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Semiconvergence of alternating iterative methods for singular linear systems

Jing Wu

Abstract—In this paper, we discuss semiconvergence of the alternating iterative methods for solving singular systems. The semiconvergence theories for the alternating methods are established when the coefficient matrix is a singular matrix. Furthermore, the corresponding comparison theorems are obtained.

Keywords—Alternating iterative method; Semiconvergence; Singular matrix.

I. Introduction

CONSIDER a linear system

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$ is a singular matrix, $x \in R^n$ is an unknown vector and $b \in R^n$ is a given vector. We assume that the system (1) is solvable, i.e., it has at least a solution.

It is customary to consider a splitting $A = M - N$, where M is a nonsingular matrix, then iterative formula for solving the system (1) can be described as follows

$$x^{(k+1)} = Tx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \cdots, \tag{2}$$

where $T = M^{-1}N$ is the iterative matrix. In 1997, Benzi and Szyld [1] have introduced the following general alternating method.

The alternating iterative method. If we consider two splittings

$$A = M - N = P - Q, \tag{3}$$

we obtain the general class of the iterative methods of the form

$$\left.\begin{array}{l} x^{(k+1/2)} = M^{-1}Nx^{(k)} + M^{-1}b \\ x^{(k+1)} = P^{-1}Qx^{(k+1/2)} + P^{-1}b \end{array}\right\} \text{ for } k = 0, 1, 2, \cdots. \tag{4}$$

Let us eliminating $x^{(k+1/2)}$ from (4) and obtain the iterative process (see [1])

$$x^{(k+1)} = P^{-1}QM^{-1}Nx^{(k)} + P^{-1}(QM^{-1} + I)b.$$

Let

$$R = P^{-1}(QM^{-1} + I), \quad T = P^{-1}QM^{-1}N, \tag{5}$$

then

$$x^{(k+1)} = Tx^{(k)} + Rb, \tag{6}$$

where $T = I - RA$.

J. Wu is with the Department of Applied Mathematics, Southwestern University of Finance and Economics, Chengdu, 610074 China. e-mail:wujing8119@163.com.

It is well-known that iteration method (2) converges to the unique solution of linear system with nonsingular coefficient matrices if and only if $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the iteration matrix T. However, for the singular systems we have $\rho(T) = 1$, so that we can only require the semiconvergence of the iterative method (2), which means that for every $x^{(0)}$ the sequence defined by (2) converges to a solution of system (1).

From the convergence and comparison results introduced by Benzi and Szyld [1] for P-regular splittings of a symmetric positive definite matrix and weak nonnegative splittings of the first type of a monotone matrix, many authors such as Young and Kincaid [3], Marchuk [4], Climent and Perea [5], [6], Wang and Huang [7] have established convergence theories for the alternating method when the coefficient matrix is an H-matrix or a monotone matrix. Recently, the investigations for singular linear systems have arisen attentions of many scientists. For example, Song [8] establish some semiconvergence results for solving singular linear systems with singular M-matrix. Huang and Song [10] have proved semiconvergence of the block AOR method for solving singular linear systems with P-cyclic matrices. Wang [19] study the semiconvergence of two-stage iterative methods for solving nonsymmetric singular linear systems. Approaches different from the those used in [8], [10], [19] will be used in this paper. We will study the semiconvergence of the alternating method for solving the singular linear systems. Furthermore, we will present the comparison theorems for the alternating iterative methods with singular coefficient matrices.

II. Conclusion

A. Notations and preliminaries

For convenience we shall now briefly explain some of the notations and preliminaries used in the next sections.

As usual, a square matrix T is said to be convergent if $\lim_{j\to\infty} T^k = 0$ and semiconvergent if $\lim_{j\to\infty} T^k$ exists. In the meantime, iteration (2) is said to be semiconvergent if the iteration matrix T is semiconvergent(see [8], [9], [10]).

We denote the spectrum of T by $\lambda(T)$. Moreover,

$$\vartheta(T) \equiv \max\left\{|\mu| \,|\mu \in \lambda(T), \mu \neq 1\right\}.$$

In addition, $index(A)$ denotes the index of the matrix A, that is defined as the smallest nonnegative integer k such that $rank(A^{(k+1)}) = rank(A^{(k)})$.

Since the Drazin inverse is important tools for singular linear system analysis, we state the definitions as follows.

Definition 2.1.([11]). For every matrix A, $index(A) = k$, the Drazin inverse A^D satisfies the following conditions

$$A^D A A^D = A^D, \quad AA^D = A^D A, \quad A^{k+1} A^D = A^k.$$

If A is a nonsingular matrix, then $A^D = A^{-1}$.

In the following definition, we present the different types of splittings that appear in this papers [6], [9], [12], [13], [14].

Definition 2.2. Let $A \in R^{n\times n}$, a splitting of A is a pair of matrices M, N such that M is nonsingular and $A = M - N$. The splitting is called
- Regular if $M^{-1} \geq 0$ and $N \geq 0$.
- Nonnegative if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$.
- Weak nonnegative of the first type if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$; Weak nonnegative of the second type if $M^{-1} \geq 0$ and $NM^{-1} \geq 0$.

We denote

$$T = M^{-1}N, \quad K = (I-T)(I-T)^D,$$

$$G = NM^{-1}, \quad L = (I-G)^D(I-G).$$

Definition 2.3. Let $A \in R^{n\times n}$. Assume that the splitting $A = M - N$ satisfies

$$index(I-T) \leq 1, \quad index(I-G) \leq 1.$$

Then the splitting is called
- Quasi-regular if $M^{-1} \geq 0$ and $NK \geq 0$.
- Quasi-nonnegative if $M^{-1} \geq 0$, $M^{-1}NK \geq 0$ and $LNM^{-1} \geq 0$.
- Weak quasi-nonnegative of the first type if $M^{-1} \geq 0$ and $M^{-1}NK \geq 0$; Weak quasi-nonnegative of the second type if $M^{-1} \geq 0$ and $LNM^{-1} \geq 0$.

Clearly, if $index(I - T) = 0$ (respectively, $index(I - G) = 0$), then $K = I$ (respectively, $L = I$) and the concepts in Definition 2.3 concord with the corresponding ones in Definition 2.2.

We present several lemmas that will be used.

Lemma 2.1. (Theorem 2.3.1 of [15]). *Iterative method* (2) *semiconvergence* $\Longleftrightarrow$ *The following two conditions are satisfied*:
- $\vartheta(T) < 1$,
- $index(I-T) \leq 1$.

Lemma 2.2. ([16]). *If the splitting* $A = M - N$ *satisfies* $index(I-T) \leq 1$, *then* $\vartheta(T) = \rho(TK)$.

Denote

$$\hat{A} = M - NK, \quad \hat{A}' = M - LN.$$

Then in the case when $index(I-T) \leq 1$ or $index(I-G) \leq 1$ we have

$$\hat{A}^{-1} = (I-TK)^{-1}M^{-1}, \quad \hat{A}'^{-1} = M^{-1}(I-LG)^{-1}.$$

Lemma 2.3. (Theorem 2.4 of [9]). *Let* $A = M - N$ *be a weak quasi-nonnegative of the first type . Then the following statements are equivalent*
- $(I-TK)^{-1} \geq 0$.
- T *is semiconvergent.*

Lemma 2.4. *Let* $A = M - N$ *be a weak quasi-nonnegative of the second type . Then the following statements are equivalent*
- $(I-LG)^{-1} \geq 0$.
- G *is semiconvergent.*

Lemma 2.5. (Lemma 3 of [5]). *Let A be a nonsingular matrix. For the splitting* (3), *consider matrix* $S = QP^{-1}NM^{-1}$, *then*
- $S = ATA^{-1}$, *where matrix T is defined by* (5), *and consequently,* $\rho(S) = \rho(T)$.
- *If* $\rho(T) < 1$, *the matrix T and S induce the same splitting* $A = B - C$ *of* A.

Lemma 2.6. ([16]). *Let A and T be square matrices such that A and $I-T$ are nonsingular. Then, there exists a unique pair of matrices (B, C), such that B is nonsingular, $T = B^{-1}C$ and $A = B - C$. The matrices are $B = A(I-T)^{-1}$ and $C = B - A$.*

B. Semiconvergence

In this section, we will prove the semiconvergence of the classical alternating iterations when the coefficient matrix A is a singular matrix.

Lemma 3.1 *Let* $A = M - N = P - Q$, $index(I - M^{-1}N) \leq 1$, $index(I - P^{-1}Q) \leq 1$ *and* $\hat{A} = M - NK_1$, $\hat{A}' = P - QK_2$, *where* $K_1 = (I-M^{-1}N)(I-M^{-1}N)^D$, $K_2 = (I-P^{-1}Q)(I-P^{-1}Q)^D$, *then* $\hat{A} = \hat{A}'$.

Proof. Using $index(I - M^{-1}N) \leq 1$, we have

$$(I - M^{-1}N)K_1 = (I - M^{-1}N)$$

and

$$\begin{aligned} \hat{A} &= M - NK_1 \\ &= M + M(I - M^{-1}N) - MK_1 \\ &= A + M(I - K_1). \end{aligned}$$

Since $A = M - N = P - Q$, we know

$$(I - M^{-1}N) = M^{-1}P(I - P^{-1}Q).$$

Using the definition of Drazin inverse, we have

$$I - K_1 = M^{-1}P(I - K_2).$$

By $\hat{A} = M - NK_1$ and $\hat{A}' = P - QK_2$, it is easy to prove that $\hat{A} = \hat{A}'$.

We have the following conclusion similar to that obtained in Lemma 3.1.

Lemma 3.2 *Let* $A = M - N = P - Q$. $index(I - NM^{-1}) \leq 1$, $index(I - QP^{-1}) \leq 1$ *and* $\hat{A} = M - L_1 N$, $\hat{A}' = P - L_2 Q$, *where* $L_1 = (I-NM^{-1})^D(I-NM^{-1})$, $L_2 = (I-QP^{-1})^D(I-QP^{-1})$, *then* $\hat{A} = \hat{A}'$.

Theorem 3.1. *Let* A *be a singular matrix,* $A = M - N$ or $A = P - Q$ *be semiconvergence splitting,* $index(I - T) \leq 1$. *Then* $T = P^{-1}QM^{-1}N$ *is semiconvergence matrix if one of the following conditions is satisfied.*

- $A = M - N = P - Q$ *are weak quasi-nonnegative splittings of the first type and* $index(I - M^{-1}N) \leq 1$, $index(I - P^{-1}Q) \leq 1$.
- $A = M - N = P - Q$ *are weak quasi-nonnegative splittings of the second type and* $index(I - NM^{-1}) \leq 1$, $index(I - QP^{-1}) \leq 1$.

Furthermore, the splitting $A = B - C$ *induced by* T *is weak quasi-nonnegative of the same type.*

Proof. We will show that $\rho(TK) < 1$.

(i) From Lemma 3.1, we note

$$\hat{A} = M - NK_1 = P - QK_2, \tag{7}$$

where

$$K_1 = (I - M^{-1}N)(I - M^{-1}N)^D$$

and

$$K_2 = (I - P^{-1}Q)(I - P^{-1}Q)^D.$$

If the splittings (3) are weak quasi-nonnegative of the first type, then

$$M^{-1} \geq 0, \quad P^{-1} \geq 0, \quad M^{-1}NK_1 \geq 0, \quad P^{-1}QK_2 \geq 0.$$

It is shown that the splittings (7) are weak nonnegative of the first type of $\hat{A}$.

In order to prove the conclusion, let's assume that $A = M - N$(or $A = P - Q$) is semiconvergence. By Lemma 2.3, we have

$$(I - M^{-1}NK_1)^{-1} \geq 0,$$

and

$$\hat{A}^{-1} = (I - M^{-1}NK_1)^{-1}M^{-1} \geq 0.$$

From Lemma 2.6, we obtain that

$$TK = P^{-1}QK_2M^{-1}NK_1 \quad (K = (I-T)(I-T)^D)$$

is the iterative matrix of the classical alternating methods induced by (7). Clearly,

$$\begin{aligned} TK &= P^{-1}QK_2M^{-1}NK_1 \\ &= (I - P^{-1}\hat{A})(I - M^{-1}\hat{A}) \\ &= I - P^{-1}\hat{A} - M^{-1}\hat{A} + P^{-1}\hat{A}M^{-1}\hat{A}, \end{aligned}$$

then

$$\begin{aligned} (I - TK)\hat{A}^{-1} &= P^{-1} + M^{-1} - P^{-1}AM^{-1} \\ &= P^{-1} + (I - P^{-1}\hat{A})M^{-1} \\ &= P^{-1} + P^{-1}QK_2M^{-1}. \end{aligned}$$

Since the splittings (7) are weak nonnegative of the first type, it follows that

$$TK \geq 0, \quad (I - TK)\hat{A}^{-1} \geq 0.$$

Hence, for every nonnegative integer m, we have

$$\begin{aligned} 0 &\leq (I + TK + (TK)^2 + \cdots + (TK)^m)(I - TK)\hat{A}^{-1} \\ &= (I - (TK)^{m+1})\hat{A}^{-1} \leq \hat{A}^{-1}. \end{aligned}$$

Therefore, $\lim_{m\to\infty}(TK)^m = 0$(see, e.g.,[17]). Furtherore,

$$\rho(TK) < 1.$$

By Lemma 2.2 we know that $\vartheta(T) = \rho(TK) < 1$. So, T is semiconvergence.

(ii) Now, we assume that the splittings (3) are weak quasi-nonnegative of the second type. By Lemma 3.2, we note

$$\hat{A} = M - L_1N = P - L_2Q, \tag{8}$$

where $L_1 = (I - NM^{-1})^D(I - NM^{-1})$ and $L_2 = (I - QP^{-1})^D(I - QP^{-1})$. If the splittings (3) are weak quasi-nonnegative of the second type, then

$$M^{-1} \geq 0, \quad P^{-1} \geq 0, \quad L_1NM^{-1} \geq 0, \quad L_2QP^{-1} \geq 0.$$

It is shown that the splittings (8) are weak nonnegative of the second type of $\hat{A}$.

In order to prove the conclusion, let's assume that $A = M - N$(or $A = P - Q$) is semiconvergence. By Lemma 2.4, we have

$$(I - L_1NM^{-1})^{-1} \geq 0,$$

then

$$\hat{A}^{-1} \geq 0, \quad LS = L_2QP^{-1}L_1NM^{-1} \geq 0,$$

and

$$\begin{aligned} \hat{A}^{-1}(I - LS) &= \hat{A}^{-1}(I - L_2QP^{-1}L_1NM^{-1}) \\ &= \hat{A}^{-1}[I - (I - \hat{A}P^{-1})(I - \hat{A}M^{-1})] \\ &= \hat{A}^{-1}[\hat{A}M^{-1} + \hat{A}P^{-1} - \hat{A}P^{-1}\hat{A}M^{-1}] \\ &= M^{-1} + P^{-1} - P^{-1}\hat{A}M^{-1} \\ &= M^{-1} + P^{-1}(I - \hat{A}M^{-1}) \\ &= M^{-1} + P^{-1}L_1NM^{-1} \geq 0, \end{aligned}$$

where$L = (I - S)^D(I - S)$.

For every nonnegative integer m, we have

$$\begin{aligned} 0 &\leq \hat{A}^{-1}(I - LS)(I + LS + (LS)^2 + \cdots + (LS)^m) \\ &= \hat{A}^{-1}(I - (LS)^{m+1}) \leq \hat{A}^{-1}, \end{aligned}$$

then $\lim_{m\to\infty}(LS)^m = 0$ and $\rho(LS) < 1$.

According to Lemma 2.5, we know that $\rho(LS) = \rho(TE)$ and matrix LS and TK induce the same splitting. So we have $\rho(TK) < 1$ and T is semiconvergence.

Let $\hat{A} = B - CK$ be the unique splitting induced by $TK = P^{-1}QK_2M^{-1}NK_1$, Using Lemma 2.6 and (7), we have

$$B^{-1} = (I - TK)\hat{A}^{-1} \geq 0, \quad B^{-1}CK = TK \geq 0. \tag{9}$$

Thus, $\hat{A} = B - CK$ is weak nonnegative splitting of the first type and $A = B - C$ is weak quasi-nonnegative splitting of the first type.

Let $\hat{A} = B - LC$ be the unique splitting induced by $LS = L_2QP^{-1}L_1NM^{-1}$, Applying Lemma 2.6 and (7), we have

$$B^{-1} = \hat{A}^{-1}(I - LS) \geq 0, \quad LCB^{-1} = LS \geq 0. \tag{10}$$

Thus, $\hat{A} = B-LC$ is weak nonnegative splitting of the second type and $A = B - C$ is weak quasi-nonnegative splitting of the second type.

Therefore, $A = B - C$ and $A = M - N = P - Q$ are the same type of splittings. It completes the proof.

Remark 3.1. Theorem 3.1 can be extended to alternating schemes involving more than two splittings of the coefficient matrix A. For example, considering three splittings $A = M - N = P - Q = R - S$ and corresponding three-step alternating procedure, we can study the semiconvergence of the iteration matrix $T = R^{-1}SP^{-1}QM^{-1}N$ to extend the alternating iteration methods.

C. A comparison theorem

We know that the semiconvergence of the two splittings $A = M - N = P - Q$ is not to insure the semiconvergence of the classical alternating iteration (4). Even if the classical alternating iteration semiconverges, there is no guarantee that it will converge faster than either of the two basic splittings. Hence, we will give the following comparison theorem.

Theorem 4.1. *Let A be a singular matrix and $A = M - N$ be quasi-regular semiconvergence splitting, $index(I-T) \leq 1$. Then $\vartheta(T)$ of $T = P^{-1}QM^{-1}N$ holds*

$$\vartheta(T) \leq \vartheta(M^{-1}N) < 1 \tag{11}$$

if one of the following conditions is satisfied.

• *$A = P - Q$ are weak quasi-nonnegative splittings of the first type and $index(I-M^{-1}N) \leq 1$, $index(I-P^{-1}Q) \leq 1$.*

• *$A = P - Q$ are weak quasi-nonnegative splittings of the second type and $index(I - NM^{-1}) \leq 1$, $index(I - QP^{-1}) \leq 1$.*

Proof. From Lemma 2.6, let T be the iteration matrix corresponding to the induced splitting $A = B - C$. Noting

$$\hat{A} = B - CK = M - NK_1 = P - QK_2,$$

and assuming $A = P - Q$ be a weak quasi-nonnegative splitting of the first type, from Theorem 3.1, we know that $A = B - C$ is a weak quasi-nonnegative splitting of the first type. It is shown that $\hat{A} = B - CK$ and $\hat{A} = P - QK_2$ are weak nonnegative splitting of the first type. We have

$$B^{-1} = M^{-1} + P^{-1}NK_1M^{-1} \geq M^{-1}.$$

We can now apply the comparison theorem for weak regular splittings due to Elsner [18] to get

$$\rho(TK) \leq \rho(M^{-1}NK_1) < 1,$$

thus

$$\vartheta(T) \leq \vartheta(M^{-1}N) < 1.$$

Similarly, if $A = P - Q$ is a weak quasi-nonnegative splitting of the second type, we obtain

$$\rho(LS) \leq \rho(M^{-1}NK_1) < 1,$$

thus

$$\vartheta(T) \leq \vartheta(M^{-1}N) < 1.$$

In a similar way we obtain the result of Theorem 4.2.

Theorem 4.2. *Let A be a singular matrix and $A = P - Q$ be quasi-regular semiconvergence splitting, $index(I-T) \leq 1$. Then $\vartheta(T)$ of $T = P^{-1}QM^{-1}N$ holds*

$$\vartheta(T) \leq \vartheta(P^{-1}Q) < 1 \tag{12}$$

if one of the following conditions is satisfied.

• *$A = M - N$ are weak quasi-nonnegative splittings of the first type and $index(I-M^{-1}N) \leq 1$, $index(I-P^{-1}Q) \leq 1$.*

• *$A = M - N$ are weak quasi-nonnegative splittings of the second type and $index(I - NM^{-1}) \leq 1$, $index(I - QP^{-1}) \leq 1$.*

Remark 3.1. From Theorem 4.1 and 4.2, we know that under appropriate conditions the asymptotic rate of semiconvergence of the iteration (4) is at least as good as the rate of semiconvergence of the fastest of the two basic iterations, i.e.,

$$\vartheta(T) \leq \min(\vartheta(M^{-1}N), \vartheta(P^{-1}Q)).$$

In short, the alternating between two splittings can be advantageous over iterating with a single splitting.

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