

Reliability Approximation through the Discretization of Random Variables using Reversed Hazard Rate Function

Tirthankar Ghosh, Dilip Roy, and Nimai Kumar Chandra

Abstract—Sometime it is difficult to determine the exact reliability for complex systems in analytical procedures. Approximate solution of this problem can be provided through discretization of random variables. In this paper we describe the usefulness of discretization of a random variable using the reversed hazard rate function of its continuous version. Discretization of the exponential distribution has been demonstrated. Applications of this approach have also been cited. Numerical calculations indicate that the proposed approach gives very good approximation of reliability of complex systems under stress-strength set-up. The performance of the proposed approach is better than the existing discrete concentration method of discretization. This approach is conceptually simple, handles analytic intractability and reduces computational time. The approach can be applied in manufacturing industries for producing high-reliable items.

Keywords—Discretization, Reversed Hazard Rate, Exponential distribution, reliability approximation, engineering item.

I. INTRODUCTION

DESIGNING of items in manufacturing industries is a crucial task. An item should be designed in such way that its reliability is very high. Now, to assess the same one has to examine the behavior of the response function of the item which is random in nature. The response is a function of its random component values. Moreover, there are other factors also which makes the response function a random variable. These factors are like measurement errors, uncertainty in the environmental situations, change of operators etc. So, one should study the probability distribution of the system response function. But mostly the task is difficult due to complexity of the form of the system response function. As way out we have some alternative techniques. Monte-Carlo simulation, Taylor series expansion, Numerical integration and Discretization of variables are the existing alternative techniques in literatures. Evans [1] has studied relative advantages and disadvantages of the first three methods. Taguchi [2] has introduced the discretization of random variables method in the line of experimental design approach. Discretization of random variables may also be resulted from characterization results.

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It is an important alternative in approximating system reliabilities of complex systems. Principal step of such approximation is the discretization of the variable(s) under consideration. Since the component lives may follow different probability distributions, we should consider, if possible, the discretization of all the continuous random variables and study their properties in this regard. We should also study different discretization approaches with respect to relative advantages or disadvantages. The reversed Hazard Rate (RHR) is an important concept with nice applications in reliability. We are, therefore, interested in studying discretization of continuous distributions using the RHR function. For demonstration purpose we have taken the case of the exponential distribution. Applications of the proposed discretization have also been demonstrated. Reliability approximation of an important engineering item has been carried out. The paper is organized as follows. Brief reviews of earlier discretizations have been done in section II. Section III deals with the proposed discretization approach. Application of the proposed method has been studied, with one example, in section IV. Section V concludes the paper.

II. EARLIER WORKS

Let us consider a function, $f(X_1, X_2, \dots, X_n)$, of n random variables. Taguchi [2] has claimed that the random behavior of f can be studied through the 3^n values of the function corresponding to 3-level combinations of values taken by each random variable with equal probabilities. The three level combinations of a random variable are $\mu_i - \sqrt{3/2}\sigma_i$, μ_i and $\mu_i + \sqrt{3/2}\sigma_i$, $i = 1, 2, \dots, n$ where $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$. D'Errico & Zaino [3] have established the mathematical basis of the Taguchi's [2] approach in the normal setup. They have shown that the first three central moments of the continuous distribution and of the discrete distribution are same. They have also generalized the discretization into k points. According to their approach if X_i follows $N(\mu_i, \sigma_i^2)$ then each X_i can be discretized into K nodes $\alpha_1, \alpha_2, \dots, \alpha_k$ with respective probabilities $\omega_1, \omega_2, \dots, \omega_k$ such that first $(2K - 1)$ central moments of the continuous and of the discretized distribution are equal. In this approach 3-point discrete Normal distribution has the probability mass points $\mu_i - \sqrt{3}\sigma_i$, μ_i and $\mu_i + \sqrt{3}\sigma_i$ and with probabilities $\frac{1}{6}$, $\frac{4}{6}$, and $\frac{1}{6}$ respectively. First five central moments of this particular discrete distribution are equal with the first five central moments of the continuous $N(\mu_i, \sigma_i^2)$. Taguchi's solution is a particular case of this approach. English et al. [4] have extended this

approach up to six points. They have also demonstrated application of the discretization in designing well known engineering items e.g. Hollow Cylinder, Hollow rectangular Tube and Power resistor. This method is referred as the method of moment equalization. Recently, Ghosh & Roy [5] have used this approach to discretize the Normal distribution in a simpler way. Discretized distributions can also be resulted through study of characterizing properties of continuous distributions. Constancy of failure rate in the discrete domain characterizes the Geometric (discretized Exponential) distribution (See Roy & Gupta [6]). Roy [7] has introduced the concept of discrete concentration in terms of survival (distribution) function. He has also characterized the Bivariate Geometric distribution (BVGd) through local constancy of the failure rate function in the discrete domain. A discrete version of the Normal distribution was obtained through the characterizing property of maximum entropy subject to specific mean and variance in the discrete domain by Kemp [8]. Inusha and Kozubowski [9] have studied the discrete version of the Laplace distribution following Kemp [8]. Use of shifted (location shift) distribution function to discretize a random variable was made by Roy & Dasgupta [10]. Using distribution function Roy [11] has introduced the discrete version of the normal distribution. Some non-normal distributions like Weibull, Uniform and Rayleigh were also discretized by Roy & Dasgupta [12], Roy [13] and Roy [14] respectively. A linear transformation of the discretized distribution for equality of first two central moments of the continuous and of the discretized distribution was introduced by Roy [14]. It produces better results in the cases of applications. Roy & Ghosh [15] have discretized the Rayleigh distribution by using the failure rate function. Ghosh et al. [16] have examined this approach for the Weibull distribution also. In the present paper we have used the reversed hazard rate function of a continuous random variable to discretize it. The linear transformation proposed by Roy [14] has also been used.

III. PROPOSED DISCRETIZATION

Let Y be a non-negative absolutely continuous random variable. Then we know that

$$S(y) = 1 - F(y) = \exp\left[-\int_0^y r(u)du\right] \quad (1)$$

where $r(y)$ is the corresponding failure rate, $S(y)$ and $F(y)$ are the corresponding survival function and the distribution function respectively. The reversed hazard rate (RHR) of Y is defined by

$$a(y) = \frac{f(y)}{F(y)} \quad (2)$$

where $f(y)$ is the pdf of Y . Hence, $a(y)dy$ can be interpreted as an approximate probability of a failure in $(y - dy, y]$ given that the failure had occurred in $[0, y]$. Thus, we have the following relationship

$$f(y) = a(y) \exp\left[-\int_y^\infty r(u)du\right] \quad (3)$$

Now let X be a discrete random variable with support $N = 1, 2, \dots$ or a subset of it. Also let $p(x) = P[X = x]$,

$F(x) = P[X \leq x] = \sum_{i=1}^x P(i)$ and $S(x) = P[X > x] = \sum_{x+1}^\infty p(i)$, respectively, denote the probability mass function, distribution function and survival function of X . For this discrete random variable the hazard rate and the reversed hazard rate are given, respectively, by

$$r(x) = \frac{p(x)}{S(x-1)}, x = 1, 2, \dots \quad (4)$$

and

$$a(x) = \frac{p(x)}{F(x)}, x = 1, 2, \dots \quad (5)$$

Unlike continuous case in discrete situation both the hazard rate and the reversed hazard rate can be interpreted as probabilities. When X represents the lifetime of a component the $r(x)$ is said to be the probability that the component will fail at time $X = x$ given that it has survived up to time before x . On the other hand $a(x)$ is the probability that the component will fail at time $X = x$, given it is known to have failed before x . We know that the distribution of X can be uniquely determined by $r(x)$ using the relationship

$$S(x) = \prod_{i=0}^x [1 - r(i)] \quad (6)$$

This relationship was used for discretization of the continuous distribution using the failure rate. We rewrite (5) as follows

$$a(x) = \frac{F(x) - F(x-1)}{F(x)} \quad (7)$$

Thus, we have

$$F(x-1) = [1 - a(x)]F(x) \quad (8)$$

Now putting $x = 0, 1, 2, \dots, t$ in (8) we have

$$F(0) = \prod_{i=0}^{t-1} [1 - a(i)]F(t) \quad (9)$$

Combining (2) and (9) we can write

$$p(t) = \frac{a(t)F(0)}{\prod_{i=1}^t [1 - a(i)]} \quad (10)$$

Since X is a non-negative discrete random variable hence at $t = 0$, $F(0) = p(0)$. We, therefore, write

$$p(t) = \frac{a(t)p(0)}{\prod_{i=1}^t [1 - a(i)]} \quad (11)$$

Thus (11) can be used to discretize a continuous random variable if we know the form of the RHR function of its distribution. The value of $p(0)$ can be determined from the condition of sum of total probability.

A. Example

Let us consider the discretization of the exponential distribution. We know that discretization of the exponential distribution yields the geometric distribution except in the method of moment equalization. Let the continuous random variable Y follows the exponential distribution with the survival function $S(y) = \exp(-\lambda y)$. We denote it by $\exp(\lambda)$. Then the failure rate function and the reversed hazard rate function

of the distribution are respectively given by $r(y) = \lambda$ and $a(y) = \frac{\lambda \exp(-\lambda y)}{1 - \exp(-\lambda y)}$. Substituting the expression of in (11) we have the discretized version of the exponential distribution. The discretized exponential distribution have the recursive relationship

$$p(t + 1) = \frac{[\exp(-\lambda)1 - \exp(-\lambda t)]}{[1 - \exp(-t - 1) - \lambda \exp(-\lambda t)]} p(t), t = 1, 2, \dots$$

Now for different choices of λ we shall have different discrete exponential distribution. Therefore, we shall search for a λ to get a discrete exponential distribution which will be treated as the standard discrete exponential distribution. Here our first choice is $\lambda = 1.0$. The corresponding discrete probability distribution is given in the following table. Graph of the

TABLE I
 DISCRETE EXPONENTIAL DISTRIBUTION WITH $\lambda = 1.0$

$X_d = x$	Probability
0	0.32439
1	0.45162
2	0.14400
3	0.05087
4	0.01846
5	0.00676
6	0.00248
7	0.00091
8	0.00034
9	0.00012
10	0.00005
TOTAL	1.00000

above discrete distribution does not follow the nature of the exponential distribution. Thus, we search for another λ which will have that nature. Calculation shows that for all $\lambda \geq 1.3$, the obtained discrete distribution has the exponential nature. Moreover, we have considered the absolute deviations between means of the discrete and of the continuous distribution to fix the value of λ . Various choices of λ and corresponding deviations are shown in Table II. From Table II we note that

TABLE II
 CHOICE OF λ FOR DISCRETE EXPONENTIAL DISTRIBUTION

λ	True mean	Observed mean	Deviations
1.3	0.76923	0.74420	0.02503
1.4	0.71429	0.67446	0.03983
1.5	0.66667	0.61305	0.05362
1.6	0.62500	0.55853	0.06647
1.7	0.58824	0.50997	0.07826
1.8	0.55556	0.46649	0.08907
1.9	0.52632	0.42727	0.09905
2.0	0.50000	0.39192	0.10808
2.1	0.47619	0.35991	0.11628
2.2	0.45455	0.33077	0.12378
2.3	0.43478	0.30422	0.13056
2.4	0.41667	0.28003	0.13664
2.5	0.40000	0.25793	0.14207
2.6	0.38462	0.23776	0.14686
2.7	0.37037	0.21916	0.15121
2.8	0.35714	0.20212	0.15502
2.9	0.34483	0.18645	0.15838
3.0	0.33333	0.17204	0.16129

as the value of λ increases the discrete distribution gives larger absolute deviation with respect to mean. We, therefore, suggest to take $\lambda = 1.3$ and the corresponding discrete distribution,

given in Table III table, may be called as the standard discrete exponential distribution. To sharpen the discretized distribution

TABLE III
 STANDARD DISCRETE EXPONENTIAL DISTRIBUTION WITH $\lambda = 1.3$

$X_d = x$	Probability
0	0.44272
1	0.42032
2	0.10050
3	0.02659
4	0.00719
5	0.00196
6	0.00053
7	0.00015
8	0.00004
TOTAL	1.00000

we have taken a linear transformation of the above discrete distribution. The transformation has been taken in such a way that the first two central moments of the continuous and of the discretized distribution are equal. Following Roy [14] we take

$$X_d^* = \alpha + \beta X_d$$

where α and β are such that

$$E(X_d^*) = E(X)$$

and

$$Var(X_d^*) = Var(X)$$

solving for α and β we have the following final form

$$X_d^* = 0.08541 + 0.9188X_d$$

where probability distribution of X_d is described in Table III. Any non-standard exponential random variable can be discretized using that standard discretized distribution through a transformation. Suppose Z has the survival function $\exp(-\lambda_1 z)$. Then its discretized version, Z_d , is given by

$$Z_d = \left(\frac{1.3}{\lambda}\right) X_d^*$$

IV. APPLICATION

Let $f_1(x_1, x_2, \dots, x_n)$ and $f_2(y_1, y_2, \dots, y_m)$, respectively, be the strength and stress functions of a complex system where x_i & y_i are random component values. Then the reliability, R , of the system is given by

$$R = P[f_1(x_1, x_2, \dots, x_n) > f_2(y_1, y_2, \dots, y_m)]$$

To find out R we have to know the probability distributions of f_1 and f_2 . Mostly, due to the complex nature of the functions, the exact distributions are difficult to obtain. Here we can apply the above discretization method to overcome the problem. Using the same we approximate the R as follows

$$R \simeq \sum \dots \sum \prod_{i=1}^n P[X_{d_i} = x_{d_i}] \times \prod_{j=1}^m P[Y_{d_j} = y_{d_j}] \times I[f_1 > f_2]$$

where $I[E]$ is an indicator function which takes the value 1 if the event E is true, and the value 0 otherwise. The summation is over the all possible values of x_{d_i} and y_{d_j} 's which are the

values taken by the discretized versions of the random variables X_i and Y_j , $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$; respectively. We can use this discretization method to approximate the distribution of functions of random variables which are complex in nature. That is, where we can not find the exact distribution functions in conventional ways. This discretization may be used in manufacturing industries like in Engineering Design by Reliability (EDBR). Here we demonstrate the application in approximating reliabilities under the stress-strength set-up through the following worked out example.

A. Workedout example

We consider reliability approximation for designing of a Tension element (see [17]). The load P acting on the element is a random variable. The element has a circular cross-section and its diameter is a random variable because of the manufacturing tolerance. The ultimate tensile strength of the material used for the element is a random variable because the properties of the material vary. The tensile stress function of the element is given by

$$s = \frac{P}{\pi r^2}$$

where r is the radius of the circular cross-section. Then under the assumptions that P follows exponential distribution with mean 20 lb., r follows exponential distribution with mean 0.15in. and the tensile strength (s) follows exponential distribution with mean λ we have approximated the reliabilities of the element at different values of λ using the proposed discretization. We assumed statistical independence of the distributions of s , P and r . A simulation study, with 10^5 replications, was also conducted for judging the performance of our approximation. For comparison purpose the same approximation was also conducted using discrete concentration method (DCM) of discretization. The TableIV gives the calculated results. The absolute deviations (AD)

TABLE IV
 RELIABILITY VALUES OF A TENSION ELEMENT OBTAINED VIA SIMULATION AND DISCRETIZATIONS BASED ON RHR & DCM UNDER THE EXPONENTIAL SET-UP

λ	Simulation	Proposed method	DCM
20	0.091	0.145	0.050
80	0.216	0.208	0.168
200	0.332	0.396	0.305
320	0.397	0.420	0.362
1200	0.576	0.479	0.529
2000	0.640	0.641	0.830
5000	0.739	0.660	0.850
25000	0.864	0.864	0.997
40000	0.889	0.865	1.000
50000	0.899	0.884	1.000
100000	0.926	0.890	1.000
500000	0.964	0.973	1.000
1000000	0.975	0.998	1.000

between simulated and approximated reliabilities by different discretization approaches were computed. Following TableV describes the absolute deviations mean absolute deviations (MAD). From the TableV it may be noted that the DCM of discretization produces 127% more MAD than the proposed

TABLE V
 ABSOLUTE DEVIATIONS OF SIMULATED AND APPROXIMATED RELIABILITIES OF A TENSION ELEMENT OBTAINED VIA DISCRETIZATIONS BASED ON RHR & DCM UNDER THE EXPONENTIAL SET-UP

λ	Proposed method	DCM
20	0.054	0.041
80	0.008	0.048
200	0.064	0.027
320	0.023	0.035
1200	0.097	0.047
2000	0.001	0.190
5000	0.079	0.111
25000	0.000	0.133
40000	0.024	0.111
50000	0.015	0.101
100000	0.036	0.074
500000	0.009	0.036
1000000	0.023	0.025
MAD	0.033	0.075

RHR based discretization. This indicates that the proposed method works better than DCM.

V. CONCLUSION

Discretization of a random variable based on RHR has been demonstrated. Exponential distribution was examined for discretization. In the proposed method discretized exponential distribution is not the geometric distribution unlike other approaches viz. discrete concentration, failure rate. Obtained discrete distribution may be applied to approximate system reliabilities of complex systems. Probability distributions of complex functions with exponential variables may also be approximated using this discretized distribution. For setting design parameters of different components of a complex system in manufacturing industries (Engineering Design by Reliability) this discretization may also be used. From the worked out example it may be said that in approximating system reliability, under the stated set-up, the discretization using RHR performs better than the earlier introduced DCM of discretization. Thus the proposed approach will be a preferred choice for discretization in these situations. Efficacy of this discretization may also be examined by applying it to other complex systems as well. This approach has the advantages like simplicity, lesser time consumption for computation, handling of analytically intractable situations etc.

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