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An Analytical Method to Analysis of Foam Drainage Problem

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***Abstract*—**In this study, a new reliable technique use to handle the foam drainage equation. This new method is resulted from VIM by a simple modification that is Reconstruction of Variational Iteration Method (RVIM). The drainage of liquid foams involves the interplay of gravity, surface tension, and viscous forces. Foaming occurs in many distillation and absorption processes. Results are compared with those of Adomian's decomposition method (ADM).The comparisons show that the Reconstruction of Variational Iteration Method is very effective and overcome the difficulty of traditional methods and quite accurate to systems of non-linear partial differential equations.

***Keywords*—**Reconstruction of Variational Iteration Method (RVIM), Foam drainage; nonlinear partial differential equation.

I. INTRODUCTION

FOAMS [1,2] are a prime example of a multiphase "soft condensed matter" system. They have important applications in the food and chemical industries, firefighting, mineral processing, and structural material science [2], and their properties are subject of intensive studies from both practical and scientific points of view [3]. Foams are common in personal care products such as creams and lotions, and foams often occur, even when not desired, during cleaning (clothes, dishes, scrubbing) and dispensing processes (c.f. [4]). They have important applications in the food and chemical industries, firefighting, mineral processing, and structural material science (c.f. [5]). Less obviously they appear in acoustic cladding, lightweight mechanical components, and impact absorbing parts on cars, heat exchangers and textured wallpapers (incorporated as foaming inks) and even have an analogy in cosmology. History connects foams with a number of eminent scientists, and foams continue to excite imaginations [6].although there are now many applications of polymeric foams [7] and more recently metallic foams, which are foams made out of metals such as aluminum [8]. In addition, industrial applications of polymeric foams and porous metals include their use for structural purposes and as heat exchange media analogous to common "finned" structures [9]. Recent research in foams and emulsions has centered on three topics which are often treated separately, but are in fact interdependent: drainage, coarsening, and rheology; see Fig. 1. We focus here on a quantitative description of the coupling of drainage and coarsening. The flow of liquid relative to the bubbles is called drainage. Drainage plays an important role in foam stability: indeed, when foam dries, its structure becomes more fragile; the liquid films between adjacent bubbles being thinner, then can break, leading to foam collapse. In the case of aqueous foams, surfactant is added in to water and it adsorbs at the surface of the films, protecting them against rupture (c.f. [10]).

Foam drainage is the flow of liquid through channels (Plateau borders) and nodes (intersections of four channels) between the bubbles, driven by gravity and capillarity [11-13]

The foam drainage equation models the dynamics of the liquid volume fraction e in the foam on length scales larger than the bubble size Generally drainage is driven by gravity and/or capillary (surface tension) forces and is resisted by viscous forces (c.f. [5]).

Recent theoretical studies by Verbist and Weaire describe the main features of both free drainage [14, 15], where liquid drains out of a foam due to gravity, and forced drainage [16], where liquid is introduced to the top of a column of foam Forced foam drainage may well be the best prototype or certain general phenomena described by non-linear differential equations, particularly the type of solitary wave which is most familiar in tidal bores.

The aim of current study is analytically investigation non-linear foam drainage equation in the form of Eq. (1), using Reconstruction of Variational Iteration Method (RVIM). In recent years, several such techniques have drawn special attention, such as Hirota's bilinear method [17], the homogeneous balance method [18], inverse scattering method [19], Adomian's decomposition method ADM [20], the variational iteration method [21] and the δ-expansion method [22], Variational iteration method VIM [23, 24], Energy Balance Method [25], as well as Homotopy analysis method (HAM).After that, many types of nonlinear problems were solved by the HAM by others [26-28].

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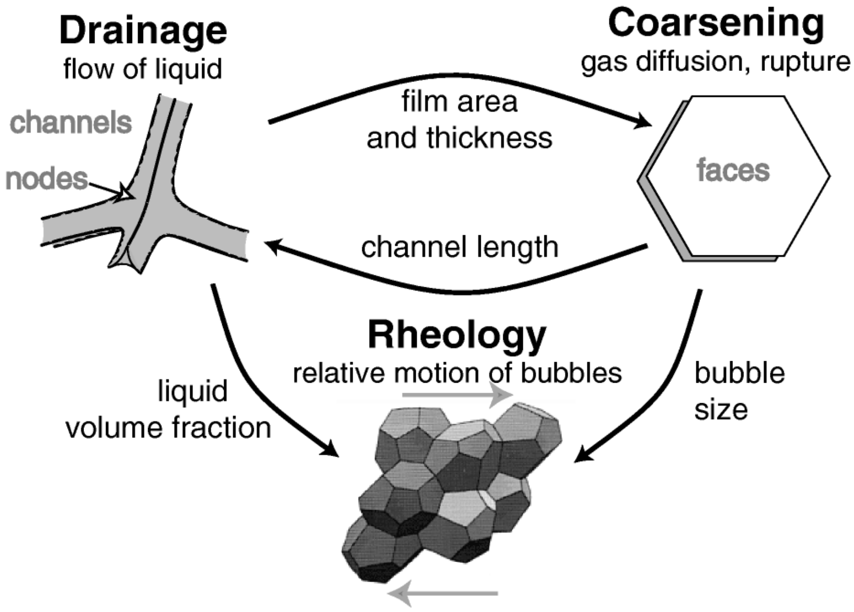

Fig. 1 Schematic of the interdependence of drainage, coarsening, and rheology of foams [6].

In this paper, we will apply new algorithm that is a powerful and efficient technique in finding the approximate solutions for the following foam drainage equation:

$$\frac{\partial A}{\partial t}+\frac{\partial}{\partial x}(A^2-\frac{\sqrt{A}}{2}\frac{\partial A}{\partial x})=0, \qquad (1)$$

where x and t are scaled position and time coordinates respectively.

In the case of forced drainage, the solution can be expressed as [29]:

$$A(x,t)=c\tanh^2(\sqrt{c}(x-ct)) \qquad (2)$$

where c is the velocity of the wave front [16].

II. Basic Concept of RVIM

In the following section, an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform [30],[31] will be investigated a large of problems in science and engineering involve the solution of partial differential equations. Suppose *x, t* are two independent variables; consider *t* as the principal variable and *x* as the secondary variable. If *u(x,t)* is a function of two variables *x* and *t* , when the Laplace transform is applied with *t* as a variable, definition of Laplace transform is

$$\ell[u(x,t);s]=\int_0^\infty e^{-st}u(x,t)dt \qquad (3)$$

We have some preliminary notations as

$$\ell[\frac{\partial u}{\partial t};s]=\int_0^\infty e^{-st}\frac{\partial u}{\partial t}dt=sU(x,s)-u(x,0) \qquad (4)$$

$$\ell[\frac{\partial^2 u}{\partial t^2};s]=s^2U(x,s)-su(x,0)-u_t(x,0) \qquad (5)$$

where

$$U(x,s)=\ell[u(x,t);s] \qquad (6)$$

We often come across functions which are not the transform of some known function, but then, they can possibly be as a product of two functions, each of which is the transform of a known function. Thus we may be able to write the given function as *U(x,s),V(x,s)* where *U(s)* and *V(s)* are known to the transform of the functions *u(x,t),v(x,t)* respectively. The convolution of *u(x,t)* and *v(x,t)* is written *u(x,t)*v(x,t)*. It is defined as the integral of the product of the two functions after one is reversed and shifted.

Convolution Theorem: if *U(x,s),V(x,s)* are the Laplace transform of *u(x,t),v(x,t)*, when the Laplace transform is applied to *t* as a variable, respectively; then *U(x,t).V(x,t)* is the Laplace Transform of $\int_0^t u(x,t-\varepsilon)v(x,\varepsilon)d\varepsilon$

$$\ell^{-1}[U(x,s).V(x,s)]=\int_0^t u(x,t-\varepsilon)v(x,\varepsilon)d\varepsilon \qquad (7)$$

To facilitate our discussion of Reconstruction of Variational Iteration Method, introducing the new linear or nonlinear function $h(u(t,x))=f(t,x)-N(u(t,x))$ and considering the new equation, rewrite $h(u(t,x))=f(t,x)-N(u(t,x))$

as

$$L(u(x,t)) = h(t,x,u) \qquad (8)$$

Now, for implementation the correctional function of VIM based on new idea of Laplace transform, applying Laplace Transform to both sides of the above equation so that we introduce artificial initial conditions to zero for main problem, then left hand side of equation after transformation is featured as

$$\ell[L\{u(x,t)\}] = U(x,s)\,p(s) \qquad (9)$$

where *P(s)* is polynomial with the degree of the highest order derivative of the selected linear operator.

$$\ell[L\{u(x,t)\}] = U(x,s)\,p(s) = \ell[\{h(x,t,u)\}] \qquad (10)$$

Then

$$U(x,s) = \frac{\ell[\{h(x,t,u)\}]}{p(s)} \qquad (11)$$

Suppose that $D(s) = \frac{1}{p(s)}$ and $\ell[\{h(x,t,u)\}] = H(x,s)$. Therefore using the convolution theorem we have

$$U(x,s) = H(x,s).D(s) = \ell\{(d(t) * h(x,t,u)\} \qquad (12)$$

Taking the inverse Laplace transform on both side of eq. (12)

$$u(x,t) = \int_0^t d(t-\varepsilon)h(x,\varepsilon,u)d\varepsilon \qquad (13)$$

Thus the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t d(t-\varepsilon)h(x,\varepsilon,u_n)d\varepsilon \qquad (14)$$

And $u_0(x,t)$ is initial solution with or without unknown parameters. In absence of unknown parameters, $u_0(x,t)$ should satisfy initial/ boundary conditions.

III. Application of RVIM

In this section, we will apply the RVIM to solve foam drainage equation.

Foam drainage equation (1) can be written as [29]:

$$u_t + 2uu_x - \frac{\sqrt{u}}{2}u_{xx} - \frac{1}{4\sqrt{u}}(u_x)^2 = 0 \qquad (15)$$

with initial conditions

$$u_0(x,t) = 3\tanh^2(\sqrt{3}x) \qquad (16)$$

The exact solution for this problem is

$$u(x,t) = c\tanh^2(\sqrt{c}(x-ct)) \qquad (17)$$

At first rewrite eq. (17) based on selective linear operator as

$$\ell\{u(x)\} = u_t = \overbrace{-(2uu_x - \frac{\sqrt{u}}{2}u_{xx} - \frac{1}{4\sqrt{u}}u_x^2)}^{h(x,t,u)} \qquad (18)$$

Now Laplace transform is implemented with respect to independent variable x on both sides of eq. (18) and by using the new artificial initial condition (which all of them are zero) we have

$$s \cup (x,t) = \ell\{h(x,t,u)\} \qquad (19)$$

$$\cup(x,t) = \frac{\ell\{h(x,t,u)\}}{s} \qquad (20)$$

and whereas Laplace inverse transform of *1/s* is as follows

$$\ell^{-1}[\frac{1}{s}] = 1 \qquad (21)$$

Therefore by using the Laplace inverse transform and convolution theorem it is concluded that

$$u(x,t) = \int_0^t h(x,\varepsilon,u)d\varepsilon \qquad (22)$$

Hence, we arrive at the following iterative formula for the approximate solution of subject to the initial condition (16).

So, in exchange with applying recursive algorithm, following relations are achieved

$$u_{n+1} = u_0 + \int_0^t (-2uu_x + \frac{\sqrt{u}}{2}u_{xx} + \frac{1}{4\sqrt{u}}(u_x)^2)d\varepsilon \qquad (23)$$

Now we start with an arbitrary initial approximation $u_0(x,t) = 3\tanh^2(\sqrt{3}x)$ that satisfies the initial condition and by using the RVIM iteration formula (23), we have the following successive approximation

$$u_{n+1} = u_0 + \int_0^t (-2u_0 u_{0x} + \frac{\sqrt{u_0}}{2} u_{0xx} + \frac{1}{4\sqrt{u_0}} (u_{0x})^2) d\varepsilon \qquad (24)$$

$$\begin{aligned} u_1 &= 3\tanh(\sqrt{3}x)(\tanh(\sqrt{3}x)) - 12\tanh^2(\sqrt{3}x)\sqrt{3}t \\ &+ 12\tanh^4(\sqrt{3}x)\sqrt{3}t + 6\sqrt{3}c\,\mathrm{sgn}(\tanh(\sqrt{3}x)t \\ &- 18\sqrt{3}c\,\mathrm{sgn}(\tanh(\sqrt{3}x)t\tanh^2(\sqrt{3}x) \\ &+ 12\sqrt{3}c\,\mathrm{sgn}(\tanh(\sqrt{3}x)t\tanh^4(\sqrt{3}x) \end{aligned} \qquad (25)$$

whereas, the RVIM method admits the use of

$$\mathrm{u} = \lim_{n\to\infty} \mathrm{u}_n$$

Tables I, II, and III investigated comparison between errors of ADM [29].

TABLE I
COMPARISON BETWEEN ERRORS OF ADM AND RVIM FOR *T*=0.1 AND C=3.

X	u_{exact}-u_{ADM}	u_{exact}-u_{RVIM}
0	1.42109E-14	0
-2	1.40941E-11	0
-4	1.43842E-8	-5E-9
-6	0.00000146727	-4.527E-6
-8	0.0064218	-4.59239E-3
-10	-3.71367	0.68374

TABLE II
COMPARISON BETWEEN ERRORS OF ADM AND RVIM FOR *T*=0.01 AND C=3.

X	u_{exact}-u_{ADM}	u_{exact}-u_{RVIM}
0	-1.77636E-15	0
-2	-1.86962E-12	0
-4	-1.9087E9	0
-6	-1.94811E-6	0
-8	-0.00197296	-6.0881E-0.5
-10	0.00051592	8.8544E-3

TABLE III
COMPARISON BETWEEN ERRORS OF ADM AND RVIM FOR *T*=0.001 AND C=3.

X	u_{exact}-u_{ADM}	u_{exact}-u_{RVIM}
0	4.44089E-16	0
-2	2.24265E-13	0
-4	-2.29754E-10	0
-6	-2.34498E-7	0
-8	-0.000236656	0
-10	5.2479E-8	0

IV. CONCLUSION

In this paper, an explicit analytical solution is obtained for foam drainage equation by means of the Reconstruction of Variational Iteration Method (RVIM), which is a powerful mathematical tool in dealing with nonlinear equations. Comparison of results in tables proved that RVIM can be used in applied mathematics as a trustworthy and explicit method. The accuracy of the method is acceptable and the resulting solutions are close to the numerical solutions that are shown in tables.

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