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Jacobi-Based Methods in Solving Fuzzy Linear Systems

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Abstract**—**Linear systems are widely used in many fields of science and engineering. In many applications, at least some of the parameters of the system are represented by fuzzy rather than crisp numbers. Therefore it is important to perform numerical algorithms or procedures that would treat general fuzzy linear systems and solve them using iterative methods. This paper aims are to solve fuzzy linear systems using four types of Jacobi based iterative methods. Four iterative methods based on Jacobi are used for solving a general $n \times n$ fuzzy system of linear equations of the form $Ax = b$, where A is a crisp matrix and b an arbitrary fuzzy vector. The Jacobi, Jacobi Over-Relaxation, Refinement of Jacobi and Refinement of Jacobi Over-Relaxation methods was tested to a five by five fuzzy linear system. It is found that all the tested methods were iterated differently. Due to the effect of extrapolation parameters and the refinement, the Refinement of Jacobi Over-Relaxation method was outperformed the other three methods.

Keywords**—**Fuzzy linear systems, Jacobi, Jacobi Over-Relaxation, Refinement of Jacobi, Refinement of Jacobi Over-Relaxation.

I. Introduction

FUZZY linear system was firstly introduced by Friedman et al [11]. They have proposed a general model for solving a system of n fuzzy linear equations with n variables. The original system with a matrix A is replaced by a $2n \times 2n$ crisp linear system with a matrix S which may be singular even if A is nonsingular. In 1999, Ma et al [18] proposed and solved the general model for $n \times n$ fuzzy linear system whose coefficients matrix is crisp and the right hand side column is an arbitrary fuzzy number vector by using embedding method. In the following years, many literatures suggest fuzzy linear system of equations and various methods were proposed to solve these systems [1,2,3,5,20]. Allahviranloo [3] have considered numerical algorithms for solving fuzzy system of linear equations. Schemes based on the iterative Jacobi and Gauss Seidel methods were thoroughly discussed and followed by convergence theorems. In 2006, Allahviranloo [5] has applied the Adomian and Jacobi iterative methods for approximate of the unique solution of fuzzy system of linear equation. They assumed that the proposed matrix S by Friedman et.al. [11] be nonsingular and $a_{ii} > 0$, then it was shown that the Adomian method is equal to the Jacobi method.

Another class of methods for solving fuzzy linear systems is iterative methods. To the best of authors' knowledge, Allahviranloo [3] introduced the Jacobi method for solving fuzzy linear systems for the first time. He applied Jacobi and Gauss Seidel iterative methods for approximate of the unique solution of fuzzy system of linear equation (FSLE), since solving FSLE as analytically on the hole is difficult. He finally found that if the unique solution is a strong or weak fuzzy number, then the approximate solution of iterative method also would be a strong or weak fuzzy number. Besides, Allahviranloo also proposed the Gauss Seidel method and Successive Over-Relaxation method in [3,4]. Then Dehgan and Hashemi [10] have modified these methods and proposed other iterative schemes for solving fuzzy linear systems. One of them is Jacobi Over-Relaxation (JOR) method, which will be discussed in this paper.

Jacobi method is one of the methods with a few computations, but its rate of convergence is low. Based on this postulation, Dafchahi [9] have successfully introduced the Refinement of Jacobi method in linear equations. This refinement method increases its rate of convergence up to the rate of convergence of Successive Over-Relaxation (SOR) method. In his work, the Jacobi and Refinement of Jacobi methods were used in solving linear systems. As a result, it can be seen that the Refinement of Jacobi method have less number of iterations compared to the Jacobi method. Therefore, his works motivates us to perform Refinement of Jacobi method as a solution in fuzzy linear systems. Hence, by using this method it could conceivably be hypothesized that the rate of convergence is increase and the number of required iterations for this method is shorter when be applied in fuzzy linear systems.

Based on the success of Refinement of Jacobi method in solving fuzzy linear systems successfully, then it motivates us to extend this method to another one new iterative method. It is known as Refinement of Jacobi Over-Relaxation (RJOR) method. The difference of both methods depends on extrapolation parameter used in RJOR method. Surely RJOR method is presented to overcome the weakness of Jacobi and Jacobi Over-Relaxation method as well as reduces number of iterations shorter than Refinement of Jacobi method. This paper aims are to provide evidence that iterative methods based on Jacobi can be used to solve fuzzy linear systems. Specifically, the objective of this paper is to solve fuzzy linear systems using Jacobi method, Jacobi Over-Relaxation method, Refinement of Jacobi method and Refinement of Jacobi Over-Relaxation method.

The structure of this paper is organized as follows. Some basic definitions of linear system, fuzzy linear systems and fuzzy numbers are given in Section II. In Section III, four Jacobi-based methods are presented clearly provided with step-wise algorithm. Section IV represents 5×5 fuzzy

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systems as a numerical application for all of the Jacobi based methods. Comparison result and Conclusions comes in Section V.

II. Preliminaries

In this section, some necessary backgrounds and notions of linear systems, fuzzy linear equation, fuzzy linear systems and fuzzy number were reviewed.

Definition 2.1 Linear systems

Ujevic [21] has defined linear systems as follows.

The linear system of equations $Ax = b$ was considered. A is a positive definite matrix of order n and $b \in R^n$ is a given element.

For a linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

or equivalently in matrix or vector notation,

$$(A)_{n\times n}(x)_{n\times 1} = b_{n\times 1}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

find $x = [x_1, x_2, \dots, x_n]^T$ such that the relation $Ax = b$ holds. It can be seen that nearly all methods for numerical solution of differential equations require the solution of a system of linear equations of the form $Ax = b$.

Definition 2.2 Fuzzy linear equation

For a linear equation system,

$$\sum_{i=1}^{n} a_{ij}x_i = b_j, \; j = 1,2,\dots,m \qquad (1)$$

If a_{ij} and b_j are fuzzy numbers, then the equation is said to be a fuzzy linear equation. Then the following theorem and proof is produced to show the stability of fuzzy linear equations.

Theorem 2.1

If the fuzzy linear equation has a solution, then the following conditions hold [18]:

1. $\sum_{i=1}^{n}(a_{ij})_0 x_i = (b_i)_0$ has a solution as a crisp linear equation system.
2. $\max_{1\le i\le n}\{(a_{ij})_*\} \le (b_j)_*$, $j = 1,2,\dots,m$.
3. $\max_{1\le i\le n}\{(a_{ij})^*\} \le (b_j)^*$, $j = 1,2,\dots,m$.

Definition 2.3 Fuzzy linear systems

Allahviranloo [3,4], Allahviranloo and Kermani [6], Asady et al [7], Babolian & Paripour [8], Ghoncheh & Paripour [12] and Kandel et al [16] have provided definition of fuzzy linear systems.

The $n \times n$ linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= y_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= y_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= y_n \end{aligned} \qquad (2)$$

The matrix form of the above equations is

$$AX = Y \qquad (3)$$

where the coefficient matrix $A = (a_{ij}), 1 \le i, j \le n$ is a crisp $n \times n$ matrix and $y_i \in E^1$, $1 \le i \le n$ is called fuzzy linear systems (FLS).

Definition 2.4 Fuzzy number

A fuzzy number is a fuzzy set $u : R^1 \to I = [0,1]$ which satisfies

i. u is upper semicontinuous.
ii. $u(x) = 0$ outside some interval $[c, d]$.
iii. There are real numbers $a, b : c \le a \le b \le d$ for which
    1. $u(x)$ is monotonic increasing on $[c, a]$.
    2. $u(x)$ is monotonic decreasing on $[b, d]$.
    3. $u(x) = 1$, $a \le x \le b$.

Also u is called symmetric fuzzy number if $u(u^c + x) = u(u^c - x)$ for $\forall x \in R$, where $u^c = \frac{a+b}{2}$, Klir et al [17]. The set of all the fuzzy numbers is denoted by E^1. Goetschel & Voxman [13], and Ma et al. [19]. An alternative definition which yields the same E^1 is given by Kaleva [15].

III. Jacobi Based Methods

In this section, four types of Jacobi methods are briefly presented. They are Jacobi method, Jacobi Over-Relaxation method, Refinement of Jacobi method and the last one is Refinement of Jacobi Over-Relaxation method.

A. Jacobi Method

Allahviranloo [3] has introduced the Jacobi method to solve fuzzy linear system for the first time. Using this method, he described that without loss of generality, suppose that $s_{ii} > 0$ for all $i = 1,\ldots\ldots,2n.$. Let $S = D + L + U$ where

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}, L = \begin{bmatrix} L_1 & 0 \\ S_2 & L_1 \end{bmatrix}, U = \begin{bmatrix} U_1 & S_2 \\ 0 & U_1 \end{bmatrix}.$$

$(D_1)_{ii} = s_{ii} > 0,\ i = 1,\ldots.,n$ and suppose $S_1 = D_1 + L_1 + U_1$. In Jacobi method, from the structure of $SX = Y$,

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}\begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} + \begin{bmatrix} L_1 + U_1 & S_2 \\ S_2 & L_1 + U_1 \end{bmatrix}\begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} = \begin{bmatrix} \underline{Y} \\ \overline{Y} \end{bmatrix}$$

So the Jacobi iterative technique will be

$$\begin{aligned} \underline{X}^{(k+1)} &= D_1^{-1}\underline{Y} - D_1^{-1}(L_1 + U_1)\underline{X}^k - D_1^{-1}S_2\overline{X}^k \\ \overline{X}^{(k+1)} &= D_1^{-1}\overline{Y} - D_1^{-1}(L_1 + U_1)\overline{X}^k - D_1^{-1}S_2\underline{X}^k \end{aligned} \tag{4}$$

The elements of $X^{k+1} = \left(\underline{X}^{k+1}, \overline{X}^{k+1}\right)^t$ are

$$\begin{aligned} \underline{x}_i^{k+1}(r) &= \frac{1}{s_{i,i}}\left[\underline{y}_i(r) - \sum_{j=1, j\neq i}^{n} s_{i,j}\,\underline{x}_j^k(r) - \sum_{j=1}^{n} s_{i,n+j}\,\overline{x}_j^k(r)\right], \\ \overline{x}_i^{k+1}(r) &= \frac{1}{s_{i,i}}\left[\overline{y}_i(r) - \sum_{j=1, j\neq i}^{n} s_{i,j}\,\overline{x}_j^k(r) - \sum_{j=1}^{n} s_{i,n+j}\,\underline{x}_j^k(r)\right] \\ k &= 0,1,\ldots.., \quad i = 1,\ldots..n. \end{aligned}$$

The results in the matrix form of the Jacobi iterative technique are $X^{k+1} = PX^k + C$ where

$$P = \begin{bmatrix} -D_1^{-1}(L_1 + U_1) & -D_1^{-1}S_2 \\ -D_1^{-1}S_2 & -D_1^{-1}(L_1 + U_1) \end{bmatrix}, C = \begin{bmatrix} D_1^{-1}\underline{Y} \\ D_1^{-1}\overline{Y} \end{bmatrix},$$

$$X = \begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix}$$

When this method used to solve fuzzy linear systems, it needs longer computing time. [10]

B. Jacobi Over-Relaxation Method

Young [22] has defined that JOR method is an extrapolated of Jacobi method.

$$\begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} = \begin{bmatrix} \omega D_1^{-1}\underline{Y} + \left(I_n - \omega D_1^{-1}B\right)\underline{X} + \omega D_1^{-1}C\overline{X} \\ \omega D_1^{-1}\overline{Y} + \left(I_n - \omega D_1^{-1}B\right)\overline{X} + \omega D_1^{-1}C\underline{X} \end{bmatrix}$$

So the JOR iterative method will be

$$\begin{cases} \underline{X}^{(m+1)} = \omega D_1^{-1}\underline{Y} - \omega D_1^{-1}\left[\left(1 - \dfrac{1}{\omega}\right)D_1 + L_1 + U_1\right]\underline{X}^{(m)} + \\ \qquad \omega D_1^{-1}C\overline{X}^{(m)} \\ \overline{X}^{(m+1)} = \omega D_1^{-1}\overline{Y} - \omega D_1^{-1}\left[\left(1 - \dfrac{1}{\omega}\right)D_1 + L_1 + U_1\right]\overline{X}^{(m)} + \\ \qquad \omega D_1^{-1}C\underline{X}^{(m)} \end{cases}$$

Hence the JOR method in matrix form is as follows:

$$X^{(m+1)} = D^{-1}(D - \omega S)X^{(m)} + \omega D^{-1}Y \tag{5}$$

Evidently the JOR method is reduced to Jacobi method for $\omega = 1$ [14].

C. Refinement of Jacobi Method

The aim of Refinement of Jacobi method is to increase rate of convergence. Hence, Jacobi method can be simplified as follows [9]:

$$X^{(k+1)} = \left(-D^{-1}(L + U)\right)X^{(k)} + D^{-1}b$$

Then, Refinement of Jacobi method which has been applied in linear equations as before then will be applied to fuzzy linear systems as an iterative method.

$$\begin{aligned} \overline{A}\,\overline{X} &= \overline{b} \\ \left(\overline{L} + \overline{D} + \overline{U}\right)\overline{X} &= \overline{b} \\ \overline{D}\,\overline{X} &= -\left(\overline{L} + \overline{U}\right)\overline{X} + \overline{b} \\ \overline{D}\,\overline{X} &= \left(\overline{D} - \overline{A}\right)\overline{X} + \overline{b} \\ \overline{D}\,\overline{X} &= \overline{D}\,\overline{X} + \left(\overline{b} - \overline{A}\,\overline{X}\right) \\ X &= \overline{X} + \overline{D}^{-1}\left(\overline{b} - \overline{A}\,\overline{X}\right) \end{aligned}$$

From the above form, the iterative refinement of formula in matrix form is given by

$$X^{(k+1)} = \overline{X}^{(k+1)} + \overline{D}^{-1}\left(\overline{b} - \overline{A}\,\overline{X}^{(k+1)}\right) \tag{6}$$

Then from Equation (6), the RJ method in matrix form is written as

$$X^{(k+1)} = \left(\overline{D}^{-1}\left(\overline{L}+\overline{U}\right)\right)^2 \overline{X}^{(k)} + \left(I - \overline{D}^{-1}\left(\overline{L}+\overline{U}\right)\right)\overline{D}^{-1}\overline{b} \qquad (7)$$

In order to understand clearly how to get the solution of fuzzy linear system by Refinement of Jacobi method, an algorithm is proposed as follows:

Step 1: Determine L, U, D, D^{-1} and b of the systems.
Step 2: After that, sum up L and U and multiply them with the inverse of D.
Step 3: Then evaluate $\left((L+U)D^{-1}\right)^2$.
Step 4: Identify the number of variables of the systems. Make sure they are including lower bound and upper bound variables.
Step 5: Then, multiply the variables with the $\left((L+U)D^{-1}\right)^2$.
Step 6: Identify an identity matrix for the systems and then subtract it to the $\left((L+U)D^{-1}\right)$.
Step 7: At the same time, multiply b with the inverse of diagonal matrix of the systems.
Step 8: Multiply $D^{-1}b$ with the $I-(L+U)D^{-1}$.
Step 9: Lastly, substitute all the terms to the Equation (7).
Step 10: When $\dfrac{x_i^{(k+1)} - x_i(k)}{x_i^{(k+1)}} < \varepsilon$, stop the calculation.

Note that, the decision to stop the iterations was based on the criterion. For this purpose, any convenient norm can be used, the usual being was 10^{-3}.

Therefore, Equation (7) can be used to solve FLS. Based on the formula, $\overline{D}$ is diagonal for the system, $\overline{L}$ is strict lower part of the system, $\overline{U}$ is strict upper part of the systems, $\overline{b}$ is fuzzy vector for the system which including fuzzy lower bound and fuzzy upper bound, and $\overline{X}$ are the variables of the system, where they are lower variable and upper variable.

D. Refinement of Jacobi Over-Relaxation Method

The RJOR method is a generation of ideas which stick to over-relaxation. The SOR and JOR methods are samples of over-relaxation method, Allahviranloo [4] and Dehgan and Hashemi [10]. Therefore RJ method which proposed in the Section III.C will be expanded to RJOR method. This method is proposed to solve FLS. Therefore, RJ method is an extension of Jacobi by adding an extrapolation parameter (ω) in the Equation (7). Hence, RJOR method in matrix form is given as follows:

$$X^{(k+1)} = \left(\overline{D}^{-1}\left(\overline{L}+\omega\overline{U}\right)\right)^2 \overline{X}^{(k)} + \left(I - \overline{D}^{-1}\left(\overline{L}+\overline{U}\right)\right)\omega\,\overline{D}^{-1}\overline{b} \qquad (8)$$

According to Dehgan & Hashemi [10], the JOR method is reduced to Jacobi method for $\omega = 1$. However in this study, three values of extrapolation parameter will be used for RJOR method where $\omega = 0.80$, $\omega = 0.90$ and $\omega = 1.0$.

Based on Equation (8), this method is simplified to eleven of step-wise. Hence in order to solve FLS by RJOR method, the following algorithm is summarized as below.

Step 1: Determine $\overline{L}$, $\overline{U}$, $\overline{D}^{-1}$ and $\overline{b}$ of the systems.
Step 2: Multiply extrapolation parameter (ω) with $\overline{U}$ and sum up to $\overline{L}$.
Step 3: Multiply $\left(\overline{L}+\omega\overline{U}\right)$ with the inverse of $\overline{D}$.
Step 4: Evaluate $\left(\left(\overline{L}+\omega\overline{U}\right)\overline{D}^{-1}\right)^2$.
Step 5: Identify the number of variables $(\overline{X})$ of the systems. Make sure they are lower bound and upper bound variables.
Step 6: Multiply the variables with the $\left(\left(\overline{L}+\omega\overline{U}\right)\overline{D}^{-1}\right)^2$.
Step 7: Identify an identity matrix for the systems and then subtract $\left(\left(\overline{L}+\overline{U}\right)\overline{D}^{-1}\right)$.
Step 8: Multiply $\overline{b}$ with the inverse of diagonal matrix of the systems and extrapolation parameter.
Step 9: Multiply $\omega\overline{D}^{-1}\overline{b}$ with the $I-\left(\overline{L}+\overline{U}\right)\overline{D}^{-1}$.
Step 10: Substitute all the terms to the Equation (8).
Step 11: When $\dfrac{x_i^{(k+1)} - x_i(k)}{x_i^{(k+1)}} < \varepsilon$, stop the calculation. Note that, the decision to stop the iterations was based on the criterion. For this purpose, any convenient norm can be used, the usual being was 10^{-3}.

The numerical application is presented to test the feasibility of the four Jacobi-based methods in solving fuzzy linear systems.

IV. Numerical Application

Example below was taken from Dehgan and Hashemi [10]. Consider the 5×5 fuzzy system

$$\begin{aligned}
&8x_1 + 2x_2 + x_3 - 3x_5 = (r, 2-r)\\
&-2x_1 + 5x_2 + x_3 - x_4 + x_5 = (4+r, 7-2r)\\
&x_1 - x_2 + 5x_3 + x_4 + x_5 = (1+2r, 6-3r)\\
&-x_3 + 4x_4 + 2x_5 = (1+r, 3-r)\\
&x_1 - 2x_2 + 3x_5 = (3r, 6-3r)
\end{aligned}$$

Based on [10], the exact solution of this system is as follows:

$$x_1 = (0.7287 - 0.33057r,\ 0.044135 + 0.35399r)$$
$$x_2 = (0.61418 + 0.16626r,\ 1.0773 - 0.29682r)$$
$$x_3 = (0.12628 + 0.29059r,\ 0.91822 - 0.50136r)$$
$$x_4 = (0.24192 - 0.33149r,\ -0.4158 + 0.32622r)$$
$$x_5 = (0.47528 + 0.91231r,\ 2.3947 - 1.0072r)$$

In order to view graphically, the exact solution is illustrated by the triangular fuzzy number as Fig. 1.

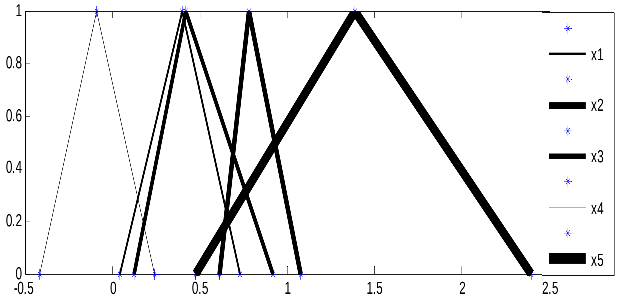

Fig. 1 Exact solution of 5×5 fuzzy system.

This fuzzy system is solved using four types of Jacobi methods.

A. Jacobi Method

By using Jacobi method, the approximation solution is obtained after completing 107 iterations.

$$x_1 = (0.72871 - 0.33058r,\ 0.04413 + 0.35400r)$$
$$x_2 = (0.61419 + 0.16625r,\ 1.07726 - 0.29682r)$$
$$x_3 = (0.12628 + 0.29058r,\ 0.91821 - 0.50135r)$$
$$x_4 = (0.24192 - 0.33150r,\ -0.41581 + 0.32623r)$$
$$x_5 = (0.47530 + 0.91230r,\ 2.39473 - 1.00714r)$$

This approximation solution then illustrated in triangular fuzzy number as in Fig. 2.

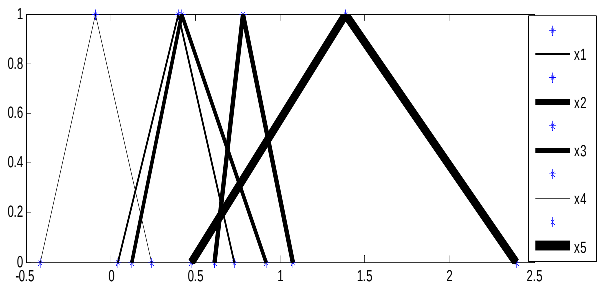

Fig. 2 Approximation solution of 5×5 fuzzy system by Jacobi method.

B. Jacobi Over-Relaxation Method

The following approximation solution obtained after 59 iterations.

$$x_1 = (0.728 - 0.330r,\ 0.045 + 0.354r)$$
$$x_2 = (0.613 + 0.166r,\ 1.078 - 0.297r)$$
$$x_3 = (0.125 + 0.291r,\ 0.919 - 0.501r)$$
$$x_4 = (0.241 - 0.331r,\ -0.415 + 0.326r)$$
$$x_5 = (0.475 + 0.912r,\ 2.396 - 1.007r)$$

This approximation solution then illustrated in triangular fuzzy number as in Fig. 3.

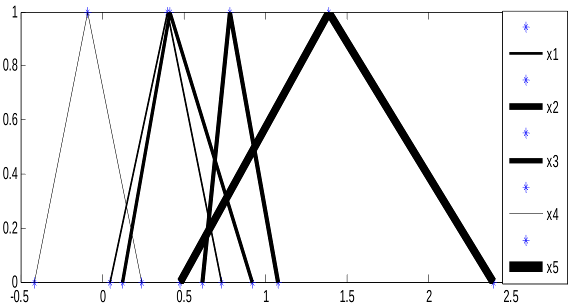

Fig. 3 Approximate solution of 5×5 fuzzy system by Jacobi Over-Relaxation method.

C. Refinement of Jacobi Method

After 46 iterations, the following approximation solution was defined.

$$x_1 = (0.72870 - 0.33058r,\ 0.04413 + 0.35400r)$$
$$x_2 = (0.61419 + 0.16626r,\ 1.07726 - 0.29682r)$$
$$x_3 = (0.12628 + 0.29058r,\ 0.91822 - 0.50135r)$$
$$x_4 = (0.24192 - 0.33150r,\ -0.41581 + 0.32623r)$$
$$x_5 = (0.47529 + 0.91230r,\ 2.39474 - 1.00715r)$$

The triangular fuzzy number for this approximation solution is shown in Fig. 4.

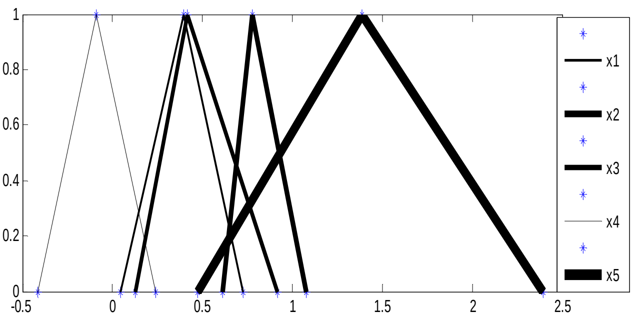

Fig. 4 Approximate solution of 5×5 fuzzy system by Refinement of Jacobi method.

D. Jacobi Over-Relaxation Method

The approximation solution of this fuzzy system obtained after 42 iterations with extrapolation parameter $\omega = 1.0$.

$$x_1 = (0.72872 - 0.33059r,\ 0.04412 + 0.35401\,r)$$
$$x_2 = (0.61421 + 0.16623r,\ 1.07725 - 0.29680r)$$
$$x_3 = (0.12630 + 0.29056r,\ 0.91820 - 0.50134r)$$
$$x_4 = (0.24194 - 0.33152r,\ -0.41582 + 0.32625r)$$
$$x_5 = (0.47531 + 0.91228r,\ 2.39472 - 1.00712r)$$

This approximation solution then illustrated in triangular fuzzy number as in Fig. 5.

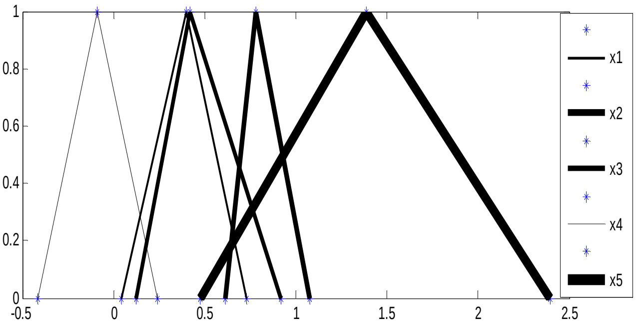

Fig. 5 Approximate solutions of 5×5 fuzzy system by RJOR method with $\omega = 1.0$.

However, with extrapolation parameter $\omega = 0.90$, the approximation solution obtained after 30 iterations.

$$x_1 = (0.71853 - 0.35425r,\ -0.07937 + 0.38449r)$$
$$x_2 = (0.71694 + 0.04370r,\ 0.84216 - 0.17459r)$$
$$x_3 = (0.23213 + 0.17540r,\ 0.70718 - 0.37734r)$$
$$x_4 = (0.34631 - 0.36182r,\ -0.44706 + 0.35291\,r)$$
$$x_5 = (0.58807 + 0.72872r,\ 2.05005 - 0.83470r)$$

Hence the approximation solution is illustrated in triangular fuzzy number as in Fig. 6.

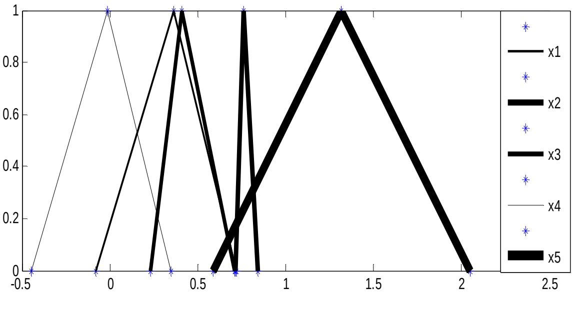

Fig. 6 Approximate solutions of 5×5 fuzzy system by RJOR method with $\omega = 0.90$.

With extrapolation parameter $\omega = 0.80$, the approximation solution obtained after 20 iterations as follows:

$$x_1 = (0.64988 - 0.33263r,\ -0.13244 + 0.36945r)$$
$$x_2 = (0.72647 - 0.01498r,\ 0.68311 - 0.10869r)$$
$$x_3 = (0.38247 + 0.09505r,\ 0.56059 - 0.29995r)$$
$$x_4 = (0.37196 - 0.34444r,\ -0.41585 + 0.33509r)$$
$$x_5 = (0.61041 + 0.60555r,\ 1.76940 - 0.71311\,r)$$

This approximation solution then illustrated in triangular fuzzy number as in Fig. 7.

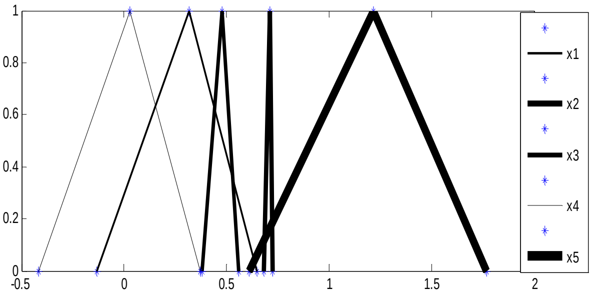

Fig. 7 Approximate solutions of 5×5 fuzzy system by RJOR method with $\omega = 0.80$.

V. Comparison of Results and Conclusions

In this paper, four types of Jacobi methods have been applied to solve fuzzy linear systems. From the numerical application in Section IV, it can be seen that the approximation solutions were successfully obtained for all the Jacobi methods. The Jacobi method has the greatest number of iterations compared to the others three Jacobi methods. It is found that Jacobi, JOR, RJ and RJOR methods need 107, 59, 46 and 42 iterations respectively to produce approximation solution. There is no doubt that JOR method is better than Jacobi method, and RJ method is better than JOR method in solving fuzzy linear systems. Otherwise, the RJOR method is much better than the RJ method.

The RJOR method has an extrapolation parameter (ω), which is used to advance the solution more quickly. With that expression, RJOR method is tested with $\omega = 0.80$, $\omega = 0.90$ and $\omega = 1.0$ in numerical application. Similarly to the RJ method, the RJOR method with $\omega = 1.0$ has successfully solved FLS and produced approximate solution which are achieved the exact solutions. Besides, RJOR method with $\omega = 1.0$ is reduced to the RJ method. Apparently, the RJOR method with $\omega = 0.80$ and $\omega = 0.90$ failed to solve properly fuzzy linear systems since their approximate solutions obtained did not achieve the exact solutions. Hence, it can be concluded that the RJOR method with $\omega = 1.0$ is valid as an iterative method of fuzzy linear systems. All in all, this paper has provided a clear evidence to show the feasibility of all the four Jacobi based methods in solving fuzzy linear system. Further research could be conducted to prove the feasibility of these Jacobi based with various sizes of fuzzy linear systems.

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