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Maximum Likelihood Estimation of Burr Type V Distribution under Left Censored Samples

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Abstract**—**The paper deals with the maximum likelihood estimation of the parameters of the Burr type V distribution based on left censored samples. The maximum likelihood estimators (MLE) of the parameters have been derived and the Fisher information matrix for the parameters of the said distribution has been obtained explicitly. The confidence intervals for the parameters have also been discussed. A simulation study has been conducted to investigate the performance of the point and interval estimates.

Keywords**—**Fisher information matrix, confidence intervals, censoring.

I. INTRODUCTION

BURR family of distributions consists of a dozen of distributions these can be used to fit almost any given set of unimodal data. Burr [1] proposed these distributions. From these twelve distributions Burr type X and XII have received the maximum attention of the analysts. The authors considering the analysis of Burr type X and XII include: Surles and Padgett [2], Mousa and Jaheen [3], Soliman [4], Shao [5], Shao et al. [6], Soliman [7], Wu and Yu [8], Amjad and Ayman [9], Wahed [10], Wu et al. [11], Aludaat et al. [12], Silva et al. [13], Yarmohammadi and Pazira [14], Dasgupta [15], Makhdoom and Jafari [16], Panahi and Asadi [17] and Feroze and Aslam [18]. The remaining types of the Burr family of distributions haven't received a considerable interest of the analysts; same is the case with Burr type V distributions. The Burr Type V distribution can be used to model the lifetime data. The probability density function (pdf) of the distribution is:

$$f(y)=\theta\lambda e^{-\tan y}\sec^2 y\left(1+\lambda e^{-\tan y}\right)^{-\theta-1},\ -\frac{\pi}{2}<y<\frac{\pi}{2},\ \lambda,\theta>0 \tag{1}$$

The cumulative distribution function of this distribution is:

$$F(y)=\left(1+\lambda e^{-\tan y}\right)^{-\theta} \tag{2}$$

where λ and θ are the location parameters of the distribution.

This distribution is still waiting for the attention of the researchers may be due to its complex pdf. Many properties of the parameters of the distribution under different estimation procedures are still to be revealed. To deal with characteristics of such deprived distributions is always important for the researchers. The investigation of properties of such distributions can be beneficial to the professionals looking to use those distributions as models. The rare consideration of the Burr type V distribution in the literature is a motivation for the resent study.

Censoring is useful procedure when the value of a measurement or observation is only partially known. That is, all information regarding a portion of the sample/population is omitted or do not exist. In practice, it occurs when an observed value is outside the range of a measuring instrument or the measure outside a range is not desired. Censoring has many types; however, we will concentrate on the left censored samples for the estimation of the said parameters. The left censored data is very likely to occur in survivor analysis. It can happen where and event of interest has already occurred at the observation time, but it is not known exactly when. For example, the situations including: the infection with a sexually-transmitted disease such as HIV/AIDS, onset of a pre-symptomatic illness such as cancer and time at which teenagers begin to drink alcohol can lead to left censored data. In case of left censored samples, we can only observe those individuals whose event time is greater than some truncation point. This truncation point may or may not be the same for all individuals. For example, in case of actuarial life studies, the individuals those died in the womb are often ignored. Another example: suppose you wish to study how long patients who have been hospitalized for a heart attack survive taking some treatment at home. In such situations, the starting time is often considered to be the time of the heart attack. Only those patients who survive their stay in hospital are able to be included in the study. The more illustrations on left censoring can be seen from Jerald and Lawless [19], Sinha [20], Asselineau et al. [21], Antweller and Taylor [22], Thompson et al. [23] and Feroze and Aslam [24].

We have considered the maximum likelihood estimation (MLE) of the Burr type V distribution under left censored samples. As the explicit expressions for the maximum likelihood estimators of the parameters cannot be obtained, a fixed point iteration technique has been used to obtain the MLE of shape parameter λ . Once the MLE of λ has been obtained the MLE of the second shape parameter θ become possible to be solved explicitly. In addition, the Fisher information matrix has been derived explicitly and the variance covariance matrix has been obtained by inverting the information matrix. The approximate confidence intervals for both the parameters have also been constructed. A

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comprehensive simulation study has been carried out to assess the behavior and performance of the estimates under different sample sizes, parametric space and various degrees of censoring rate.

II. Maximum Likelihood Estimation

Based on the left censored sample the maximum likelihood function along with maximum likelihood estimators of the parameters of the Burr type V distribution have been discussed in the following. Let $Y_{(r+1)}...Y_{(n)}$ be the last $n-r$ ordered statistics from the Burr type V distribution. Then, the likelihood function for the sample of $n-r$ left censored sample is:

$$L(\theta,\lambda) \propto \left\{F\left(y_{(r+1)}\right)\right\}^r f\left(y_{(r+1)}\right)...f\left(y_{(n)}\right) \quad (3)$$

$$L(\theta,\lambda) \propto \left(1+\lambda e^{-\tan y_{(r+1)}}\right)^{-\theta r} (\theta\lambda)^{n-r} e^{-\sum_{i=r+1}^{n} \tan y_{(i)}} \times\left\{\prod_{i=r+1}^{n} \sec^2 y_{(i)}\right\} \prod_{i=r+1}^{n}\left(1+\lambda e^{-\tan y_{(i)}}\right)^{-\theta-1} \quad (4)$$

The logarithmic of the likelihood function can be written as:

$$l(\theta,\lambda) \propto -\theta r \ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+(n-r)\ln(\theta\lambda) -(\theta+1)\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right) \quad (5)$$

The normal equations for the derivation of the MLE of the λ and θ parameters are:

$$\frac{\partial l}{\partial\theta} = -r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\frac{n-r}{\theta}-\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)=0 \quad (6)$$

$$\frac{\partial l}{\partial\lambda} = \frac{-\theta r e^{-\tan y_{(r+1)}}}{1+\lambda e^{-\tan y_{(r+1)}}}+\frac{n-r}{\lambda}-(\theta+1)\sum_{i=r+1}^{n}\frac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}}=0 \quad (7)$$

From (6), the MLE of θ can be derived as a function of λ that can be denoted as

$$\hat{\theta}(\lambda) = \frac{n-r}{r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)} \quad (8)$$

It is immediate from (7) that the MLE of λ cannot be obtained in an explicit form. So, we have to play some mathematical/numerical tricks to find out the approximate MLE of λ. Firstly, the parameter θ in log-likelihood (5) has been replaced by its MLE given in (8) the resultant log-likelihood becomes:

$$l(\lambda) \propto \frac{-r(n-r)\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)}{r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)}+(n-r)\ln(\lambda) +(n-r)\ln\left\{\frac{(n-r)}{r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)}\right\} -\left\{\frac{(n-r)}{r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)}+1\right\}\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)$$

After some simplifications it can be presented as:

$$l(\lambda) = (n-r)\ln(\lambda)-\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right) -(n-r)\ln\left\{r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)\right\} \quad (9)$$

For MLE of λ, the normal equation can be given as:

$$\frac{\partial l(\lambda)}{\partial\lambda} = \frac{n-r}{\lambda}-\sum_{i=r+1}^{n}\frac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}} -(n-r)\left\{\frac{\dfrac{r e^{-\tan y_{(r+1)}}}{1+\lambda e^{-\tan y_{(r+1)}}}+\sum_{i=r+1}^{n}\dfrac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}}}{r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)}\right\}=0 \quad (10)$$

Again the explicit solution for MLE of λ is not possible. We have used the fixed point iteration method to have the approximate solution MLE of λ by considering the following function.

$$\lambda = g(\lambda) = \left\{\frac{1}{n-r}\sum_{i=r+1}^{n}\frac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}}+\frac{\dfrac{r e^{-\tan y_{(r+1)}}}{1+\lambda e^{-\tan y_{(r+1)}}}+\sum_{i=r+1}^{n}\dfrac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}}}{r\ln\left(1+\lambda e^{-\tan y_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1+\lambda e^{-\tan y_{(i)}}\right)}\right\}^{-1} \quad (11)$$

The final result of the above function has been considered as an MLE of λ and denoted by $\hat{\lambda}$. Now, the value of $\hat{\lambda}$ facilitated to find out the solution for MLE of θ given in (8), that can be denoted by $\hat{\theta}\left(\hat{\lambda}\right)$.

III. Approximate Fisher Information Matrix

In this section, the elements of the Fisher information matrix for the parameters of the Burr type V distribution based on left censored samples have been derived explicitly. The variance covariance matrix for the parameters of the Burr type

V distribution can be obtained by inverting the Fisher information matrix which has been used construct the confidence intervals for the said parameters. The Fisher information matrix can be defined as:

$$I(\theta,\lambda) = -\mathrm{E}\begin{bmatrix} \frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2 l}{\partial\theta\partial\lambda} \\ \frac{\partial^2 l}{\partial\lambda\partial\theta} & \frac{\partial^2 l}{\partial\lambda^2} \end{bmatrix} \tag{12}$$

The equations for the elements of the Fisher information matrix can be written as:

$$\frac{\partial^2 l}{\partial\theta^2} = -\frac{n-r}{\theta^2} \tag{13}$$

$$\frac{\partial^2 l}{\partial\lambda^2} = \frac{r\theta e^{-2\tan y_{(r+1)}}}{\left\{1+\lambda e^{-\tan y_{(r+1)}}\right\}^2} - \frac{n-r}{\lambda^2} + (\theta+1)\sum_{i=r+1}^{n}\frac{e^{-2\tan y_{(i)}}}{\left\{1+\lambda e^{-\tan y_{(i)}}\right\}^2} \tag{14}$$

$$\frac{\partial^2 l}{\partial\theta\partial\lambda} = \frac{-re^{-\tan y_{(r+1)}}}{1+\lambda e^{-\tan y_{(r+1)}}} - \sum_{i=r+1}^{n}\frac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}} \tag{15}$$

Now, the expected values of the (14) and (15) require the distribution of the i^{th} order statistics from the Burr type V distribution which can be written as:

$$g\left(y_{(i)}\right) = C_{n,i}\theta\lambda e^{-\tan y_{(i)}}\sec^2 y_{(i)}\left(1+\lambda e^{-\tan y_{(i)}}\right)^{-\theta i-1} \times\left\{1-\left(1+\lambda e^{-\tan y_{(i)}}\right)^{-\theta}\right\}^{n-i},$$

$$-\frac{\pi}{2} < y_{(i)} < \frac{\pi}{2} \qquad \text{where } C_{n,i} = \frac{n!}{(i-1)!(n-i)!}$$

Here, the expectations necessary to derive the elements of the Fisher information matrix are:

$$\mathrm{E}\left[\frac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}}\right] = C_{n,i}\theta\lambda\int_{-\pi/2}^{\pi/2} e^{-\tan y_{(i)}}\sec^2 y_{(i)}\left(1+\lambda e^{-\tan y_{(i)}}\right)^{-\theta i-1} \times\left\{1-\left(1+\lambda e^{-\tan y_{(i)}}\right)^{-\theta}\right\}^{n-i} dy_{(i)}$$

After simplifications it becomes

$$\mathrm{E}\left[\frac{e^{-\tan y_{(i)}}}{1+\lambda e^{-\tan y_{(i)}}}\right] = \frac{2\theta C_{n,i}}{\lambda}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}B\left(\frac{5}{2},\theta(i+j)-\frac{1}{2}\right) \tag{16}$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is a Beta function.

Similarly,

$$\mathrm{E}\left[\frac{e^{-2\tan y_{(i)}}}{\left\{1+\lambda e^{-\tan y_{(i)}}\right\}^2}\right] = \frac{2\theta C_{n,i}}{\lambda^2}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}B\left(3,\theta(i+j)-1\right) \tag{17}$$

Hence, the elements of the Fisher information matrix becomes

$$-\mathrm{E}\left(\frac{\partial^2 l}{\partial\lambda^2}\right) = \frac{n-r}{\lambda^2} - \frac{2\theta(\theta+1)C_{n,i}}{\lambda^2}\sum_{i=r+1}^{n}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}B\left(3,\theta(i+j)-1\right) - \frac{2r\theta^2 C_{n,i}}{\lambda^2}\sum_{j=0}^{n-r-1}(-1)^j\binom{n-r-1}{j}B\left(3,\theta(r+1+j)-1\right)$$

$$-\mathrm{E}\left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right) = \frac{2\theta C_{n,i}}{\lambda}\sum_{i=r+1}^{n}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}B\left(\frac{5}{2},\theta(i+j)-\frac{1}{2}\right) + \frac{2r\theta C_{n,i}}{\lambda}\sum_{j=0}^{n-r-1}(-1)^j\binom{n-r-1}{j}B\left(\frac{5}{2},\theta(r+1+j)-\frac{1}{2}\right)$$

The variance covariance can be obtained by inverting the Fisher information matrix as:

$$I^{-1}(\theta,\lambda) = \begin{bmatrix} V(\hat{\theta}) & Cov(\hat{\lambda},\hat{\theta}) \\ Cov(\hat{\lambda},\hat{\theta}) & V(\hat{\lambda}) \end{bmatrix}$$

where, the diagonal elements of the matrix are the variances of the MLEs of θ and λ respectively. The approximate confidence intervals for θ and λ as discussed by Wu and Kus [25] are:

$$\hat{\theta} \pm Z_{\alpha/2}\sqrt{V(\hat{\theta})} \text{ and } \hat{\lambda} \pm Z_{\alpha/2}\sqrt{V(\hat{\lambda})}$$

IV. Limiting Fisher Information Matrix

This section discusses the asymptotic efficiencies and limiting information matrix when r/n converges to, say, p which lies in $(0,1)$. According to Gupta et al. [26], for the left censored observations at the time point T, the limiting Fisher information matrix can be written as

$$I(\theta,\lambda) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where

$$b_{ij} = \int_T^\infty \left(\frac{\partial}{\partial\psi_i}\ln r(y,\psi)\right)\left(\frac{\partial}{\partial\psi_j}\ln r(y,\psi)\right) f(y;\psi)\,dy$$

and $\psi = (\theta,\lambda)$, $r(y,\psi) = \frac{f(y;\psi)}{F(y;\psi)}$ the reversed hazard function. Zheng and Gastwirth [27] have shown that for location and scale family, the Fisher information matrix for

Type-I and Type-II (both for left and right censored data) are asymptotically equivalent. They further described that for general case (not for location and scale family) the results for Type-II censored data (both for left and right) of the asymptotic Fisher information matrices are very difficult to obtain. We cannot obtain the explicit expression for the limiting Fisher information matrix for Burr type V distribution under left censored samples as it does not belong to the location and scale family. Numerically, we have studied the limiting behavior of the Fisher information matrix by taking $n=5000$ (assuming it is very large) and compare them with the different small samples and different 'p' values. The numerical results have been presented in Section (5).

V. Results and Discussions

This section covers the discussions regarding the results of the simulation study for n = 30, 50, 70, 100 and 150 using parametric space $(\theta,\lambda)=\{(0.5,1.5,2,2.5,3),(0.5,1.5,2,2.5,3)\}$ under 10% and 20% left censored samples. The purpose of the simulation study is to assess the behavior of the MLEs and confidence intervals for the parameters of the Burr type V distribution. As the MLE of parameter λ cannot be obtained in the explicit form, a fixed point iteration scheme has been proposed to have the approximate MLE of the parameter λ. The performance of the MLEs have been evaluated in terms of their mean square errors (MSEs); while, the performance of the confidence intervals have been discussed on the basis of the widths of the intervals along with corresponding coverage probabilities. The inverse transformation method has been used to generate the random samples from the distribution. The function used for the generation of the random numbers is: $Y=\tan^{-1}\left[-\ln\left\{\lambda^{-1}\left(U^{-1/\theta}-1\right)\right\}\right]$ where U is the random variable following the uniform distribution. For the whole parametric space of the θ we have assumed $\lambda=2$ and for the entire parametric space of λ we assumed $\theta=2$. The entries in the tables below are the average of the results under 1000 replications. The average relative estimate (A.R.E) defined as the ratio of MLE to the true parametric value, MSE, lower confidence limits (LCL), upper confidence limits (UCL), width of the confidence limits and associated coverage probabilities (C.P) calculated by the proportion of the intervals containing the true parametric values to the total (1000) intervals, have been presented in the tables.

TABLE I
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Θ WHEN N = 30

θ	A.R.E	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.1987	0.0232	0.3733	0.8255	0.4522	0.9520
1.00	1.2587	0.1256	0.7839	1.7335	0.9496	0.9540
1.50	1.2713	0.3002	1.1876	2.6262	1.4386	0.9580
2.00	1.2738	0.5403	1.5867	3.5086	1.9219	0.9560
2.50	1.2776	0.8596	1.9893	4.3989	2.4096	0.9620
3.00	1.2789	1.2453	2.3895	5.2840	2.8945	0.9610
	20% Censored Samples					
0.50	1.2280	0.0287	0.3684	0.8597	0.4913	0.9430
1.00	1.2894	0.1531	0.7736	1.8053	1.0318	0.9470
1.50	1.3023	0.3647	1.1719	2.7351	1.5631	0.9500
2.00	1.3049	0.6558	1.5657	3.6540	2.0883	0.9460
2.50	1.3089	1.0423	1.9630	4.5813	2.6183	0.9540
3.00	1.3102	1.5095	2.3580	5.5030	3.1450	0.9550

TABLE II
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Θ WHEN N = 50

θ	A.R.E	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.1250	0.0109	0.3981	0.7268	0.3287	0.9570
1.00	1.1812	0.0638	0.8361	1.5263	0.6903	0.9590
1.50	1.1930	0.1550	1.2667	2.3124	1.0457	0.9630
2.00	1.1954	0.2798	1.6923	3.0894	1.3971	0.9610
2.50	1.1990	0.4472	2.1217	3.8733	1.7516	0.9670
3.00	1.2002	0.6488	2.5486	4.6526	2.1040	0.9680
	20% Censored Samples					
0.50	1.1525	0.0141	0.3977	0.7548	0.3571	0.9490
1.00	1.2101	0.0807	0.8351	1.5851	0.7500	0.9510
1.50	1.2222	0.1951	1.2651	2.4014	1.1363	0.9520
2.00	1.2246	0.3518	1.6902	3.2083	1.5181	0.9550
2.50	1.2283	0.5615	2.1191	4.0224	1.9033	0.9580
3.00	1.2295	0.8143	2.5455	4.8317	2.2862	0.9600

TABLE III
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Θ WHEN N = 70

θ	A.R.E	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0736	0.0059	0.4042	0.6693	0.2651	0.9470
1.00	1.1272	0.0364	0.8489	1.4056	0.5567	0.9500
1.50	1.1385	0.0895	1.2861	2.1295	0.8434	0.9540
2.00	1.1408	0.1619	1.7182	2.8450	1.1268	0.9600
2.50	1.1442	0.2599	2.1542	3.5669	1.4127	0.9660
3.00	1.1454	0.3776	2.5876	4.2846	1.6970	0.9750
	20% Censored Samples					
0.50	1.0998	0.0079	0.4059	0.6939	0.2881	0.9380
1.00	1.1548	0.0478	0.8523	1.4572	0.6049	0.9410
1.50	1.1663	0.1169	1.2913	2.2077	0.9164	0.9420
2.00	1.1687	0.2113	1.7251	2.9495	1.2244	0.9500
2.50	1.1722	0.3386	2.1629	3.6979	1.5351	0.9580
3.00	1.1733	0.4917	2.5981	4.4420	1.8439	0.9670

TABLE IV
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Θ WHEN N = 100

θ	A.R.E	MSE	LCL	UCL	Width	C.P
	10% Censored Samples					
0.50	1.0286	0.0031	0.4081	0.6206	0.2125	0.9570
1.00	1.0801	0.0194	0.8569	1.3032	0.4463	0.9600
1.50	1.0909	0.0483	1.2983	1.9744	0.6761	0.9720

2.00	1.0931	0.0877	1.7345	2.6378	0.9033	0.9690
2.50	1.0963	0.1415	2.1746	3.3071	1.1325	0.9760
3.00	1.0974	0.2059	2.6121	3.9725	1.3604	0.9850
			20% Censored Samples			
0.50	1.0538	0.0042	0.4114	0.6424	0.2309	0.9490
1.00	1.1065	0.0266	0.8640	1.3489	0.4849	0.9520
1.50	1.1175	0.0662	1.3090	2.0436	0.7347	0.9620
2.00	1.1198	0.1201	1.7488	2.7303	0.9815	0.9610
2.50	1.1231	0.1933	2.1925	3.4231	1.2306	0.9650
3.00	1.1243	0.2811	2.6337	4.1119	1.4782	0.9770

TABLE V
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Θ WHEN N = 150

θ	A.R.E	MSE	LCL	UCL	Width	C.P
			10% Censored Samples			
0.50	1.0119	0.0019	0.4206	0.5913	0.1707	0.9590
1.00	1.0624	0.0123	0.8832	1.2417	0.3584	0.9620
1.50	1.0731	0.0312	1.3381	1.8811	0.5430	0.9650
2.00	1.0752	0.0569	1.7877	2.5132	0.7255	0.9730
2.50	1.0784	0.0923	2.2413	3.1509	0.9096	0.9780
3.00	1.0795	0.1346	2.6922	3.7849	1.0926	0.9870
			20% Censored Samples			
0.50	1.0366	0.0026	0.4256	0.6110	0.1855	0.9480
1.00	1.0884	0.0177	0.8937	1.2831	0.3895	0.9510
1.50	1.0993	0.0448	1.3539	1.9440	0.5901	0.9540
2.00	1.1015	0.0816	1.8088	2.5971	0.7883	0.9600
2.50	1.1048	0.1322	2.2678	3.2562	0.9884	0.9670
3.00	1.1059	0.1927	2.7241	3.9113	1.1872	0.9750

TABLE VI
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Λ WHEN N = 30

λ	A.R.E	MSE	LCL	UCL	Width	C.P
			10% Censored Samples			
0.50	0.9527	0.0090	0.2967	0.6561	0.3594	0.9470
1.00	0.9481	0.0360	0.5905	1.3057	0.7153	0.9570
1.50	0.9472	0.0810	0.8848	1.9567	1.0718	0.9550
2.00	0.9491	0.1438	1.1821	2.6141	1.4320	0.9590
2.50	0.9519	0.2242	1.4821	3.2774	1.7953	0.9660
3.00	0.9529	0.3226	1.7803	3.9368	2.1565	0.9750
			20% Censored Samples			
0.50	0.9385	0.0101	0.2815	0.6570	0.3755	0.9370
1.00	0.9339	0.0407	0.5603	1.3075	0.7473	0.9480
1.50	0.9330	0.0917	0.8395	1.9593	1.1198	0.9430
2.00	0.9348	0.1626	1.1216	2.6177	1.4960	0.9470
2.50	0.9376	0.2533	1.4062	3.2819	1.8756	0.9590
3.00	0.9386	0.3643	1.6892	3.9422	2.2530	0.9680

TABLE VII
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Λ WHEN N = 50

λ	A.R.E	MSE	LCL	UCL	Width	C.P
			10% Censored Samples			
0.50	0.9625	0.0055	0.3406	0.6219	0.2812	0.9510
1.00	0.9578	0.0222	0.6780	1.2377	0.5597	0.9610
1.50	0.9569	0.0500	1.0159	1.8547	0.8387	0.9590
2.00	0.9588	0.0885	1.3573	2.4778	1.1205	0.9630
2.50	0.9616	0.1376	1.7017	3.1066	1.4049	0.9700
3.00	0.9626	0.1979	2.0441	3.7316	1.6875	0.9780
			20% Censored Samples			
0.50	0.9481	0.0063	0.3271	0.6209	0.2938	0.9430
1.00	0.9435	0.0255	0.6511	1.2358	0.5848	0.9540
1.50	0.9425	0.0574	0.9756	1.8519	0.8763	0.9480
2.00	0.9444	0.1016	1.3034	2.4741	1.1707	0.9550
2.50	0.9472	0.1576	1.6342	3.1019	1.4677	0.9600
3.00	0.9482	0.2265	1.9630	3.7260	1.7630	0.9690

TABLE VIII
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Λ WHEN N = 70

λ	A.R.E	MSE	LCL	UCL	Width	C.P
			10% Censored Samples			
0.50	0.9694	0.0040	0.3650	0.6044	0.2394	0.9480
1.00	0.9646	0.0160	0.7264	1.2028	0.4764	0.9510
1.50	0.9637	0.0361	1.0886	1.8025	0.7139	0.9550
2.00	0.9656	0.0639	1.4543	2.4081	0.9538	0.9600
2.50	0.9685	0.0993	1.8234	3.0192	1.1958	0.9670
3.00	0.9695	0.1427	2.1902	3.6266	1.4364	0.9760
			20% Censored Samples			
0.50	0.9548	0.0046	0.3524	0.6024	0.2501	0.9360
1.00	0.9502	0.0186	0.7013	1.1990	0.4977	0.9430
1.50	0.9492	0.0420	1.0509	1.7968	0.7459	0.9460
2.00	0.9511	0.0742	1.4040	2.4005	0.9965	0.9510
2.50	0.9540	0.1148	1.7603	3.0096	1.2493	0.9550
3.00	0.9549	0.1648	2.1145	3.6151	1.5007	0.9660

TABLE IX
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Λ WHEN N = 100

λ	A.R.E	MSE	LCL	UCL	Width	C.P
			10% Censored Samples			
0.50	0.9829	0.0028	0.3899	0.5930	0.2031	0.9530
1.00	0.9781	0.0111	0.7760	1.1802	0.4041	0.9560
1.50	0.9771	0.0250	1.1629	1.7685	0.6056	0.9600
2.00	0.9791	0.0444	1.5536	2.3627	0.8091	0.9660
2.50	0.9820	0.0690	1.9478	2.9622	1.0144	0.9720
3.00	0.9830	0.0992	2.3397	3.5582	1.2185	0.9810
			20% Censored Samples			
0.50	0.9681	0.0032	0.3780	0.5901	0.2121	0.9440
1.00	0.9634	0.0129	0.7523	1.1745	0.4222	0.9470
1.50	0.9625	0.0292	1.1273	1.7600	0.6327	0.9520
2.00	0.9644	0.0516	1.5061	2.3514	0.8453	0.9590
2.50	0.9673	0.0798	1.8883	2.9481	1.0598	0.9630
3.00	0.9682	0.1145	2.2682	3.5412	1.2730	0.9740

TABLE X
AVERAGE RELATIVE ESTIMATES, MSES, CONFIDENCE LIMITS AND COVERAGE PROBABILITY OF Λ WHEN N = 150

λ	A.R.E	MSE	LCL	UCL	Width	C.P
			10% Censored Samples			
0.50	0.9912	0.0018	0.4120	0.5792	0.1672	0.9610
1.00	0.9864	0.0074	0.8200	1.1528	0.3328	0.9640
1.50	0.9854	0.0167	1.2287	1.7274	0.4987	0.9680
2.00	0.9874	0.0295	1.6416	2.3078	0.6662	0.9740

2.50	0.9903	0.0460	2.0581	2.8934	0.8353	0.9810
3.00	0.9913	0.0662	2.4722	3.4756	1.0033	0.9900
20% Censored Samples						
0.50	0.9763	0.0021	0.4008	0.5755	0.1747	0.9520
1.00	0.9716	0.0087	0.7977	1.1454	0.3477	0.9550
1.50	0.9706	0.0196	1.1954	1.7164	0.5210	0.9590
2.00	0.9725	0.0345	1.5971	2.2931	0.6960	0.9660
2.50	0.9755	0.0533	2.0023	2.8750	0.8727	0.9730
3.00	0.9764	0.0765	2.4052	3.4534	1.0482	0.9860

It is immediate from the above analysis that the shape parameter θ has been over estimated; while the parameter λ has been under estimated for all sample sizes and under each censoring rate. The degree of over/under estimation is inversely proportional to sample size, while it is directly proportional to the true parametric value and the censoring rate. It has also been assessed that the estimates of parameter λ are comparatively closer to the actual values. However, the magnitudes of the mean square error (MSE) associated with the estimates of both the parameters tend to decrease by increasing the sample size. Again, greater sample sizes lead to the smaller widths of the confidence intervals and larger coverage probabilities. This simply indicates that the estimators of the parameters are consistent. It is interesting to note that for higher true parametric values, the coverage probabilities are relatively bigger due to the larger widths of the concerned confidence intervals. On the other hand, larger levels of the censoring rate result in the slower convergence (towards the true parametric values) of the estimates with inflated amounts of MSEs, hence providing wider confidence intervals. So, the performance of the estimators has been negatively affected by the increased censoring rates. It is a natural consequence of the censoring. However, it has been observed that the affects of the left censored observations are not that much severe in case of bigger sample sizes. Further for fixed sample size and censoring rate, the higher actual values of the parameters impose a negative impact on the performance (in terms of MSEs, convergence rate and widths of confidence intervals) of the estimates. It leads to the conclusion that the estimation of extremely large values of the parameters of the Burr type V distribution may become difficult and the Fisher information matrix may be the decreasing function of the parameters. But the moderate to huge sample sizes can face off this problem.

In the tables 11-14, we have discussed the limiting behavior of the variance covariance matrix obtained by inverting the fisher information matrix given in (12). As the analytical results of the Fisher information matrix for $n \to \infty$ cannot be obtained, we have calculated the entries of the Fisher information/variance covariance matrix by taking n = 5000 (extremely large). Different levels of the censoring rate have been employed for the analysis. Each table contains the variance of the concerned estimator along with their covariance. The covariance terms have been presented in the parenthesis.

TABLE XI

ELEMENTS OF VARIANCE-COVARIANCE MATRIX INCLUDING $V\left(\hat{\theta}\right)$ AND $Cov\left(\hat{\lambda},\hat{\theta}\right)$ FOR 10% CENSORED DATA ($\Lambda = 2$)

θ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0133	0.0070	0.0045	0.0029	0.0018	0.0015
	(0.0157)	(0.0164)	(0.0107)	(0.0068)	(0.0044)	(0.0035)
1.00	0.0586	0.0310	0.0201	0.0129	0.0083	0.0066
	(0.0692)	(0.0780)	(0.0507)	(0.0326)	(0.0210)	(0.0168)
1.50	0.1346	0.0711	0.0462	0.0297	0.0191	0.0153
	(0.1590)	(0.1741)	(0.1132)	(0.0727)	(0.0469)	(0.0375)
2.00	0.2403	0.1270	0.0826	0.0531	0.0342	0.0274
	(0.2838)	(0.3488)	(0.2269)	(0.1458)	(0.0940)	(0.0752)
2.50	0.3778	0.1996	0.1298	0.0834	0.0538	0.0430
	(0.4461)	(0.5683)	(0.3697)	(0.2376)	(0.1532)	(0.1226)
3.00	0.5452	0.2880	0.1874	0.1204	0.0776	0.0621
	(0.6436)	(0.8488)	(0.5521)	(0.3548)	(0.2289)	(0.1831)

TABLE XII

ELEMENTS OF VARIANCE-COVARIANCE MATRIX INCLUDING $V\left(\hat{\theta}\right)$ AND $Cov\left(\hat{\lambda},\hat{\theta}\right)$ FOR 20% CENSORED DATA ($\Lambda = 2$)

θ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0157	0.0083	0.0054	0.0034	0.0022	0.0017
	(0.0368)	(0.0194)	(0.0126)	(0.0081)	(0.0052)	(0.0042)
1.00	0.0692	0.0366	0.0238	0.0153	0.0098	0.0078
	(0.1743)	(0.0921)	(0.0599)	(0.0385)	(0.0248)	(0.0198)
1.50	0.1590	0.0840	0.0546	0.0351	0.0226	0.0181
	(0.3890)	(0.2055)	(0.1337)	(0.0859)	(0.0554)	(0.0443)
2.00	0.2838	0.1499	0.0975	0.0626	0.0404	0.0323
	(0.4000)	(0.2449)	(0.1774)	(0.1277)	(0.0865)	(0.0692)
2.50	0.4461	0.2357	0.1533	0.0985	0.0635	0.0508
	(1.2698)	(0.6710)	(0.4365)	(0.2805)	(0.1809)	(0.1447)
3.00	0.6436	0.3401	0.2212	0.1421	0.0917	0.0733
	(1.8966)	(1.0022)	(0.6519)	(0.4189)	(0.2702)	(0.2162)

TABLE XII

ELEMENTS OF VARIANCE-COVARIANCE MATRIX INCLUDING $V\left(\hat{\lambda}\right)$ AND $Cov\left(\hat{\lambda},\hat{\theta}\right)$ FOR 10% CENSORED DATA ($\Theta = 2$)

λ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0084	0.0051	0.0037	0.0026	0.0018	0.0014
	(0.0197)	(0.0120)	(0.0087)	(0.0062)	(0.0042)	(0.0034)
1.00	0.0332	0.0203	0.0147	0.0106	0.0072	0.0057
	(0.0837)	(0.0513)	(0.0371)	(0.0267)	(0.0181)	(0.0145)
1.50	0.0747	0.0457	0.0331	0.0238	0.0161	0.0129
	(0.1829)	(0.1120)	(0.0811)	(0.0583)	(0.0395)	(0.0316)
2.00	0.1334	0.0817	0.0592	0.0426	0.0288	0.0231
	(0.2838)	(0.3488)	(0.2269)	(0.14584)	(0.0940)	(0.0752)
2.50	0.2097	0.1284	0.0930	0.0669	0.0454	0.0363

	(0.5970)	(0.3656)	(0.2648)	(0.1906)	(0.1292)	(0.1033)
3.00	0.3026	0.1853	0.1342	0.0966	0.0655	0.0524
	(0.8917)	(0.5460)	(0.3956)	(0.2847)	(0.1930)	(0.1544)

TABLE XIV

ELEMENTS OF VARIANCE-COVARIANCE MATRIX INCLUDING $V\left(\hat{\lambda}\right)$ AND $Cov\left(\hat{\lambda},\hat{\theta}\right)$ FOR 20% CENSORED DATA ($\Theta = 2$)

λ	Sample Size					
	30	50	70	100	150	5000
0.50	0.0091	0.0056	0.0040	0.0029	0.0019	0.0015
	(0.0215)	(0.0131)	(0.0095)	(0.0068)	(0.0046)	(0.0037)
1.00	0.0363	0.0222	0.0161	0.0116	0.0078	0.0062
	(0.0914)	(0.0559)	(0.0405)	(0.0291)	(0.0197)	(0.0158)
1.50	0.0816	0.0499	0.0362	0.0260	0.0176	0.0141
	(0.1996)	(0.1222)	(0.0885)	(0.0637)	(0.0432)	(0.0345)
2.00	0.1456	0.0891	0.0646	0.0465	0.0315	0.0252
	(0.4000)	(0.2449)	(0.1774)	(0.1277)	(0.0865)	(0.0692)
2.50	0.2289	0.1401	0.1015	0.0730	0.0495	0.0396
	(0.6517)	(0.3990)	(0.2891)	(0.2080)	(0.1410)	(0.1128)
3.00	0.3303	0.2022	0.1465	0.1054	0.0715	0.0572
	(0.9733)	(0.5960)	(0.4318)	(0.3107)	(0.2106)	(0.1685)

VI. CONCLUSIONS AND RECOMMENDATIONS

It is evident from the above tables that even the small samples sizes with higher censoring rates are closely related to the limiting figures of the variance covariance matrix. It implies that the approximate variance covariance matrix can be used for the analysis of the unknown parameters of the Burr type V distribution. It further indicates that the proposed maximum likelihood point and interval estimates can effectively be applied to the real life situations using moderate sample sizes. The findings are useful for researchers dealing with left censored data especially in the field of medical sciences.

REFERENCES

[1] W. I. Burr, “Cumulative frequency distribution,” *Annals of Mathematical Statistics*, vol. 13, pp. 215–232, 1942.

[2] J. G. Surles, and W.J. Padgett, “Inference for reliability and stress-length for a scaled burr type x distribution,” *Lifetime Data analysis*, vol. 7, pp. 187–202, 2001.

[3] M. A. M. Mousa, and Z. F. Jaheen, “Statistical inference for the burr model based on progressively censored data,” *Computers & Mathematics with Applications*, vol. 10-11, pp. 1441–1449, 2002.

[4] A. A. Soliman, “Reliability estimation in a generalized life model with application to the burr-xii,” *IEEE Tran. on Reliability*, Vol. 51, pp. 337–343, 2002.

[5] Q. Shao, “Notes on maximum likelihood estimation for the three-parameter burr xii distribution,” *Computational Statistics and Data Analysis*, vol. 45, pp. 675–687, 2004a.

[6] Q. Shao, H. Wong, and J. Xia, “Models for extremes using the extended three parameter burr xii system with application to flood frequency analysis,” *Hydrological Sciences Journal des Sciences Hydrologiques*, vol. 49, pp. 685–702, 2004b.

[7] A. A. Soliman, “Estimation of parameters of life from progressively censored data using burr-xii model,” *IEEE Transactions on Reliability*, vol. 54, pp. 34–42, 2005.

[8] J. W. Wu, and H. Y. Yu, “Statistical inference about the shape parameter of the burr type xii distribution under the failure-censored sampling plan,” *Applied Mathematics and Computation*, vol. 163, no. 1, pp. 443–482, 2005.

[9] A. Amjad, and B. Ayman, “Interval estimation for the scale parameter of burr type x distribution based on grouped data,” *Journal of Modern Applied Statistical Methods*, vol. 3, pp. 386–398, 2006.

[10] A. S. Wahed, “Bayesian inference using burr model under asymmetric loss function: an application to carcinoma survival data,” *Journal of Statistical Research*, vol. 40, no. 1, pp. 45–57, 2006.

[11] S. J. Wu, Y. J. Chen, and C. T. Chang, “Statistical inference based on progressively censored samples with random removals from the burr type xii distribution,” *Journal of Statistical Computation and Simulation*, vol. 77, pp. 19–27, 2007.

[12] K. M. Aludaat, M. T. Alodat, and T. T. Alodat, “Parameter estimation of burr type x distribution for grouped data,” *Journal of Applied Mathematical Sciences*, vol. 2, no. 9, pp. 415–423, 2008.

[13] G. O. Silva, E. M. M. Ortega, V. C. Garibay et al., “Log-burr xii regression models with censored data,” *Computational Statistics and Data Analysis*, vol. 52, pp. 3820–3842, 2008.

[14] M. Yarmohammadi, and H. Pazira, “Minimax estimation of the parameter of the burr type xii distribution,” *Australian Journal of Basic and Applied* Sciences, vol. 4, no. 12, pp. 6611–6622, 2010.

[15] R. Dasgupta, “On the distribution of burr with applications,” *Sankhya B*, vol. 73, pp. 1–19, 2011.

[16] I. Makhdoom, and A. Jafari, “Bayesian estimations on the burr type xii distribution using grouped and un-grouped data,” *Australian Journal of Basic and Applied Sciences*, vol. 5, no. 6, pp. 1525–1531, 2011.

[17] H. Panahi, and S. Asadi, “Analysis of the type-ii hybrid censored burr type xii distribution under linex loss function,” *Applied Mathematical Sciences*, vol. 5, no. 79, pp. 3929–3942, 2011.

[18] N. Feroze, and M. Aslam, “Bayesian analysis of burr type x distribution under complete and censored samples,” *International Journal of Pure and Applied Sciences and Technology*, vol. 11, no. 2, pp. 16–28, 2012.

[19] F. Jerald, and J. F. Lawless, *“Statistical models and methods for lifetime data*,” Second Edition, University of Waterloo, 2003.

[20] P. Sinha, M. B. Lambert, and V. L. Trumbull, “Evaluation of statistical methods for left-censored environmental data with nonuniform detection limits,” *Environ Toxicol Chem.*, vol. 25, no. 9, pp. 2533–40, 2006.

[21] J. Asselineau, R. Thiebaut, P. Perez, et al., “Analysis of left-censored quantitative outcome: example of procalcitonin level,” *Rev Epidemiol Sante Publique*, vol. 55, no. 3, pp. 213–20, 2007.

[22] R. C. Antweller, and H. E. Taylor, “Evaluation of statistical treatments of left-censored environmental data using coincident uncensored data sets: I. Summary statistics,” *Environ. Sci. Technol.*, vol. 42, pp. 3732–3738, 2008.

[23] E. M. Thompson, J. B. Hewlett, and L. G. Baise, “The Gumbel hypothesis test for left censored observations using regional earthquake records as an example,” *Nat. Hazards Earth Syst. Sci.*, vol. 11, pp. 115–126, 2011.

[24] N. Feroze and M. Aslam, “On Bayesian analysis of burr type vii distribution under different censoring schemes,” *International Journal of Quality, Statistics, and Reliability*, vol. 3, pp. 1–5, 2012.

[25] S. Wu, and C. Kus, “On estimation based on progressive first-failure-censored sampling,” *Computational Statistics and Data Analysis*, vol. 53, pp. 3659–3670. 2009.

[26] G. Zheng, and J. L. Gastwirth, “Where is the Fisher information in an ordered sample?,” *Statistica Sinica,* vol. 10, pp. 1267–1280, 2000.

[27] R. D. Gupta, R. C. Gupta, and P. G. Sankaran, “Fisher information in terms of the (reversed) hazard rate function,” *Communication in Statistics: Theory and Methods,* vol. 33, pp. 3095–3102, 2004.