0\leq\|x\|_c\leq R\}$ and $K_{r,r_0}=\{x\in K\mid r\leq\|x\|_c\leq r_0\}$, by $(17)-(19)$ and lemma 1.2, $A$ has fixed point $x_1$ in $K_{r_0,R}$ and $x_2$ in $K_{r,r_0}$ respectively, they are all positive solutions of problem (1), by (19), $\|x_1\|_c\neq r_0,\|x_2\|_c\neq r_0$, so problem (1) has at least two positive solutions.

**Theorem 2.3** Suppose $E$ is a cone in $P$, conditions $(H_1),(H_3),(H_4),(H_7)$ are all satisfied, then problem (1) has at least two positive solutions.

**Proof:** Take cone $K$ in $E$, similar to theorem 2.1, we can show $A(K)\subset K$, by $(H_2)$ and $(H_5)$, take $R>r_0>r>0$ such that

$$Ax\ngeq x,\forall x\in K,\|x\|_c\leq r, \tag{20}$$

$$Ax\ngeq x,\forall x\in K,\|x\|_c\leq R, \tag{21}$$

On the other hand, we can show

$$Ax\nleq x,\forall x\in K,\|x\|_c\leq r_0. \tag{22}$$

In fact, if there exist $x_0\in K$ with $\parallel x_0\parallel=r_0$ such that $Ax_0\geq x_0$, so $\theta\leq(Ax_0)(t)\leq x_0(t),t\in I$, consequently

$$\varphi((Ax_0)(t))\leq\varphi(x_0(t))\leq r_0, \tag{23}$$

since $x_0\in\partial B_{r_0}\bigcap K$, we have $q(t)r_0\leq\parallel x_0(t)\parallel\leq r_0$, take $\tau$ satisfying $\tau<\eta<1-\tau$, for $t\in(\tau,1-\tau)$, by $(H_7)$

$$\begin{aligned}\varphi((Ax_0)(t)) &= \int_0^1 G(t,s)a(s)\varphi(f(x_0(s)))ds\\ &> \alpha r_0 q(t)\int_\tau^{1-\tau} J(s)a(s)\varphi(f(x_0(s)))ds\\ &> \alpha r_0\{\eta\tau,2\eta\tau^2\}_{max}\int_\tau^{1-\tau} J(s)a(s)\varphi(f(x_0(s)))ds\\ &= r_0.\end{aligned}$$

This is in contradiction with (23), so (22) is proved.

Since $A$ is strictly set contraction in $K_{r_0,R}=\{x\in K\mid r_0\leq\|x\|_c\leq R\}$ and $K_{r,r_0}=\{x\in K\mid r\leq\|x\|_c\leq r_0\}$, by (20)-(22) and lemma 1.2, $A$ has fixed point $x_1$ in $K_{r_0,R}$ and $x_2$ in $K_{r,r_0}$, by (22), $\|x_1\|_c\neq r_0,\|x_2\|_c\neq r_0$, so problem (1) has at least two positive solutions.

## References

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