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Positive solutions for boundary value problems of fourth-order nonlinear singular differential equations in Banach space

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Abstract—In this paper, by constructing a special non-empty closed convex set and utilizing $M\ddot{o}nch$ fixed point theory, we investigate the existence of solution for a class of fourth-order singular differential equation in Banach space, which improved and generalized the result of related paper.

Keywords—Banach space, cone, fixed point index, singular differential equation.

I. Introduction

THE singular differential equation arises in a variety of applied mathematics and physics, in recent years, some results concerning the boundary value problems of fourth-order nonlinear singular differential equations have been obtained by a variety of method. Thesis [1-2] investigated the following equation:

$$\begin{cases} x^{(4)}(t)+f(t,x(t))=\theta \quad t\in(0,1) \\ x(0)=x^{''}(0)=x(1)=x^{''}(1)=\theta \end{cases}$$

In thesis [1], the sufficient and necessary condition of solution in $C^2[0,1]\bigcap C^3[0,1]$ was got, where $f(t,x)$ is sub-linear with respect to x. In thesis [2], $f(t,x)=a(t)g(x)$, this is a form of variable separation. Thesis [3] investigated the equation with integral boundary condition:

$$\begin{cases} x^{(4)}(t)+f(t,x(t))=\theta \quad t\in(0,1) \\ x(0)=x(1)=\displaystyle\int_0^1 g(t)x(t)dt \\ x^{''}(0)=x^{''}(1)=\displaystyle\int_0^1 h(t)x^{''}(t)dt \end{cases}$$

Motivated by the work of [3-4], this paper investigates the following equation in Banach space:

$$\begin{cases} x^{(4)}(t)+f(t,x(t))=\theta, \quad t\in(0,1), \\ a_1x(0)-b_1x^{'}(0)=\theta, \\ c_1x(1)+d_1x^{'}(1)=\theta, \\ a_2x^{''}(0)-b_2x^{'''}(0)=\theta, \\ c_2x^{''}(1)+d_2x^{'''}(1)=\theta, \end{cases} \qquad (1)$$

where $f\in[J\times P\setminus\{\theta\},P], J=(0,1)$, the nonlinear term $f(t,x)$ may be singular at $t=0,1$ and $x=\theta$, i.e. $\|f(t,x)\|\to\infty(t\to 0^+,1^-$, or $x\to\theta^+)$, $a_i>0, c_i>0, b_i\geq 0, d_i\geq 0 (i=1,2)$, by constructing a special non-empty closed convex set and utilizing $M\ddot{o}nch$ fixed point theory, we get the existence of positive solution for problem (1). Comparing with the paper above mentioned, this paper is different. First, the result is more generally; second, the method is fixed point theory about cone, this is different from thesis [4] completely. Last, the result exists in abstract space, this is different from thesis [1,2,4]. The organization of this paper is as follows, we shall introduce some definitions and lemmas in the rest of this section, the main result will be stated and proved in section 2, finally, we give a examples to demonstrate our result.

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Let $(E,\|.\|)$ be a Banach space, P is a normal cone in E, the normal constant of cone P is 1, the partial order induced by cone $\leq$: $x\leq y\Leftrightarrow y-x\in P$. Let $I=[0,1], R^+=[0,+\infty)$, we consider problem (1) in $C[I,E]$. Obviously $(C[I,E],\|.\|_c)$ is also a Banach space, where $\|.\|_c=\max_{t\in I}\|x(t)\|$. $x(t)$ is solution of problem (1) if and only if $x\in C^2(I)\bigcap C^4(J)$ and satisfies all equations of (1). If $x^{(3)}(0+)$ and $x^{(3)}(1-0)$ exist, we call it a solution of $C^{(3)}(I)$, if $x(t)>\theta$ for $\forall t\in J$, we call it positive solution.

Suppose $x(t):[0,1]\to E$ is continuous and $\lim_{\delta\to 0+}\int_\delta^1 x(s)ds$ exists, we call general integration $\int_0^1 x(s)ds$ is convergent in abstract space, similarly we can define the convergence of other general integration in abstract space.

Denote α the $Kuratowski$ non-compactness measure. In this paper, we use $\alpha(.)$ and $\alpha_c(.)$ to denote the $Kuratowski$ non-compactness measure in space E and space $C[I,E]$ respectively

For convenience, we list the following lemmas:

Lemma 1.1[5] Let $S\subset C[I,E]$ be bounded and equicontinuous in J, then $\alpha_c(S)=\sup_{t\in I}\alpha(S(t))$, where $S(t)=\{x(t):x\in S\}(t\in I)$.

Lemma 1.2[5] Let H be a countable set of strongly measurable function $x:I\to E$ such that there exists a $M\in L[I,R^+]$ such that $\|x(t)\|\leq M, a.e.t\in I$ for all $x\in H$. Then $\alpha(H(t))\in L[I,R^+]$ and $\alpha(\{\int_I x(t)dt|x\in H\})\leq 2\int_I\alpha(H(t))dt$.

Lemma 1.3[5] Let E be a Banach space, and Q be a closed and convex subset of E and $x_0\in Q$. Assume that the continuous operator $A:Q\to Q$ has the following property: $V\subset Q$ is countable, $V\subset\overline{co}(A(V)\bigcup\{x_0\})\Rightarrow V$ is relatively compact. Then A has a fixed point in Q.

Let $G_i(t,s)$ be the Green function of the following equation:

$$\begin{cases} x''(t)=0, \qquad t\in(0,1),\\ a_ix(0)-b_ix^{'}(0)=0,\\ c_ix(1)+d_ix^{'}(1)=0,\end{cases}$$

then

$$G_i(t,s)=\begin{cases}\dfrac{\varphi_i(t)\psi_i(s)}{\rho_i}, 0\le s\le t\le 1\\ \dfrac{\varphi_i(s)\psi_i(t)}{\rho_i}, 0\le t\le s\le 1\end{cases} \qquad (2)$$

where $\rho_i=a_ic_i+a_id_i+b_ic_i, \varphi_i(t)=c_i+d_i-c_it, \psi_i(t)=b_i+a_it(i=1,2), 0\le t\le 1$. Obviously we have

$$\delta_iG_i(t,t)G_i(s,s)\le G_i(t,s)\le G_i(s,s), G_i(t,s)\le G_i(t,t),$$

where $\delta_i=\frac{\rho_i}{(a_i+b_i)(c_i+d_i)}(i=1,2)$.

For convenience, we list the following assumptions:

$(\mathbf{H_1})$ $f\in C[J\times P\setminus\{\theta\},P]$, and there exists a non-negative measurable function K and $m\in C[R^+,R^+]$ such that $\forall t\in J, x\in P\setminus\{\theta\}$ we have $\|f(t,x)\|\le K(t)m(\|x\|)$.

$(\mathbf{H_2})$ $\int_0^1 G_2(s,s)K(s)m[C\delta_1\delta_2^2G_1(s,s)G_2(s,s)r,R]ds < +\infty$, $\forall R>r>0$. where $m[a,b]:=\sup_{a\le v\le b} m(v)$, $C=\int_0^1 G_1(s,s)G_2(s,s)ds$.

$(\mathbf{H_3})$ There exists $\phi^*\in P^*$ (P^* is a dual cone of P), $e\in P\setminus\{\theta\}, \|\phi^*\|=1=P^*(e)$ such that for $t\in J, x\in P\setminus\{\theta\}$ we have $\phi^*(f(t,x))\ge\phi(t)$ and $0<\int_0^1 G_2(s,s)\phi(s)ds<+\infty$.

$(\mathbf{H_4})$ There exists a $L\in L^1[0,1]$ such that for any $b>a>0, B\subset\overline{P_b}\setminus P_a, t\in J$ we have $\alpha(f(t,B))\le L(t)\alpha(B)$ and $0<4\eta\int_0^1 G_2(s,s)L(s)ds<1$, where $P_a=\{x\in P|\|x\|<a\}, \eta=\int_0^1 G_1(s,s)ds$.

We define operator $T_k: C[I,R]\to C[I,R]$ as following:

$$(T_ky)(t)=\int_0^1 G_2(t,s)K(s)y(s)ds, y\in C[I,R],$$

obviously $T_k: C[I,R]\to C[I,R]$ is completely continuous.

Lemma 1.4[6] Suppose that conditions$(H_1)-(H_3)$ hold and $r(T_K)<1$, then for $y\in C[I,R], \|y\|_c=\max_{t\in I}|y(t)|$, there exists $\|.\|_c^*$ which is equivalent to $\|.\|_c$ satisfying
(1)$\|T_Ky\|_c^*\le\frac{r(T_K)+1}{2}\|y\|_c^*, \forall y\in C[I,R]$;
(2) $\|\varphi\|_c^*\le\|\psi\|_c^*, \varphi,\psi\in C[I,R], \varphi(t)\le\psi(t), \forall t\in I$.

We define operator $S: C[I,E]\to C[I,E]$ as following:

$$(Sv)(t)=\int_0^1 G_1(t,\tau)v(\tau)d\tau,$$

by (2) we have

$$\begin{cases}(Sv)^{''}(t)=-v(t), 0\le s\le t\le 1,\\ a_1(Sv)(0)-b_1(Sv)^{'}(0)=\theta,\\ c_1(Sv)(1)-d_1(Sv)^{'}(1)=\theta.\end{cases} \qquad (3)$$

Lemma 1.5 Problem (1) has a solution if and only if the following problem (4)

$$\begin{cases}(v)^{''}(t)+f(t,(Sv)(t))=\theta, 0\le s\le t\le 1,\\ a_2(v)(0)-b_2(v)^{'}(0)=\theta,\\ c_2(v)(1)-d_2(v)^{'}(1)=\theta,\end{cases} \qquad (4)$$

has a solution.

Proof: In fact, by (3), if x is solution of problem (1), then $v=x^{''}$ is a solution of problem (4); Conversely, if v is a solution of problem (4), let $x=Sv$, by (3) $x^{''}=(Sv)^{''}=-v$, so $x=Sv$ is a solution of problem (1).

By lemma 1.5, we only need to consider problem (4). In order to overcome the difficulty caused by singularity, we construct a cone

$$Q=\{x\in C[I,P]|x(t)\ge\delta_2G_2(t,t)x(s), \forall t,s\in I\}, \qquad (5)$$

it is easy to see $Q\ne\theta$ and Q is a cone in $C[J,E]$, since the normal constant of cone is 1, $\forall x\in Q, t\in I$, we have $x(t)\ge\delta_2\frac{\varphi_2(s)\psi_2(t)}{\rho_2}\|x\|_c$.

In order to use $M\ddot{o}nch$ fixed point theory, we construct a special closed convex set:

$$W=\{x\in Q|\phi^*(x(t))\ge\int_0^1 G_2(t,s)\phi(s)ds, t\in I\},$$

by H_3, it is easy to see $x(t)=\int_0^1 G_2(t,s)\phi(s)ds.e\in W$, so $W\ne\emptyset$. Obviously, W is a closed convex set, $\forall x\in W$ we have $x\ne\theta$, so there exists a $r>0$ such that

$$P_{C_r}\bigcap W=\emptyset, \qquad (6)$$

where $P_{C_r}=\{x\in C[I,P]|\|x\|_c<r\}$. We define operator A as following:

$$(Av)(t)=\int_0^1 G_2(t,s)f(s,(Sv)(s))ds, t\in[0,1]. \qquad (7)$$

II. Conclusion

Theorem 1 Suppose that conditions $(H_1)-(H_3)$ hold, then $A: W\to W$ is continuous.

Proof: We first show $\forall v\in W, Av$ is reasonable. For $v\in W$, it is easy to see

$$\begin{aligned}\|Sv\| &= \|\int_0^1 G_1(t,\tau)v(\tau)d\tau\|\\ &\le\int_0^1 G_1(\tau,\tau)d\tau\|v\|_c\\ &\le\eta\|v\|_c,\end{aligned} \qquad (8)$$

where $\eta=\int_0^1 G_1(\tau,\tau)d\tau$. On the other hand, by (5) we can get

$$\begin{aligned}(Sv)(t) &= \int_0^1 G_1(t,\tau)v(\tau)d\tau\\ &\ge\int_0^1 G_1(t,\tau)\delta_2G_2(\tau,\tau)v(t)d\tau\\ &\ge\delta_1\delta_2G_1(t,t)\int_0^1 G_1(\tau,\tau)G_2(\tau,\tau)d\tau v(t)\\ &= C\delta_1\delta_2G_1(t,t)v(t),\end{aligned}$$

since the cone is normal, we have

$$\begin{aligned}\|(Sv)(t)\| &\geq C\delta_1\delta_2 G_1(t,t)\|v(t)\|\\ &\geq C\delta_1\delta_2^2 G_1(t,t)G_2(t,t)\|v\|_c,\end{aligned}$$

combine $(6),(8),(9)$ and (H_1), we get

$$\|f(t,(Sv)(t))\| \leq K(t)m[C\delta_1\delta_2^2G_1(t,t)G_2(t,t)r,\eta\|v\|_c].$$

By (H_2), we have

$$\begin{aligned}&\|Av)(t)\|\\ &= \|\int_0^1 G_2(t,s)f(s,(Sv)(s))ds\|\\ &\leq \int_0^1 G_2(t,s)K(t)m[C\delta_1\delta_2^2G_1(t,t)G_2(t,t)r,\eta\|v\|_c]ds\\ &< +\infty,\end{aligned}$$

so $\forall t \in I,(Av)(t)$ is reasonable and $(Av)(t) \in P$, by Lebesgue dominated convergence theorem, $Av \in C[I,P]$. Now we show $\forall v \in W$, we have $Av \in Q$. By (7),

$$\begin{aligned}(Av)(t) &= \int_0^1 G_2(t,\tau)f(\tau,(Sv)(\tau))d\tau\\ &\geq \delta_2G_2(t,t)\int_0^1 G_2(s,\tau)f(\tau,(Sv)(\tau))d\tau\\ &= \delta_2G_2(t,t)(Av)(s),\forall t,s\in I,\end{aligned}$$

this implies $AW \subset Q$. Next we show $AW \subset W$. $\forall v \in W$, by (H_3),

$$\begin{aligned}\phi^*((Av)(t)) &= \phi^*(\int_0^1 G_2(t,s)f(s,(Sv)(s))ds)\\ &= \int_0^1 G_2(t,s)\phi^*(f(s,(Sv)(s)))ds\\ &\geq \int_0^1 G_2(t,s)\phi(s)ds,\end{aligned}$$

this implies $AW \subset W$. Last, we show A is continuous. Let $V_n, v \in W$ satisfying $\|V_n - v\|_c \to 0(n \to +\infty)$, take $R_2 = \max\{\sup_n \|v_n\|_c, \|v\|_c\}$, so $\forall t \in I, \|V_n(t) - v(t)\| \to 0(n \to +\infty)$.

By (7), we have

$$\begin{aligned}\|(Av_n)^{'}(t)\| &= \|\int_0^t \frac{-c_2\psi_2(s)}{\rho_2}f(s,(Sv_n)(s))ds\\ &\quad+\int_t^1 \frac{a_2\varphi_2(s)}{\rho_2}f(s,(Sv_n)(s))ds\|\\ &\leq \int_0^t \frac{c_2\psi_2(s)}{\rho_2}\|f(s,(Sv_n)(s))\|ds\\ &\quad+\int_t^1 \frac{a_2\varphi_2(s)}{\rho_2}\|f(s,(Sv_n)(s))\|ds\\ &= g(t),\end{aligned}$$

exchange the integration order and use condition (H_2),

$$\begin{aligned}&\int_0^1 g(t)dt\\ &= \int_0^1\int_0^t \frac{c_2\psi_2(s)}{\rho_2}\|f(s,(Sv_n)(s))\|ds\\ &+ \int_0^1\int_t^1 \frac{a_2\varphi_2(s)}{\rho_2}\|f(s,(Sv_n)(s))\|ds\\ &\leq 2\int_0^1 G_2(s,s)K(s)m[C\delta_1\delta_2^2G_1(s,s)G_2(s,s)r,R]ds\\ &< +\infty,\end{aligned}$$

so $\|(Av_n)^{'}(t)\| \in L[0,1]$, for any $t_1,t_2 \in I, v_n \in W$, we have

$$\begin{aligned}&\|(Av_n)(t_1) - (Av_n)(t_2)\|\\ &= \|\int_{t_1}^{t_2}(Av_n)^{'}(t)dt\|\\ &\leq \int_{t_1}^{t_2}\|(Av_n)^{'}(t)\|dt,\end{aligned}$$

by the absolute continuity of Lebesgue integration, $\{Av_n\}$ is equicontinuous on I, by Lebesgue dominated convergence theorem,

$$\|(Av_n)(t) - (Av)(t)\| \to 0(n \to \infty), \forall t \in I,$$

so for $t \in I, \{Av_n(t)\}$ is relatively compact, by Ascoli-Arzela theorem $\{Av_n\}$ is relatively compact in $C[I,E]$, so we claim that $\|Av_n - Av\|_c \to 0(n \to \infty$. If this is false, then there exists $\varepsilon_0 > 0$ and $\{v_{n_i}\} \subset \{v_n\}$ such that $\|Av_{n_i} - Av\|_c > \varepsilon_0(i = 1,2,3...)$, since $\{Av_n\}$ is relatively compact in $C[I,E]$, then $\{Av_{n_i}\}$ has a convergent subsequence, no loss of generality, we still assume $\lim_{i\to\infty} Av_{n_i} = y$, consequently, $\lim_{i\to\infty}\|Av_{n_i} - y\|_c = 0$, contradicts to $y = Av$, so A is continuous in W.

Theorem 2 Suppose that conditions $(H_1) - (H_4)$ hold. If

$$\lim_{u\to+\infty}\frac{m[r_1,u]}{u} < \lambda_1, \forall u > r_1 > 0, \tag{9}$$

where λ_1 is the first eigenvalue of T_K, then problem (1) has at least one positive solution.

Proof: By (9), there exists $R_3 > max\{r_2, 1\}$ and $0 < \delta < 1$ such that

$$m[r_1,u] \leq \delta\lambda_1 u, \forall u \geq R_3. \tag{10}$$

Let $T_K^*y = \delta\lambda_1 T_K y, \forall y \in C[I,R]$, then $T_K^* : C[I,R] \to C[I,R]$ is a bounded linear operator, λ_1 is the first eigenvalue of T_K, $0 < \delta < 1, r(T_K^*) = \delta < 1$, by lemma (1.5), there exists $\|.\|_c^*$ which is equivalent to $\|.\|_c$ such that $\forall y \in C[I,R]$ we have $\|T_K^*(y)\|_c^* \leq \frac{\delta+1}{2}\|y\|_c^*$. Let

$$M = \int_0^1 G_2(s,s)K(s)m[C\delta_1\delta_2^2G_1(s,s)G_2(s,s)r,R]ds,$$

by (H_2), $M < +\infty$. Choose $R_4 > max\{R_3, \frac{2M^*}{1-\delta}\}$, where $M^* = \|M\|_c^*$. Let $W_1 = \{v \in W | \|y(t)\|_c^* \leq R_4, y(t) = \|Sv(t)\|\}$. Next we show $AW_1 \subset W_1$. $\forall v \in W_1$, let $y(t) = \|Sv(t)\|$, define $e(v) = \{t \in I \mid y(t) > R_3\}$, by (6) and (10), we have

$$\begin{aligned}
&\|(Av)(t)\| \\
&= \| \int_0^1 G_2(t,s)f(s,(Sv)(s))ds\| \\
&\le \int_0^1 G_2(t,s)K(s)m(\|(Sv)(s)\|)ds \\
&= \int_{I\setminus e(v)} G_2(t,s)K(s)m(\|(Sv)(s)\|)ds \\
&+ \int_{e(v)} G_2(t,s)K(s)m(\|(Sv)(s)\|)ds \\
&\le \int_{I\setminus e(v)} G_2(t,s)K(s)m[C\delta_1\delta_2^2 G_1(s,s)G_2(s,s)r_2, R_3]ds \\
&+\delta\lambda_1 \int_{e(v)} G_2(t,s)K(s)\|(Sv)(s)\|ds \\
&\le \int_0^1 G_2(t,s)K(s)m[C\delta_1\delta_2^2 G_1(s,s)G_2(s,s)r_2, R_3]ds \\
&+\delta\lambda_1 \int_0^1 G_2(t,s)K(s)\|(Sv)(s)\|ds \\
&= (T_K^* y)(t) + M.
\end{aligned}$$

By lemma (1.4), we have

$$\begin{aligned}
\|Av\|_c^* &\le \|T_K^* y + M\|_c^* \\
&\le \|T_K^* y\|_c^* + M^* \\
&\le \tfrac{\delta+1}{2}\|y\|_c^* + M^* \\
&\le \tfrac{\delta+1}{2}R_4 + \tfrac{1-\delta}{2}R_4 \\
&= R_4.
\end{aligned} \tag{11}$$

this implies that $\forall v \in W_1$ we have $Av \in W_1$. Since A is continuous, by lemma (1.5) we only need to show A has a fixed point in $C[I,P]$. Let $V \subset W_1$ be a countable set and satisfying

$$V \subset \overline{co}\{\{u\} \bigcup (AV)\}, u \in W_1 \tag{12}$$

where $AV = \{Ax | x \in V\}$. First we show V is relatively compact in $C[I,P]$. In fact, by (12), we have $\alpha_c(V) \le \alpha_c(AV)$. by theorem 1, AV is equicontinuous in I, by thesis [7] we have

$$\overline{V(t)} \subset \overline{co}\{\{u(t)\} \bigcup (AV)(t),$$

combine lemma 1, we have

$$\begin{aligned}
&\alpha_c(V) = \max_{t\in I} \alpha(V(t)), \\
&\alpha_c(AV) = \max_{t\in I} \alpha((AV)(t)), \\
&\alpha(V(t)) \le \alpha(AV(t)),
\end{aligned} \tag{13}$$

where $(AV)(t) = \{x(t) | x \in AV\}$. Next, we estimate $\alpha(AV)(t)$. by (7) and lemma 2 we have

$$\begin{aligned}
\alpha(V(t)) &\le \alpha((AV)(t)) \\
&= \alpha(\{\int_0^1 G_2(t,s)f(s,(Sv)(s))ds | v \in V\}) \\
&\le 2\int_0^1 G_2(t,s)\alpha(\{f(s,(Sv)(s)) | v \in V\})ds \\
&\le 2\int_0^1 G_2(t,s)L(s)\alpha(\{(Sv)(s) | v \in V\})ds \\
&\le 4\int_0^1 \int_0^1 G_2(s,s)L(s)G_1(s,\tau)\alpha(V(\tau))d\tau ds \\
&\le 4\int_0^1 G_2(s,s)L(s)ds \int_0^1 G_(\tau,\tau)d\tau\alpha_c(V) \\
&= 4\eta \int_0^1 G_2(s,s)L(s)ds\alpha_c(V),
\end{aligned}$$

combine (13) and (H_4) we have $\alpha_c(V) = 0$, this implies that V is relatively compact in $C[I,P]$, consequently, our conclusion follows from lemma 1.3.

Example: Let $E = l^\infty = \{x = (x_1, x_2, ..., x_n, ...) : \sup_n | x_n | < +\infty\}$, for $x \in E$, let $\|x\| = \sup_n | x_n |$, then $(E, \|.\|)$ is a Banach space, and $P = \{x \in E : x_n \ge 0, n = 1, 2, ...\}$ is a normal cone in E, the normal constant is 1. Consider the following infinite system of scalar nonlinear differential equations:

$$\begin{cases}
-x_n^{(4)}(t) = \frac{cost}{\sqrt{t(t-1)}}(1+\ln(1+x_n) \\
\qquad +\frac{1}{n}(tx_{2n} + \frac{arctant}{\sqrt{\sup_{i>1}|x_i|}})), t \in (0,1); \\
x_n(0) = x_n(1) = x_n''(0) = x_n''(1) = 0, n = 1, 2, ...
\end{cases} \tag{14}$$

Problem (14) can be regard as the form of problem (1), where $x(t) = (x_1(t), x_2(t), ...), f(t) = (f_1, f_2, ...)$;

$$\begin{aligned}
f_n(t,x) &= \tfrac{cost}{\sqrt{t(t-1)}}(1+\ln(1+x_n) \\
&\quad +\tfrac{1}{n}(tx_{2n} + \tfrac{arctant}{\sqrt{\sup_{i>1}|x_i|}})), t \in (0,1),
\end{aligned} \tag{15}$$

it is easy to see $f(t,x)$ is singular at $t = 0, 1$ and $x = \theta$, now we check (H1)-(H4) hold. Take $k(t) = \frac{1}{\sqrt{t(1-t)}}$, obviously $f \in C[I \times P \setminus \{\theta\}, P]$, by (15) we have $\|f(t,x)\| \le \frac{1}{\sqrt{t(t-1)}}(1 + 2\|x\|\| + \frac{\frac{\pi}{2}}{\sqrt{\|x\|}}$, so (H_1) holds and $K(t) = \frac{1}{\sqrt{t(1-t)}}, m(v) = 1 + 2v + \frac{\frac{\pi}{2}}{\sqrt{v}}$, $\forall R_1 > r_1 > 0$, it is easy to get

$$m[r_1, R_1] = \sup_{v\in[r_1,R_1]} m(v) \le 1 + 2R_1 + \frac{\frac{\pi}{2}}{\sqrt{r_1}}. \tag{16}$$

and $\int_0^1 \sqrt{s(1-s)}ds = \frac{\pi}{8}, \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \pi$, so

$$\begin{aligned}
&\int_0^1 s(1-s)K(s)m[s^2(1-s)^2 r_1, R_1]ds \\
&\le \int_0^1 \sqrt{s(1-s)}(1 + 2R_1 + \frac{\frac{\pi}{2}}{s(1-s)\sqrt{r_1}})ds \\
&= \tfrac{\pi}{8}(1 + 2R_1) + \tfrac{\frac{\pi}{2}}{\sqrt{r_1}}\pi \\
&< +\infty.
\end{aligned}$$

so (H_2) holds. Take $\phi^* \in P^*$ such that $\forall x \in E$ satisfying $\phi^*(x) = x_1$, take $\phi(t) = \frac{cost}{\sqrt{t(t-1)}}$, so $\phi^*(f(t,x)) \geq \phi(t) > 0, t \in (0,1), x \in P \setminus \{\theta\}$, and

$$\int_0^1 s(1-s)\phi(s)ds \quad < \int_0^1 \sqrt{s(1-s)} = \frac{\pi}{8},$$

so H_3 holds. Let $q(t,x) = (q_1(t,x), q_2(t,x)...,), p(t,x) = (p_1(t,x), p_2(t,x)...,)$ where $q_n(t,x) = \frac{1+x_n}{\sqrt{t(1-t)}}, p_n(t,x) = \frac{t}{n\sqrt{t(1-t)}}(x_{2n} + \frac{arctant}{\sqrt{\sup\limits_{i\geq 1}|x_i|}})$, so $f_n(t,x) \leq p_n(t,x) + q_n(t,x)$, for any $c > d > 0, B \subset \overline{P_c} \setminus P_d$, it is easy to see $\alpha(q(t,B)) \leq \frac{\alpha(B)}{\sqrt{t(1-t)}}, t \in (0,1)$. By using the diagonal method, we can choose a subsequence such that $\alpha(p(t,B)) = 0$, so $\alpha(f(t,B)) \leq \frac{\alpha(B)}{\sqrt{t(1-t)}}, t \in (0,1)$. $\forall y \in C[I,P]$, we have

$$\begin{aligned} (T_k y)(t) &= \int_0^1 G_2(t,s)K(s)y(s)ds \\ &\leq \int_0^1 s(1-s)\frac{1}{\sqrt{s(1-s)}}ds\|y\|_c \\ &= \tfrac{\pi}{8}\|y\|_c. \end{aligned} \qquad (17)$$

by (17), $r(T_K) < \frac{\pi}{8}$, so $\lambda_1 \geq \frac{8}{\pi}$, combine (16) we have

$$\lim_{u\to+\infty} \frac{m[r_1,u]}{u} \quad \leq \lim_{u\to+\infty} \frac{1 + 2u + \frac{\frac{\pi}{2}}{\sqrt{r_1}}}{u} = 2 < \frac{8}{\pi} \leq \lambda_1$$

so (H_4) holds. To sum up, (H1)-(H4) hold. By theorem (2), problem (14) has at least one positive solution.

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