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Existence of periodic solution for p-Laplacian neutral Rayleigh equation with sign-variable coefficient of non linear term

Aomar Anane, Omar Chakrone, Loubna Moutaouekkil.

Abstract—As p-Laplacian equations have been widely applied in field of the fluid mechanics and nonlinear elastic mechanics, it is necessary to investigate the periodic solutions of functional differential equations involving the scalar p-Laplacian. By using Mawhin's continuation theorem, we study the existence of periodic solutions for p-Laplacian neutral Rayleigh equation

$$(\varphi_p(x'(t)-c(t)x'(t-r)))' + f(x'(t)) + g_1(x(t-\tau_1(t,|x|_\infty))) + \beta(t)g_2(x(t-\tau_2(t,|x|_\infty))) = e(t),$$

It is meaningful that the functions $c(t)$ and $\beta(t)$ are allowed to change signs in this paper, which are different from the corresponding ones of known literature.

Keywords—periodic solution; neutral Rayleigh equation; variable sign; Deviating argument; p-Laplacian; Mawhin's continuation.

I. Introduction

In recent years, periodic solutions involving the scalar p-Laplacian were studied extensively by many mathematical researchers.
Cheung and Ren [1] studied the following equation

$$(\varphi_p(x'(t))' + f(x'(t)) + \beta g(x(t-\tau(t))) = e(t),$$

under the condition of constant $\beta > 0$.
Xuejun Gao [2] discussed the existence of periodic solutions for p-Laplacian functional differential equations with two deviating arguments

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g_1(t, x(t-\tau_1(t))) + g_2(t, x(t-\tau_2(t))) = e(t),$$

Feng, Lixiang and Shiping [3] investigated the existence of periodic solutions for a p-Laplacian neutral functional differential equation with the following form

$$(\varphi_p(x'(t)-c(t)x'(t-r)))' = f(x(t))x'(t) + \beta(t)g(x(t-\tau(t))) + e(t),$$

where $c(t)$ and $\beta(t)$ are allowed to change signs.
The purpose of this paper is to study the existence of periodic solutions for p-Laplacian neutral Rayleigh equation

$$(\varphi_p(x'(t)-c(t)x'(t-r)))' + f(x'(t)) + g_1(x(t-\tau_1(t,|x|_\infty))) + \beta(t)g_2(x(t-\tau_2(t,|x|_\infty))) = e(t) \quad (1)$$

University Mohamed first, Faculty of sciences, Oujda, Morocco e-mail: anane@sciences.univ-oujda.ac.ma, chakrone@yahoo.fr, loubna_anits@yahoo.fr.

where $p > 1$ is a fixed real number. The conjugate exponent of p is denoted by q, i.e $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi_p : \mathbb{R} \to \mathbb{R}$ be the mapping defined by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$, and $\varphi_p(0) = 0$, $f, g_i \in C(\mathbb{R},\mathbb{R}), f(0) = 0, c(t), \beta(t), e(t)$ are continuous T-periodic functions defined on $\mathbb{R}$ and $T > 0, c(t)$ is not a constant function, $\int_0^T \beta(t)dt \neq 0, \int_0^T e(t)dt = 0, r \in \mathbb{R}$ is a constant with $r > 0, \tau_i \in C(\mathbb{R}^2,\mathbb{R})(i = 1, 2), \tau_i(t+T, .) = \tau_i(t, .)$.
We will study the existence of periodic solutions for Eq (1) under the case $\int_0^T \beta(t)dt > 0$ ($\int_0^T \beta(t)dt < 0$ can be discussed in the same way). Obviously, $\beta(t)$ is sign-changeable. On the hand, it is meaningful that the growth degree with respect to the variable u in $g_1(u)$ is allowed to be greater than $p-1$. For constant $m_1 > p-1$, two-sided growth condition imposed on $g_1(u)$ is given as follows

$$r_1|u|^{m_1} \leq |g_1(u)| \leq r_2|u|^{m_1}, \quad \forall |u| > d.$$

On the other hand we analyze some properties of the linear difference operator $A : [Ax](t) = x(t) - c(t)x(t-r)$ in the first, and obtain new inequalities. By using the continuation theorem of coincidence degree theory and some new analysis techniques, we obtain some results on the existence of periodic solutions to Eq (1). Meanwhile, the function $c(t)$ is allowed to change sign.

II. Preliminarie

For convenience, define $\mathcal{C}_T = \{x \in \mathcal{C}(\mathbb{R},\mathbb{R}) : x(t+T) = x(t)\}$ with the norm $|x|_\infty = \max |x(t)|_{t\in[0,T]}$. Clearly $\mathcal{C}_T$ is a Banach space . In what follows, we will use $\|.\|_p$ to denote the L^P-norm. We also define a linear operator A as follows

$$A : \mathcal{C}_T \to \mathcal{C}_T, \qquad (Ax)(t) = x(t) - c(t)x(t-r) \quad (2)$$

and we have the following notation

$$D_p = \begin{cases} 1 & if \quad 1 < p < 2, \\ 3^{p-2} & if \quad p \geq 2. \end{cases}$$

Lemma 2.1: [3]
Let $B : \mathcal{C}_T \to \mathcal{C}_T, \quad (Bx)(t) = c(t)x(t-r)$,
then B satisfies the following conditions

(1) $\|B\| \leq |c|_\infty$.

(2) $(\int_0^T |[B^jx](t)|^p dt)^{\frac{1}{p}} \leq |c|^j_\infty(\int_0^T |x(t)|^p dt)^{\frac{1}{p}}, \quad \forall x \in \mathcal{C}_T, p > 1, j \geq 1.$

Lemma 2.2: [3]
If $|c|_\infty \neq 1$, then A has continuous bounded inverse A^{-1} with the following properties, where A is defined by (2.1), and

(1) $\|A^{-1}\| \leq \frac{1}{|1-|c|_\infty|}$,

(2) $(A^{-1}f)(t) = f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)r)f(t-jr), \quad \forall f \in \mathcal{C}_T$,

(3) $\int_0^T |(A^{-1}f)(t)|^p dt \leq (\frac{1}{|1-|c|_\infty|})^p \int_0^T |f(t)|^p dt, \quad \forall f \in \mathcal{C}_T$.

Now, we recall Mawhin's continuation theorem which our study is based upon.
Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero. Here $D(L)$ denotes the domain of L. This means that ImL is closed in Y and $\dim KerL = \dim(Y/ImL) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 and such that
$X = KerL \bigoplus X_1$ and $Y = ImL \bigoplus Y_1$ and let
$P : X \to KerL$ and $Q : Y \to Y_1$ be natural projections. Clearly, $KerL \bigcap (D(L) \bigcap X_1) = \{0\}$, thus the restriction
$L_p := L|_{D(L)\cap X_1}$ is invertible. Denote the inverse of L_p by K. Now, let Ω be an open bounded subset of X with
$D(L)\bigcap \Omega \neq \emptyset$, a map $N : \overline{\Omega} \to Y$ is said to be L-compact on $\overline{\Omega}$. If $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to Y$ is compact.

Lemma 2.3: (Mawhin [4])
Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthemore, $\Omega \subset X$ is an open bounded set, and
$N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all of the following conditions hold:
(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \bigcap D(L), \lambda \in]0,1]$;
(2) $Nx \notin ImL, \forall x \in \partial\Omega \bigcap KerL; and$
(3) $deg\{JQN, \Omega\cap KerL, 0\} \neq 0$, where $J : ImQ \to KerL$ is an isomorphism.
Then the equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \bigcap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of T-periodic solution for Eq (1) , we rewrite Eq (1) in the following system

$$\begin{cases} x_1'(t) = [A^{-1}\varphi_q(x_2)](t), \\ x_2'(t) = -f([A^{-1}\varphi_q(x_2)](t)) - g_1(x_1(t-\tau_1(t,|x_1|_\infty))) \\ \qquad - \beta(t)g_2(x_1(t-\tau_2(t,|x_1|_\infty))) + e(t). \end{cases} \tag{3}$$

Where $q > 1$ is constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^T$ is a T-periodic solution to equation set (3), then $x_1(t)$ must be a T-periodic solution to equation (1). Thus, in order to prove that Eq (1) has a T-periodic solution, it suffices to show that equation set (3) has a T-periodic solution. Now, we set $X = Y = \{x = (x_1(t), x_2(t))^T \in C(\mathbb{R}, \mathbb{R}^2) : x_1 \in C_T, x_2 \in C_T\}$ with the norm $\|x\| = max\{|x_1|_\infty, |x_2|_\infty\}$. Obviously, X and Y are two Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y, \qquad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}. \tag{4}$$

$$N : X \to Y,$$

$$[Nx](t) = \begin{pmatrix} [A^{-1}\varphi_q(x_2)](t) \\ -f([A^{-1}\varphi_q(x_2)](t)) - g_1(x_1(t-\tau_1(t,|x_1|_\infty)))- \\ \beta(t)g_2(x_1(t-\tau_2(t,|x_1|_\infty))) + e(t) \end{pmatrix}. \tag{5}$$

It is easy to see that equation set (3) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we see that $KerL = \mathbb{R}^2$, $ImL = \{y : y \in Y, \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero.
Let projections $P : X \to KerL$ and $Q : Y \to ImQ$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)ds, \qquad Qy = \frac{1}{T}\int_0^T y(s)ds$$

and let K represent the inverse of $L|_{KerP\cap D(L)}$. Clearly, $KerL = ImQ = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s)y(s)ds, \tag{6}$$

where $G(t,s) = \begin{cases} \frac{s}{T}, & 0 \leq s < t \leq T; \\ \frac{s-T}{T}, & 0 \leq t \leq s \leq T. \end{cases}$
From (5)and (6), it isn't hard to find that N is L-compact on $\overline{\Omega}$, where Ω is an arbitrary open bounded subset of X.

Lemma 2.4: (Borsuk [5]) $\Omega \subset \mathbb{R}^n$ is an open bounded set, and symmetric with respect to $0 \in \Omega$. If $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $f(x) \neq \mu f(-x), \forall x \in \partial\Omega, \forall \mu \in [0,1]$, then $deg(f, \Omega, 0)$ is an odd number.

Lemma 2.5: If $c(t) \in C_T$ is not a constant function, $|c|_\infty < \frac{1}{2}$,

$$(Ax)(t) = x(t) - c(t)x(t-r) \equiv d_1 \tag{7}$$

where d_1 is a nonzero constant, $x(t) \in C_T$, then
(1) $x(t) = A^{-1}d_1$ is not a constant function,
(2) $\int_0^T (A^{-1}d_1)(t)dt \neq 0$.

III. Main results

Theorem 3.1: Assume that the following conditions are satisfied.
$(H_1)-$ For $i = 1, 2$, there are positive constants d, r_i, r_i^*, m_i with $m_2 \leq p-1$ such that
(1) $\beta_1|u|^{m_i} \leq |g_i(u)| \leq \beta_2|u|^{m_i}, \quad \forall |u| > d$,
$\beta_i = \begin{cases} r_i, & if \ \ g_i(u) \equiv g_1(u), \\ r_i^*, & if \ \ g_i(u) \equiv g_2(u). \end{cases}$
(2) $g_i(w)sgn(w) < 0, \quad \forall |w| > d$.
$(H_2)-$ $A := D_{\frac{1}{m_1}+1}\left(\frac{r_2^*|\beta|_\infty}{r_1}\right)^{\frac{1}{m_1}} < 1$.
$(H_3)-$ There exists constant $\alpha \geq 0$ such that $|f(y)| \leq \alpha|y|^{p-1}, \forall y \in \mathbb{R}$.
$(H_4)-$ There are constants $r_3 > 0, \gamma$ and $k \in \mathbb{Z}$ such that $0 \leq \tau_1(t,x) - kT \leq min\left\{\frac{\gamma^{\frac{p}{p-1}}}{1+|x|_\infty^{\frac{r_3 p}{p-1}}}, T\right\}, \forall (t,x) \in [0,T] \times \mathbb{R}$.

Then, Eq (1) has at least one T-periodic solution, if one of the following conditions holds

(1) $m_2 = p-1$ and $\Delta_1 + \Delta_2 < 1$.
(2) $m_2 < p-1$ and $\Delta_1 < 1$.

where

$$\Delta_1 = \left(\frac{1}{1-|c|_\infty}\right)^p \Big[|c|_\infty(1+|c|_\infty)^{p-1} + \frac{\alpha T}{2(1-A)} + \frac{D_p r_2 \gamma T^{1+\frac{(p-1)^2}{p}}}{2^{p-1}(1-A)^{p-1}}\ \Big].$$

and

$$\Delta_2 = \left(\frac{1}{1-|c|_\infty}\right)^p \left[\frac{3^{m_2} r_2^* |\beta|_\infty T^{1+(p-1)(m_2+1)}}{2^{m_2+1}(1-A)^{m_2+1}}\right].$$

Proof 1: Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in]0,1]\}$ if $x(.) = (x_1(.), x_2(.))^T \in \Omega_1$, then from (4) and (5), we have

$$\begin{cases} x_1'(t) = \lambda[A^{-1}\varphi_q(x_2)](t), \\ x_2'(t) = -\lambda f(\lambda[A^{-1}\varphi_q(x_2)](t)) \\ \qquad - \lambda g_1(x_1(t-\tau_1(t,|x_1|_\infty))) \\ \qquad - \lambda\beta(t) g_2(x_1(t-\tau_2(t,|x_1|_\infty))) + \lambda e(t). \end{cases} \tag{8}$$

From the first equation of (8), we have $x_2(t) = \varphi_p(\frac{1}{\lambda}(Ax_1')(t)$, together with the second formula of (8), which yields

$$\begin{aligned} &[\varphi_p((Ax_1')(t))]' + \lambda^p f(\lambda[A^{-1}\varphi_q(x_2)](t)) \\ &+ \lambda^p g_1(x_1(t-\tau_1(t,|x_1|_\infty))) \\ &+ \lambda^p \beta(t) g_2(x_1(t-\tau_2(t,|x_1|_\infty))) = \lambda^p e(t). \end{aligned} \tag{9}$$

Integrating both sides of Eq.(9) on the interval $[0,T]$ and applying integral mean value theorem, then there exists a constant $\xi \in [0,T]$ such that

$$\begin{aligned} &g_1(x_1(\xi - \tau_1(\xi, |x_1|_\infty)))T \\ &= -\int_0^T \beta(t) g_2(x_1(t-\tau_2(t,|x_1|_\infty)))dt - \int_0^T f(x_1'(t))dt. \end{aligned} \tag{10}$$

Now, we claim that

$$|x_1(\xi - \tau_1(\xi,|x_1|_\infty))| \le A|x_1|_\infty + B\left(\int_0^T |x_1'(t)|^{p-1}dt\right)^{\frac{1}{m_1}} + C. \tag{11}$$

Where $\quad B = D_{\frac{1}{m_1}+1}\left(\frac{\alpha}{r_1 T}\right)^{\frac{1}{m_1}}$,

$$C = D_{\frac{1}{m_1}+1}\left(\frac{M_{g_2}|\beta|_\infty}{r_1}\right)^{\frac{1}{m_1}} + d, \quad M_{g_2} = \max_{|v|\le d}|g_2(v)|$$

.

Case(1). If $|x_1(\xi - \tau_1(\xi,|x_1|_\infty))| \le d$, then, Eq.(11) holds clearly.

Case(2). If $|x_1(\xi - \tau_1(\xi,|x_1|_\infty))| > d$, it follows from Eq.(10), $(H_1)(1)$ and (H_3) that

$$\begin{aligned} &r_1 T|x_1(\xi - \tau_1(\xi,|x_1|_\infty))|^{m_1} \\ &\le |\beta|_\infty \int_0^T |g_2(x_1(t-\tau_2(t,|x_1|_\infty)))|dt - \int_0^T |f(x_1'(t))|dt \\ &\le r_2^* T|\beta|_\infty |x_1|_\infty^{m_2} + \alpha\int_0^T |x_1'(t)|^{p-1}dt + M_{g_2}T|\beta|_\infty. \end{aligned}$$

It implies that

$$\begin{aligned} &|x_1(\xi - \tau_1(\xi,|x_1|_\infty))| \\ &\le D_{\frac{1}{m_1}+1}[\left(\frac{r_2^*|\beta|_\infty}{r_1}\right)^{\frac{1}{m_1}} |x_1|_\infty^{\frac{m_2}{m_1}} \\ &+ \left(\frac{\alpha}{r_1 T}\right)^{\frac{1}{m_1}}\left(\int_0^T |x_1'(t)|^{p-1}dt\right)^{\frac{1}{m_1}} + \left(\frac{M_{g_2}|\beta|_\infty}{r_1}\right)^{\frac{1}{m_1}}] \\ &\le D_{\frac{1}{m_1}+1}\left(\frac{r_2^*|\beta|_\infty}{r_1}\right)^{\frac{1}{m_1}} |x_1|_\infty \\ &+ D_{\frac{1}{m_1}+1}\left(\frac{\alpha}{r_1 T}\right)^{\frac{1}{m_1}}\left(\int_0^T |x_1'(t)|^{p-1}dt\right)^{\frac{1}{m_1}} \\ &+ D_{\frac{1}{m_1}+1}\left(\frac{M_{g_2}|\beta|_\infty}{r_1}\right)^{\frac{1}{m_1}}. \end{aligned}$$

Thus, it is easy to see that Eq.(11) holds.
Let

$$\xi - \tau_1(\xi,|x_1|_\infty) = kT + \bar{\xi},$$

where k is an integer and $\bar{\xi} \in [0,T]$, by (11)

$$\begin{aligned} |x_1(t)| &\le A|x_1|_\infty + B\left(\int_0^T |x_1'(s)|^{p-1}ds\right)^{\frac{1}{m_1}} \\ &+ \int_{\bar{\xi}}^t |x_1'(s)|ds + C, \qquad t \in [\bar{\xi}, \bar{\xi}+T]. \end{aligned}$$

and

$$\begin{aligned} |x_1(t)| &= |x_1(t-T)| \\ &\le A|x_1|_\infty + B\left(\int_0^T |x_1'(s)|^{p-1}ds\right)^{\frac{1}{m_1}} \\ &+ \int_{t-T}^{\bar{\xi}} |x_1'(s)|ds + C, \qquad t \in [\bar{\xi}, \bar{\xi}+T]. \end{aligned}$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x_1|_\infty &= \max_{t\in[0,T]}|x_1(t)| = \max_{t\in[\bar{\xi},\bar{\xi}+T]}|x_1(t)| \\ &\le \max_{t\in[\bar{\xi},\bar{\xi}+T]}\{A|x_1|_\infty + B\left(\int_0^T |x_1'(s)|^{p-1}ds\right)^{\frac{1}{m_1}} \\ &+ \frac{1}{2}\left(\int_{\bar{\xi}}^t |x_1'(s)|ds + \int_{t-T}^{\bar{\xi}} |x_1'(s)|ds\right) + C\} \\ &\le A|x_1|_\infty + B\left(\int_0^T |x_1'(s)|^{p-1}ds\right)^{\frac{1}{m_1}} \\ &+ \frac{1}{2}\int_0^T |x_1'(s)|ds + C. \end{aligned}$$

In view of (H_2), we have

$$|x_1|_\infty \le \frac{\int_0^T |x_1'(t)|dt}{2(1-A)} + \frac{B}{1-A}\left(\int_0^T |x_1'(t)|^{p-1}dt\right)^{\frac{1}{m_1}} + \frac{C}{1-A}. \tag{12}$$

On the hand, multiplying both sides of Eq.(9) by $x_1(t)$ and integrating it from 0 to T, we obtain

$$\begin{aligned}&\int_0^T [\varphi_p((Ax_1')(t))]'x_1(t)dt\\&+\lambda^p\int_0^T f(\lambda[A^{-1}\varphi_q(x_2)](t))x_1(t)dt\\&+\lambda^p\int_0^T g_1(x_1(t-\tau_1(t,|x_1|_\infty)))x_1(t)dt\\&+\lambda^p\int_0^T \beta(t)g_2(x_1(t-\tau_2(t,|x_1|_\infty)))x_1(t)dt\\&=\lambda^p\int_0^T e(t)x_1(t)dt.\end{aligned}\tag{13}$$

On the other hand we have

$$\begin{aligned}&\int_0^T [\varphi_p((Ax_1')(t))]'x_1(t)dt\\&=-\int_0^T \varphi_p((Ax_1')(t))x_1'(t)dt\\&=-\int_0^T \varphi_p((Ax_1')(t))[x_1'(t)-c(t)x_1'(t-r)\\&+c(t)x_1'(t-r)]dt\\&=-\int_0^T |(Ax_1')(t)|^pdt-\int_0^T c(t)x_1'(t-r)\varphi_p((Ax_1')(t))dt.\end{aligned}\tag{14}$$

Substituting Eq.(14) into Eq.(13) we get

$$\begin{aligned}&\int_0^T |(Ax_1')(t)|^pdt\\&=-\int_0^T c(t)x_1'(t-r)\varphi_p((Ax_1')(t))dt\\&+\lambda^p\int_0^T f(x_1'(t))x_1(t)dt\\&+\lambda^p\int_0^T g_1(x_1(t-\tau_1(t,|x_1|_\infty)))x_1(t)dt\\&+\lambda^p\int_0^T \beta(t)g_2(x_1(t-\tau_2(t,|x_1|_\infty)))x_1(t)dt\\&-\lambda^p\int_0^T e(t)x_1(t)dt.\end{aligned}\tag{15}$$

It follows that

$$\begin{aligned}&\int_0^T |(Ax_1')(t)|^pdt\\&\le |c|_\infty\int_0^T |\varphi_p((Ax_1')(t))||x_1'(t-r)|dt+|x_1|_\infty\int_0^T |f(x_1'(t))|dt\\&+\lambda^p\int_0^T g_1(x_1(t-\tau_1(t,|x_1|_\infty)))[x_1(t)-x_1(t-\tau_1(t,|x_1|_\infty))]dt\\&+\lambda^p\int_0^T g_1(x_1(t-\tau_1(t,|x_1|_\infty)))x_1(t-\tau_1(t,|x_1|_\infty))dt\\&+|\beta|_\infty|x_1|_\infty\int_0^T |g_2(x_1(t-\tau_2(t,|x_1|_\infty)))|dt+T|x_1|_\infty|e|_\infty.\end{aligned}\tag{16}$$

Moreover, by using Hölder's inequality and Minkowski inequality, we obtain

$$\begin{aligned}&\int_0^T |\varphi_p((Ax_1')(t))||x_1'(t-r)|dt\\&\le\left(\int_0^T |\varphi_p((Ax_1')(t))|^qdt\right)^{\frac1q}\times\left(\int_0^T |x_1'(t-r)|^pdt\right)^{\frac1p}\\&=\left(\int_0^T |(Ax_1')(t)|^pdt\right)^{\frac1q}\times\left(\int_0^T |x_1'(t)|^pdt\right)^{\frac1p}\\&=\left[\left(\int_0^T |x_1'(t)-c(t)x_1'(t-r)|^pdt\right)^{\frac1p}\right]^{\frac pq}\\&\times\left(\int_0^T |x_1'(t)|^pdt\right)^{\frac1p}\\&\le\left[\left(\int_0^T |x_1'(t)|^pdt\right)^{\frac1p}+\left(\int_0^T |c(t)x_1'(t-r)|^pdt\right)^{\frac1p}\right]^{\frac pq}\\&\times\left(\int_0^T |x_1'(t)|^pdt\right)^{\frac1p}\\&\le\left[\left(\int_0^T |x_1'(t)|^pdt\right)^{\frac1p}+|c|_\infty\left(\int_0^T |x_1'(t)|^pdt\right)^{\frac1p}\right]^{\frac pq}\\&\times\left(\int_0^T |x_1'(t)|^pdt\right)^{\frac1p}\\&=(1+|c|_\infty)^{p-1}\int_0^T |x_1'(t)|^pdt.\end{aligned}\tag{17}$$

Define

$$E_1=\{t:t\in[0,T],|x_1(t-\tau_1(t,|x_1|_\infty))|\le d\},$$

and

$$E_2=\{t:t\in[0,T],|x_1(t-\tau_1(t,|x_1|_\infty))|>d\}.$$

By the condition $(H_1(2))$, we obtain

$$\begin{aligned}&\lambda^p\int_0^T g_1(x_1(t-\tau_1(t,|x_1|_\infty)))x_1(t-\tau_1(t,|x_1|_\infty))dt\\&\le\lambda^p\int_{E_1} g_1(x_1(t-\tau_1(t,|x_1|_\infty)))x_1(t-\tau_1(t,|x_1|_\infty))dt\\&\le TdM_{g_1}.\end{aligned}\tag{18}$$

Where $M_{g_1}=\max_{|u|\le d}|g_1(u)|$.
Substituting Eqs.(18)-(17) into Eq (16) and using $(H_1(1))$ and

(H_3) yield

$$\begin{aligned}&\int_0^T |(Ax_1')(t)|^p dt\\ &\le |c|_\infty(1+|c|_\infty)^{p-1}\int_0^T |x_1'(t)|^p dt + \alpha|x_1|_\infty\int_0^T |x_1'(t)|^{p-1}dt\\ &+ r_2 \max_{t\in[0,T]} |x_1(t)-x_1(t-\tau_1(t,|x_1|_\infty))|\times\\ &\int_0^T |x_1(t-\tau_1(t,|x_1|_\infty))|^{m_1}dt + r_2^*T|\beta|_\infty|x_1|_\infty^{m_2+1}\\ &+\theta_1|x_1|_\infty + TdM_{g_1},\end{aligned} \tag{19}$$

where $\theta_1 = (2M_{g_1}+|\beta|_\infty M_{g_2}+|e|_\infty)T$.
From Eq (19) and Hölder's inequality, we obtain

$$\begin{aligned}&\alpha|x_1|_\infty\int_0^T |x_1'(t)|^{p-1}dt\\ &\le \frac{\alpha T}{2(1-A)}\int_0^T |x_1'(t)|^p dt\\ &+\frac{\alpha BT^{\frac{m_1+1}{pm_1}}}{1-A}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{(p-1)(m_1+1)}{pm_1}}\\ &+\frac{\alpha CT^{\frac{1}{p}}}{1-A}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{(p-1)}{p}}.\end{aligned} \tag{20}$$

Moreover, from (H_4) and Hölder's inequality, we have

$$\begin{aligned}&\max_{t\in[0,T]}|x_1(t)-x_1(t-\tau_1(t,|x_1|_\infty))|\\ &=\max_{t\in[0,T]}|x_1(t)-x_1(t-\tau_1(t,|x_1|_\infty))+kT|\\ &=\max_{t\in[0,T]}\left|\int_{t-\tau_1(t,|x_1|_\infty)+kT}^t x_1'(t)dt\right|\\ &\le \max_{t\in[0,T]}|\tau_1(t,|x_1|_\infty)-kT|^{\frac{p-1}{p}}\left(\int_{t-\tau_1(t,|x_1|_\infty)+kT}^t |x_1'(t)|^p dt\right)^{\frac{1}{p}}\\ &=\max_{t\in[0,T]}|\tau_1(t,|x_1|_\infty)-kT|^{\frac{p-1}{p}}\left(\int_{-\tau_1(t,|x_1|_\infty)+kT}^0 |x_1'(t)|^p dt\right)^{\frac{1}{p}}\\ &\le |\tau_1(t,|x_1|_\infty)-kT|_\infty^{\frac{p-1}{p}}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}}.\end{aligned} \tag{21}$$

$$\begin{aligned}&r_2^*T|\beta|_\infty|x_1|_\infty^{m_2+1}\\ &\le r_2^*T|\beta|_\infty\Big[\frac{\int_0^T |x_1'(t)|dt}{2(1-A)}\\ &+\frac{B}{1-A}\left(\int_0^T |x_1'(t)|^{p-1}dt\right)^{\frac{1}{m_1}}+\frac{C}{1-A}\Big]^{m_2+1}\\ &\le \frac{3^{m_2}r_2^*T|\beta|_\infty}{2^{m_2+1}(1-A)^{m_2+1}}\left(\int_0^T |x_1'(t)|dt\right)^{m_2+1}\\ &+\frac{3^{m_2}r_2^*T|\beta|_\infty B^{m_2+1}}{(1-A)^{m_2+1}}\left(\int_0^T |x_1'(t)|^{p-1}dt\right)^{\frac{m_2+1}{m_1}}\\ &+\frac{3^{m_2}r_2^*T|\beta|_\infty C^{m_2+1}}{(1-A)^{m_2+1}}\\ &\le \frac{3^{m_2}r_2^*|\beta|_\infty T^{1+\frac{(p-1)(m_2+1)}{p}}}{2^{m_2+1}(1-A)^{m_2+1}}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{m_2+1}{p}}\\ &+\frac{3^{m_2}r_2^*|\beta|_\infty B^{m_2+1}T^{1+\frac{(m_2+1)}{pm_1}}}{(1-A)^{m_2+1}}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{(p-1)(m_2+1)}{pm_1}}\\ &+\frac{3^{m_2}r_2^*T|\beta|_\infty C^{m_2+1}}{(1-A)^{m_2+1}},\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\theta_1|x_1|_\infty\\ &\le \frac{\theta_1 T^{\frac{p-1}{p}}}{2(1-A)}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}}\\ &+\frac{\theta_1 BT^{\frac{1}{pm_1}}}{1-A}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{p-1}{pm_1}}\\ &+\frac{\theta_1 C}{1-A}.\end{aligned} \tag{23}$$

In order to show that the growth degree with respect to the variable u in $g_1(u)$ is greater than $p-1$, we let $m_1 = r_3+p-1$, where r_3 is defined in (H_4).
By applying the third part of Lemma2.2, we get

$$\begin{aligned}\int_0^T |x_1'(t)|^p dt &= \int_0^T |(A^{-1}Ax_1')(t)|^p dt\\ &\le \left(\frac{1}{1-|c|_\infty}\right)^p\int_0^T |(Ax_1')(t)|^p dt.\end{aligned} \tag{24}$$

Then, substituting Eqs (20)-(24) into (19) and using (H_4)

yield

$$\begin{aligned}
&\int_0^T |(x_1')(t)|^p dt \\
&\le \frac{|c|_\infty(1+|c|_\infty)^{p-1}}{(1-|c|_\infty)^p}\int_0^T |x_1'(t)|^p dt \\
&+ \frac{\alpha|x_1|_\infty \int_0^T |x_1'(t)|^{p-1}dt}{(1-|c|_\infty)^p} \\
&+ \frac{r_2T|\tau_1(t,|x_1|_\infty)-kT|_\infty^{\frac{p-1}{p}}}{(1-|c|_\infty)^p}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}}|x_1|_\infty^{r_3}|x_1|_\infty^{p-1} \\
&+ \frac{r_2^*T|\beta|_\infty|x_1|_\infty^{m_2+1}}{(1-|c|_\infty)^p} + \frac{\theta_1|x_1|_\infty}{(1-|c|_\infty)^p} + \frac{TdM_{g_1}}{(1-|c|_\infty)^p} \\
&\le \frac{|c|_\infty(1+|c|_\infty)^{p-1}}{(1-|c|_\infty)^p}\int_0^T |x_1'(t)|^p dt \\
&+ \frac{\alpha|x_1|_\infty \int_0^T |x_1'(t)|^{p-1}dt}{(1-|c|_\infty)^p} \\
&+ \frac{r_2\gamma T}{(1-|c|_\infty)^p}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}}[\frac{\int_0^T |x_1'(t)|dt}{2(1-A)} + \\
&\frac{B}{1-A}\left(\int_0^T |x_1'(t)|^{p-1}dt\right)^{\frac{1}{m_1}} + \frac{C}{1-A}]^{p-1} \\
&+ \frac{r_2^*T|\beta|_\infty|x_1|_\infty^{m_2+1}}{(1-|c|_\infty)^p} + \frac{\theta_1|x_1|_\infty}{(1-|c|_\infty)^p} + \frac{TdM_{g_1}}{(1-|c|_\infty)^p} \\
&\le \Delta_1\int_0^T |x_1'(t)|^p dt + \Delta_2\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{m_2+1}{p}} \\
&+ \frac{\alpha BT^{\frac{m_1+1}{pm_1}}}{(1-|c|_\infty)^p(1-A)}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{(p-1)(m_1+1)}{pm_1}} \\
&+ \frac{\alpha CT^{\frac{1}{p}}}{(1-|c|_\infty)^p(1-A)}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{p-1}{p}} \\
&+ \frac{r_2T^{1+\frac{p-1}{pm_1}}\gamma D_pB^{p-1}}{(1-|c|_\infty)^p(1-A)^{p-1}}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{m_1+(p-1)^2}{pm_1}} \\
&+ \frac{3^{m_2}r_2^*|\beta|_\infty B^{m_2+1}T^{1+\frac{(m_2+1)}{pm_1}}}{(1-|c|_\infty)^p(1-A)^{m_2+1}}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{(p-1)(m_2+1)}{pm_1}} \\
&+ \frac{\theta_1BT^{\frac{1}{pm_1}}}{(1-|c|_\infty)^p(1-A)}\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{p-1}{pm_1}} \\
&+ \left[\frac{\theta_1T^{\frac{p-1}{p}}}{2(1-|c|_\infty)^p(1-A)} + \frac{r_2\gamma TD_pC^{p-1}}{(1-|c|_\infty)^p(1-A)^{p-1}}\right]\times \\
&\left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{1}{p}} + \frac{3^{m_2}r_2^*T|\beta|_\infty C^{m_2+1}}{(1-|c|_\infty)^p(1-A)^{m_2+1}} \\
&+ \frac{\theta_1C}{(1-|c|_\infty)^p(1-A)} + \frac{TdM_{g_1}}{(1-|c|_\infty)^p}.
\end{aligned} \tag{25}$$

Case1. If $m_2 = p-1$, using $\frac{m_1+(p-1)^2}{pm_1} < 1, \frac{(p-1)(m_1+1)}{pm_1} < 1, \frac{(p-1)(m_2+1)}{pm_1} < 1, \frac{1}{pm_1} < 1, \frac{p-1}{p} < 1, \frac{1}{p} < 1, \Delta_1+\Delta_2 < 1$ and Eq.(25), it is seen that $\int_0^T |x_1'(t)|^p dt$ is bounded.

Case2. If $m_2 < p-1$, noticing $\frac{m_2+1}{p} < p, \frac{m_1+(p-1)^2}{pm_1} < 1, \frac{(p-1)(m_1+1)}{pm_1} < 1, \frac{(p-1)(m_2+1)}{pm_1} < 1, \frac{1}{pm_1} < 1, \frac{p-1}{p} < 1, \frac{1}{p} < 1, \Delta_1 < 1$ and Eq.(3.18), it is also seen that $\int_0^T |x_1'(t)|^p dt$ is bounded. From the above two cases, there exists a constant $M > 0$ such that

$$\int_0^T |x_1'(t)|^p dt \le M. \tag{26}$$

Using Eq.(12) and Eq.(26) leads to

$$|x_1|_\infty \le \frac{T^{\frac{p-1}{p}}M^{\frac{1}{p}}}{2(1-A)} + \frac{BT^{\frac{1}{pm_1}}M^{\frac{p-1}{pm_1}}}{1-A} + \frac{C}{1-A} \triangleq M_1. \tag{27}$$

Again, from the first equation of (8), we have

$$\int_0^T (A^{-1}\varphi_q(x_2))(t)dt = 0,$$

then there is a constant $\eta \in [0,T]$, such that $(A^{-1}\varphi_q(x_2))(\eta) = 0$, which together with the second part of lemma2.2 gives

$$\begin{aligned}
&(A^{-1}\varphi_q(x_2))(\eta) \\
&= \varphi_q(x_2(\eta)) + \sum_{j=1}^\infty\prod_{i=1}^j c(\eta-(i-1)r)\varphi_q(x_2(\eta-jr)), \\
&= 0 \\
&|x_2(\eta)|^{q-1} = |\varphi_q(x_2(\eta))| \\
&\qquad = \left|\sum_{j=1}^\infty\prod_{i=1}^j c(\eta-(i-1)r)\varphi_q(x_2(\eta-jr))\right| \\
&\qquad \le \sum_{j=1}^\infty |c|_\infty^j|x_2|_\infty^{q-1} \\
&\qquad = \frac{|c|_\infty}{1-|c|_\infty}|x_2|_\infty^{q-1},
\end{aligned}$$

it follows that

$$|x_2(\eta)| \le \left(\frac{|c|_\infty}{1-|c|_\infty}\right)^{\frac{1}{q-1}}|x_2|_\infty. \tag{28}$$

Let $\overline{M_{g_1}} = \max_{|u|\le M_1}|g_1(u)|, \overline{M_{g_2}} = \max_{|u|\le M_1}|g_2(u)|$, and from Eq.(8), we have

$$\begin{aligned}
x_2'(t) = &-\lambda f(x_1'(t)) - \lambda g_1(x_1(t-\tau_1(t,|x_1|_\infty))) \\
&-\lambda\beta(t)g_2(x_1(t-\tau_2(t,|x_1|_\infty))) + \lambda e(t)
\end{aligned},$$

and

$$\int_0^T |x_2'(t)|dt$$
$$\leq \int_0^T |f(x_1'(t))|dt + \int_0^T |g_1(x_1(t-\tau_1(t,|x_1|_\infty)))|dt$$
$$+ \int_0^T |\beta(t)g_2(x_1(t-\tau_2(t,|x_1|_\infty)))|dt + \int_0^T |e(t)|dt$$
$$\leq \alpha \int_0^T |x_1'(t)|^{p-1}dt + T\overline{M_{g_1}} + |\beta|_\infty T\overline{M_{g_2}} + T|e|_\infty$$
$$\leq \alpha \left(\int_0^T |x_1'(t)|^p dt\right)^{\frac{p-1}{p}} T^{\frac{1}{p}} + T\overline{M_{g_1}} + |\beta|_\infty T\overline{M_{g_2}} + T|e|_\infty$$
$$\leq \alpha M^{\frac{p-1}{p}} T^{\frac{1}{p}} + T\overline{M_{g_1}} + |\beta|_\infty T\overline{M_{g_2}} + T|e|_\infty \triangleq M_2. \tag{29}$$

By (28) and (29)

$$|x_2(t)| = |x_2(\eta) + \int_\eta^t x_2'(s)ds|$$
$$\leq \left(\frac{|c|_\infty}{1-|c|_\infty}\right)^{\frac{1}{q-1}} |x_2|_\infty + \int_0^T |x_2'(s)|ds \tag{30}$$
$$\leq \left(\frac{|c|_\infty}{1-|c|_\infty}\right)^{\frac{1}{q-1}} |x_2|_\infty + M_2, \quad t \in [0,T]$$

Since $|c|_\infty < \frac{1}{2}$, $\left(\frac{|c|_\infty}{1-|c|_\infty}\right)^{\frac{1}{q-1}} < 1$, together with(30), we know there exists a positive constant M_3 such that

$$|x_2|_\infty \leq M_3. \tag{31}$$

Let $\Omega_2 = \{x | x \in KerL, QNx = 0\}$ if $x \in \Omega_2$ then $x \in \mathbb{R}^2$ is a constant vector, and

$$\begin{cases} \frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt = 0, \\ \frac{1}{T}\int_0^T [-f([A^{-1}\varphi_q(x_2)](t)) - g_1(x_1(t-\tau_1(t,|x_1|_\infty)))- \\ \beta(t)g_2(x_1(t-\tau_2(t,|x_1|_\infty))) + e(t)]dt = 0. \end{cases} \tag{32}$$

By the first formula of (32) and the second part of Lemma 2.5, we have $x_2 = 0$. Moreover, in view of $\int_0^T e(t)dt = 0, f(0) = 0$ and the second formula of (32), we know

$$g_1(x_1) + \frac{1}{T} g_2(x_1)\int_0^T \beta(t)dt = 0.$$

Which, together with $\int_0^T \beta(t)dt > 0$ and $(H_1)(2)$, yields $|x_1| \leq d$.

Now, we let $\Omega = \{x | x = (x_1, x_2)^T \in X, |x_1| < M_1 + d, |x_2| < M_3 + d\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$. So from (27) and (31), it is easy to see that conditions (1) and (2) of Lemma 2.3 are satisfied.

Next, we verify the condition (3) of Lemma 2.3. To do this, we define the isomorphism

$$J : ImQ \to KerL, \qquad J(x_1,x_2)^T = (x_1,x_2)^T,$$

then

$$JQN(x) = \begin{pmatrix} \frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt \\ \frac{1}{T}\int_0^T [-f([A^{-1}\varphi_q(x_2)](t)) - g_1(x_1)- \\ \beta(t)g_2(x_1) + e(t)]dt \end{pmatrix},$$

$x \in \overline{KerL \bigcap \Omega}$.

By Lemma 2.4, we need to prove that

$$JQN(x) \neq \mu(JQN(-x)), \forall x \in \partial(\Omega \textstyle\bigcap KerL), \mu \in [0,1]$$

Case1. If $x = (x_1,x_2)^T \in \partial(\Omega \bigcap KerL) \setminus \{(M_1+d,0)^T, (-M_1-d,0)^T\}$, then $x_2 \neq 0$ which, together with the second part of Lemma 2.5, gives us $\int_0^T [A^{-1}\varphi_q(x_2)](t)dt \neq 0$,

$$\left(\frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt\right)\left(\frac{1}{T}\int_0^T [A^{-1}\varphi_q(-x_2)](t)dt\right) < 0,$$

obviously, $\forall \mu \in [0,1] \qquad JQN(x) \neq \mu(JQN(-x))$.

Case2. If $x = (M_1+d,0)^T$ or $x = (-M_1-d,0)^T$ then

$$JQN(x) = \begin{pmatrix} 0 \\ -g_1(x_1) - \frac{1}{T}g_2(x_1)\int_0^T \beta(t)dt \end{pmatrix}$$

which, together with $(H_1)(2)$, yields $\forall \mu \in [0,1], JQN(x) \neq \mu(JQN(-x))$.

Thus, the condition (3) of Lemma 2.3 is also satisfied. Therefore, by applying Lemma 2.3, we conclude that the equation $Lx = Nx$ has at least one T-periodic solution on $\overline{\Omega}$, so Eq.(1) has at least one T-periodic solution. This completes the proof of Theorem 3.1.

IV. Example and Remark

Let us consider the following equation

$$\begin{aligned} &(\varphi_p(x'(t) - 0.1\sin(20\pi t)x'(t-r)))' + f(x'(t)) \\ &+ g_1(x(t-\tau_1(t,|x|_\infty))) \\ &+ \left(\frac{\sqrt{3}}{40} + \frac{1}{20}\sin(20\pi t)\right) g_2(x(t-\tau_2(t,|x|_\infty))) \\ &= \cos(20\pi t), \end{aligned} \tag{33}$$

where $p = 4$, $c(t) = 0.1\sin 20\pi t$, $f(u) = u^3 \sin u$, $\beta(t) = \frac{\sqrt{3}}{4} + \frac{1}{20}\sin 20\pi t$, $g_1(u) = -\frac{u^5}{3}$, $\tau_1(t,|x|_\infty))) = \frac{1}{10} - \frac{\gamma^{\frac{4}{3}}|\sin t|}{2+|x|_\infty^{\frac{4}{3}}}$, $g_2(u) = -\frac{u^3}{10}$, $e(t) = \cos 20\pi t$. Therefore we can choose $r_1 = \frac{1}{3}$, $r_2 = \frac{1}{2}$, $r_1^* = \frac{1}{10}$, $r_2^* = \frac{1}{9}$, $d = 1$, $m_1 = 5$, $m_2 = 3$, $\alpha = r_3 = 1$ and an integer k such that $(H_1)-(H_2)$ and condition (1) of Theorem 3.1 hold. By Theorem 3.1, it seen that Eq.(1) has at least one $\frac{1}{10}$-periodic solution while $\gamma < 167,6781$.

Remark Clearly, it can be seen that the growth degree $m_1 = 5 > p-1$ with respect to the variable u in $g_1(u)$ and the functions $c(t) = 0.1\sin 20\pi t, \beta(t) = \frac{\sqrt{3}}{4} + \frac{1}{20}\sin 20\pi t$ can vary signs. Furthermore, it is easy to find that all the results in [2], [3] and the references therein can not be applicable to Eq.(33), which implies that the results of this paper are essentially new.

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