World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:7, No:2, 2013

Fixed Point Theorems for Set Valued Mappings in Partially Ordered Metric Spaces

Ismat Beg and Asma Rashid Butt

Abstract—Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Assume that X satisfies; if a non-decreasing sequence $x_n \to x$ in X, then $x_n \preceq x$, for all n. Let F be a set valued mapping from X into X with nonempty closed bounded values satisfying;

(i) there exists $\kappa \in (0, 1)$ with

$$D(F(x), F(y)) \leq \kappa d(x, y), \ for\ all\ x \preceq y,$$

(ii) if $d(x, y) < \varepsilon < 1$ for some $y \in F(x)$ then $x \preceq y$,

(iii) there exists $x_0 \in X$, and some $x_1 \in F(x_0)$ with $x_0 \preceq x_1$ such that $d(x_0, x_1) < 1$.

It is shown that F has a fixed point. Several consequences are also obtained.

Keywords—Fixed point, partially ordered set, metric space, set valued mapping.

I. Introduction

Let (X, d) be a complete metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, let

$$D(A, B) := \max\{\sup_{b\in B} d(b, A), \ \sup_{a\in A} d(a, B)\},$$

where

$$d(a, B) := \inf_{b\in B} d(a, b).$$

D is said to be a *Hausdorff metric* induced by d.

Banach's contraction principle [13, Theorem 2.1] is one of the fundamental and useful tool in mathematics. A number of authors have defined contractive type mapping [24] on a complete metric space X which are generalization of the Banach's contraction principle. Because of its simplicity it has been used in solving existence problems in many branches of mathematics [25]. In 1969, Nadler [15] extended the Banach's principle to set valued mappings in complete metric spaces and proved the following result.

Theorem 1.1. [15] Let (X, d) be a complete metric space and $F : X \to CB(X)$ be a set valued mapping. If there exists $\kappa \in (0, 1)$ such that

$$D(F(x), F(y)) \leq \kappa d(x, y), \ for\ all\ x, y \in X.$$

Then F has a fixed point in X.

Various fixed point results for contractive single valued mappings have been extended to set valued mappings, see for instance [23], [14], [9], [8], [21], [12], [5] and references cited there in. Recently Ran and Reurings [22] initiated the trend of weaken the contraction condition by considering single valued mappings on partially ordered metric space. They proved the following result:

Theorem 1.2. [22] Let $(X, \preceq)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bound. Let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:

1) there exists $\kappa \in (0, 1)$ with

$$d(f(x), f(y)) \leq \kappa d(x, y) \ for\ all\ x \preceq y.$$

2) there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ or $fx_0 \preceq x_0$.

Then f has a unique fixed point $x^* \in X$ and for each $x \in X$,

$$\lim_{n\to\infty} f^n(x) = \ x^*.$$

Ran and Reurings [22] result was further extended by [17], [19], [1], [10], [20], [7], [2], [3], [4], [18], [11], [16], [6]. Aim of this paper is to obtain by following Ran and Reurings, some results on fixed point for lower set valued mappings on a partially ordered metric space with a weaker contractive condition.

II. Preliminaries

Let $F : X \to 2^X$ be a set valued mapping $i.e$, $X \ni x \mapsto F(x)$ is a subset of X.

Definition 2.1. A point $x \in X$ is said to be a *fixed point* of the set valued mapping F if $x \in F(x)$.

Definition 2.2. A *partial order* is a binary relation $\preceq$ over a set X which satisfies the following conditions:

1) $x \preceq x$ (reflexivity);
2) if $x \preceq y$ and $y \preceq x$ then $x = y$ (antisymmetry);
3) if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ (transitivity);

for all x, y and z in X.

A set with a partial order $\preceq$ is called a *partially ordered set*.

Definition 2.3. Let $(X, \preceq)$ be a partially ordered set and $x, y \in X$. x and y are said to be *comparable elements* of X if either $x \preceq y$ or $y \preceq x$.

Lemma 2.4. [15] If $A, B \in CB(X)$ with $D(A, B) < \epsilon$ then for each $a \in A$ there exists an element $b \in B$ such that $d(a, b) < \epsilon$.

Lemma 2.5. [15] Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} D(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $x \in A$.

I. Beg is with the Department of Centre for Applicable Mathematics and Statistics, University of Central Punjab, Lahore, Pakistan e-mail: begismat@yahoo.com

A. R. Butt is with the Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.

III. Main Results

We begin with the following theorem that gives the existence of a fixed point (not necessarily unique) in partially ordered metric spaces for the set valued mapping.

Theorem 3.1. Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Assume that X satisfies: if a non-decreasing sequence $x_n \to x$ in X, then $x_n \preceq x$, for all n.

Let $F : X \to CB(X)$ satisfying:

1) there exists $\kappa \in (0, 1)$ with
$$D(F(x), F(y)) \leq \kappa d(x, y), for\ all\ x \preceq y.$$
2) if $d(x, y) < \varepsilon < 1$ for some $y \in F(x)$ then $x \preceq y$.
3) there exists $x_0 \in X$, and some $x_1 \in F(x_0)$ with $x_0 \preceq x_1$ such that $d(x_0, x_1) < 1$.

Then F has a fixed point.

Proof. Let $x_0 \in X$ then by assumption 3 there exists $x_1 \in F(x_0)$ with $x_0 \preceq x_1$ such that
$$d(x_0, x_1) < 1. \tag{1}$$
By using assumption 1 and inequality 1 we have,
$$D(F(x_0), F(x_1)) \leq \kappa d(x_0, x_1) < \kappa.$$
Using assumption 2 and Lemma 2.4, we have the existence of $x_2 \in F(x_1)$ with $x_1 \preceq x_2$ such that
$$d(x_1, x_2) < \kappa. \tag{2}$$
Again by assumption 1 and inequality 2 we have,
$$D(F(x_1), F(x_2)) \leq \kappa\ d(x_1, x_2) < \ \kappa^2,$$
therefore,
$$d(x_2, F(x_2)) < \kappa^2.$$
Continuing in this way we obtain $x_n \in F(x_{n-1})$ with $x_{n-1} \preceq x_n$ such that,
$$d(x_{n-1}, x_n) < \kappa^{n-1}$$
and
$$d(x_n, F(x_n)) < \kappa^n.$$
From the above inequality and by the assumption 2 we have the existence of $x_{n+1} \in F(x_n)$ with $x_n \preceq x_{n+1}$ such that,
$$d(x_n, x_{n+1}) < \kappa^n. \tag{3}$$
Next we will show that (x_n) is a Cauchy sequence in X. Let $m > n$. Then
$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &< [\kappa^n + \kappa^{n+1} + \kappa^{n+2} + ... + \kappa^{m-1}] \\ &= \kappa^n[1 + \kappa + \kappa^2 .... + \kappa^{m-n-1}] \\ &= \kappa^n\ [\frac{1 - \kappa^{m-n}}{1 - \kappa}\ ] \\ &< \frac{\kappa^n}{1 - \kappa}\ , \end{aligned}$$
because $\kappa \in (0, 1), 1 - \kappa^{m-n} < 1$.

Therefore $d(x_n, x_m) \to 0$ as $n \to \infty$ implies that (x_n) is a Cauchy sequence and hence converges to some point (say) x in the complete metric space X.

Next we have to show that x is the fixed point of the mapping F. Since x_n is a non-decreasing sequence in X such that $x_n \to x$ therefore we have $x_n \preceq x$ for all n.

From assumption 1, it follows that
$$D(F(x_n), F(x)) \leq \kappa\ d(x_n, x) \to 0.$$
Now because $x_{n+1} \in F(x_n)$, it follows by using Lemma 2.5 that $x \in F(x)$, i.e., x is fixed under the set valued mapping F.

Remark 3.2. If we replace assumption 2 in Theorem 3.1 by the condition: if $x, y \in X$ with $x \preceq y$ and if for all $u \in F(x)$ there exists $v \in F(y)$ such that $d(u, v) < 1$ then $u \preceq v$, and assuming all other hypothesis, we obtain that F has a fixed point.

The contraction condition given by Nadler [15] is stronger than the contraction condition used in our Theorems 3.1. Also Theorem 3.1 with the condition stated in the Remark 3.2 partially generalize the result of Ran and Reurings [22] and Nieto et al. [17] .

Corollary 3.3. Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a single valued mapping satisfying

1) there exists $\kappa \in (0, 1)$ with
$$d(f(x), f(y)) \leq \kappa d(x, y), for\ all\ x \preceq y.$$
2) f is order preserving i.e., if $x, y \in X$, with $x \preceq y$ then $f(x) \preceq f(y)$.
3) there exists $x_0 \in X$ with $x_0 \preceq f(x_0) = x_1$ (say).
4) if a non-decreasing sequence $x_n \to x$ in X, then $x_n \preceq x$, for all n.

Then f has a fixed point.

Similarly we can establish the following result which is an analogue of Theorem 3.1.

Theorem 3.4. Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Assume that X satisfies: if a non-increasing sequence $x_n \to x$ in X, then $x \preceq x_n$, for all n.

Let $F : X \to CB(X)$ be a set valued mapping satisfying:

1) there exists $\kappa \in (0, 1)$ with
$$D(F(x), F(y)) \leq \kappa d(x, y), for\ all\ y \preceq x.$$
2) if $d(x, y) < \varepsilon < 1$ for some $y \in F(x)$ then $y \preceq x$.
3) there exists $x_0 \in X$, and some $x_1 \in F(x_0)$ with $x_1 \preceq x_0$ such that $d(x_0, x_1) < 1$.

Then F has a fixed point.

Proof. It follows on the similar lines as Theorem 3.1.

Remark 3.5. In Theorems 3.1 and 3.3, we can also replace the assumption of monotonicity of the terms of the sequence by the comparability.

Theorem 3.6. Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F : X \to CB(X)$ satisfying:

1) there exists $\kappa \in (0,1)$ with
$$D(F(x), F(y)) \leq \kappa d(x,y), for\ all\ x \preceq y.$$
2) if $d(x,y) < \varepsilon < 1$ for some $y \in F(x)$ then $x \preceq y$ or $y \preceq x$.
3) there exists $x_0 \in X$, and some $x_1 \in F(x_0)$ with $x_0 \preceq x_1$ or $x_1 \preceq x_0$ such that $d(x_0, x_1) < 1$.
4) if $x_n \to x$ is any sequence in X whose consecutive terms are comparable then $x_n \preceq x$ or $x \preceq x_n$ for all n.

Then F has a fixed point.

Proof. It follows on the similar line by using Theorem 3.1 and Theorem 3.4.

REFERENCES

[1] R.P. Agarwal, M.A. Elgebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Anal. 87(2008), 109-116.
[2] I. Beg and A.R. Butt, Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces, Nonlinear Anal. 71(2009), 3699-3704.
[3] I. Beg and A.R. Butt, Fixed point for weakly compatible mappings satisfying an implicit relation in partially ordered metric spaces, Carpathian J. Math. 25(1)(2009), 1-12.
[4] I. Beg and A.R. Butt, Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces, Math. Comm. 15(1)(2010), 65-76. 146.
[5] I. Beg, A.R. Butt and S. Radojevic: Contraction principle for set valued mappings on a metric space with a graph, Computers & Math. with Applications, 60(2010), 1214-1219.
[6] I. Beg and H.K. Nashine, End-point results for multivalued mappings in partially ordered metric spaces, Internat. J. Math. & Math. Sci., 2012(2012) Article ID 580250, 19 pages.
[7] T.G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(2006), 1379-1393.
[8] P.Z. Daffer, Fixed points of generalized contractive multivalued mappings, J. Math. Anal. Appl. 192 (1995), 655-666.
[9] P.Z. Daffer, H. Kaneko and W. Li, On a conjecture of S. Reich, Proc. Amer. Math. Soc. 124 (1996), 3159-3162.
[10] Z. Drici, F.A. McRae and J.V. Devi, Fixed point theorems in partially ordered metric spaces for operators with PPF dependence, Nonlinear Anal. 7(2007), 641-647.
[11] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (2009), 3403-3410.
[12] Y. Feng and S. Liu, Fixed point theorems for multivalued contractive mappings and multivaled Caristi type mappings, J. Math. Anal. Appl. 317 (2006), 103-112.
[13] W.A. Kirk and K.Goebel, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge 1990.
[14] D. Klim and D.Wardowski, Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl. 334 (2007), 132-139.
[15] S.B. Nadler, Multivalued contraction mappings, Pacific J. Math. 30 (1969), 475-488.
[16] H.K. Nashine and B. Samet, Fixed point results for mappings satisfying $(\varphi, \phi)-$weakly contractive condition in partially ordered metric spaces, Nonlinear Anal. 74(2011), 2201-2209.
[17] J.J. Nieto and R. Rodríguez-López, Contractive mapping theorms in partially ordered sets and applications to ordinary differential equations, Order 22 (2005), 223-239.
[18] D. O'Regan and A. Petrusel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2008), 1241-1252.
[19] J.J. Nieto, R.L. Pouso and R. Rodríguez-López, Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc. 135 (2007), 2505-2517.
[20] A. Petrusel and I.A. Rus, Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134(2006), 411-418.
[21] C.Y. Qing, On a fixed point problem of Reich, Proc. Amer. Math. Soc. 124 (1996), 3085-3088.
[22] A.C.M. Ran and M.C.B. Reurings, A fixed point theorm in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.
[23] S. Reich, Fixed points of contractive functions, Boll. Unione. Mat. Ital.(4) 5(1972), 26-42.
[24] B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226(1977), 257-290.
[25] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed point Theorems*, Springer Verlag, New York 1985.