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Advances on LuGre Friction Model

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Abstract**—** LuGre friction model is an ordinary differential equation that is widely used in describing the friction phenomenon for mechanical systems. The importance of this model comes from the fact that it captures most of the friction behavior that has been observed including hysteresis. In this paper, we study some aspects related to the hysteresis behavior induced by the LuGre friction model.

Keywords**—** Hysteresis, LuGre model, operator, (strong) consistency.

I. INTRODUCTION

Friction is a nonlinear phenomenon that originates from the contact of two bodies. As early as 1699, Amonton discovered that the friction force that resists relative motion between two bodies in contact is independent of the area of apparent contact surface [20]. It is only in recent times that this paradox has been solved, showing that the friction force is proportional to the true contact area [9]. As a matter of fact, friction depends on many parameters, such as surface topography, presence and type of lubrication and relative motion. The friction phenomenon is usually divided into two operating regimes, presliding friction and sliding friction. Presliding friction refers to the elastic and plastic deformations of asperities (roughness features). Sliding friction is due to the shearing resistance of the asperities. An important characteristic of presliding friction is the existence of hysteresis between the presliding friction force input and the displacement output [2], [26], [22].

The friction is decomposed into two types depending upon the nature of the two surfaces in contact, static friction and dynamic friction. The static characteristics of friction include the stiction friction, the kinetic force (the Coulomb force), the viscous force, and the Stribeck effect which are functions of steady state velocity. Therefore, static friction models are symmetric, discontinuous at zero velocity [17], with a dependence on the sign of velocity [29]. Dynamic friction models capture properties that cannot be captured by typical static friction models; for instance, presliding displacement, frictional lag (the delay in the change of friction force as a function of a change of velocity), and stick-slip motion, which is the spontaneous jerking motion that can occur while two objects are sliding over each other [6].

Dahl friction model [7] is a generalization of the Coulomb friction [8]. The steady state of the Dahl model is precisely the Coulomb friction. However, it does not capture the Stribeck effect [8]. An improvement of this model is implemented in the LuGre model [5]. This model captures the essential properties of friction such as hysteresis and Stribeck effect (and thus can describe stick-slip motion) [3], [22]. Therefore, it has been widely used to describe the friction phenomenon for mechanical systems [19], [3]. The LuGre model behaves like a linear spring/damper pair when it is linearized near zero relative velocity [5]. Necessary and sufficient conditions for the dissipativity to hold for the LuGre model are given in [4]. The model is very popular for friction compensation [11], [24], [14], [27], and its parameter identification has been studied in [31], [21], [25].

In this paper, we focus on the hysteresis behavior of the LuGre model. Following the recent research carried out in [12], this model is seen as an operator $\mathcal{H}$ that associates to an input u and initial condition x_0 an output $\mathcal{H}(u, x_0)$, all belonging to some appropriate spaces. The class of operators $\mathcal{H}$ that are considered in [12] are the causal ones, with the additional condition that a constant input leads to a constant output. For this class of operators, two properties have been defined: consistency and strong consistency. Since the LuGre model falls within the framework of [12], it is of interest to analyze its consistency and strong consistency, which is the aim of this paper.

The paper is organized as follows. Section II presents the needed mathematical background. The problem statement is introduced in Section III. The main results of this paper are presented in Section IV. Conclusions are given in Section V.

II. BACKGROUND RESULTS

This section summarizes the results obtained in [12].

A. Class of inputs

A real number x is said positive when $x > 0$, negative when $x < 0$, nonpositive when $x \leq 0$, and nonnegative when $x \geq 0$. A function $h : \mathbb{R} \to \mathbb{R}$ is said increasing when $t_1 < t_2 \Rightarrow h(t_1) < h(t_2)$, decreasing when $t_1 < t_2 \Rightarrow h(t_1) > h(t_2)$, nonincreasing when $t_1 < t_2 \Rightarrow h(t_1) \geq h(t_2)$, and nondecreasing when $t_1 < t_2 \Rightarrow h(t_1) \leq h(t_2)$.

The Lebesgue measure on $\mathbb{R}$ is denoted μ. A subset of $\mathbb{R}$ is said measurable when it is Lebesgue measurable. Consider a function $p : I \subset \mathbb{R}_+ = [0, \infty) \to \mathbb{R}^m$ where I is some interval and m a positive integer; the function p is said measurable when p is (M, B)-measurable where B is the class of Borel sets of $\mathbb{R}^m$ and M is the class of measurable sets of $\mathbb{R}_+$. For a measurable function $p : I \subset \mathbb{R}_+ \to \mathbb{R}^m$,

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$\|p\|_{\infty,I}$ denotes the essential supremum of the function $|p|$ on I where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$. When $I = \mathbb{R}_+$, it is denoted simply $\|p\|_\infty$.

Consider the Sobolev space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ of absolutely continuous functions $u : \mathbb{R}_+ \to \mathbb{R}^n$, where n is a positive integer. For this class of functions, the derivative $\dot{u}$ is defined a.e, and we have $\|u\|_\infty < \infty$, $\|\dot{u}\|_\infty < \infty$. Endowed with the norm $\|u\|_{1,\infty} = \max(\|u\|_\infty, \|\dot{u}\|_\infty)$, $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is a Banach space [1].

For $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, let $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ be the total variation of u on $[0, t]$, that is $\rho_u(t) = \int_0^t |\dot{u}(\tau)| \, \mathrm{d}\tau \in \mathbb{R}_+$. The function ρ_u is well defined as $\dot{u} \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$[1]. It is nondecreasing and absolutely continuous. Denote $\rho_{u,\max} = \lim_{t\to\infty} \rho_u(t)$ and let

- $I_u = [0, \rho_{u,\max}]$ if $\rho_{u,\max} = \rho_u(t)$ for some $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ is necessarily finite).
- $I_u = [0, \rho_{u,\max})$ if $\rho_{u,\max} > \rho_u(t)$ for all $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ may be finite or infinite).

Lemma 1. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be non-constant so that the interval I_u is not reduced to a single point. Then there exists a unique function $\psi_u \in W^{1,\infty}(I_u, \mathbb{R}^n)$ that satisfies $\psi_u \circ \rho_u = u$. The function ψ_u satisfies $\left\|\dot{\psi}_u\right\|_{\infty,I_u} = 1$ and $\mu\left[\left\{\varrho \in I_u / \dot{\psi}_u(\varrho) \text{ is not defined or } |\dot{\psi}_u(\varrho)| \neq 1\right\}\right] = 0$.*

Consider the linear time scale change $s_\gamma(t) = t/\gamma$, for any $\gamma > 0$ and $t \geq 0$.

Lemma 2. *For all $\gamma > 0$, we have $I_{u\circ s_\gamma} = I_u$ and $\psi_{u\circ s_\gamma} = \psi_u$.*

B. Class of operators

Let Ξ be a set of initial conditions. Let $\mathcal{H}$ be an operator that maps the input function $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ to an output in $L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. That is $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. The operator $\mathcal{H}$ is said to be causal if the following holds [30, p.60]: $\forall \left(u_1, \xi^0\right), \left(u_2, \xi^0\right) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$, if $u_1 = u_2$ in $[0, \tau]$, then $\mathcal{H}\left(u_1, \xi^0\right) = \mathcal{H}\left(u_2, \xi^0\right)$ in $[0, \tau]$.

Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ and let $y = \mathcal{H}\left(u, \xi^0\right) \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. In the rest of this work, only causal operators are considered.
Additionally, we consider that the operator $\mathcal{H}$ satisfies the following.

Assumption 1. *Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$; if there exists a time instant $\theta \in \mathbb{R}_+$ such that u is constant in $[\theta, \infty)$, then the corresponding output $\mathcal{H}(u, \xi^0)$ is constant in $[\theta, \infty)$.*

Assumption 1 is verified by all causal and rate-independent hysteresis operators (see for example [15, Proposition 2.1] for a proof). This includes relay hysteresis, Ishlinskii model, Preisach model, Krasnosel'skii and Pokrovskii hysteron and generalized play [16]. Assumption 1 is also verified by some causal and rate-dependent hysteresis models like the generalized Duhem model [18].

Lemma 3. *There exists a unique function $\varphi_u \in L^\infty(I_u, \mathbb{R}^m)$ that satisfies $\varphi_u \circ \rho_u = y$. Moreover, we have $\|\varphi_u\|_{\infty,I_u} \leq \|y\|_\infty$. If y is continuous on $\mathbb{R}_+$, then φ_u is continuous on I_u and we have $\|\varphi_u\|_{\infty,I_u} = \|y\|_\infty$.*

C. Definition of consistency and strong consistency

Definition 1. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ be given. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 1. The operator $\mathcal{H}$ is said to be consistent with respect to input u and initial condition ξ^0 if and only if the sequence of functions $\{\varphi_{u\circ s_\gamma}\}_{\gamma>0}$ converges in $L^\infty(I_u, \mathbb{R}^m)$ as $\gamma \to \infty$.*

Let $T > 0$. In what follows we consider that the input u is T-periodic.

Definition 2. *A T-periodic function $w : \mathbb{R}_+ \to \mathbb{R}$ is said to be wave periodic if there exists some $T^+ \in (0, T)$ such that*

- *The function w is continuous on $\mathbb{R}_+$*
- *The function w is continuously differentiable on $(0, T^+)$ and on (T^+, T)*
- *The function w is increasing on $(0, T^+)$ and is decreasing on (T^+, T)*

Lemma 4. *If the input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is non-constant and T-periodic, then $I_u = \mathbb{R}_+$ and $\psi_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is $\rho_u(T)$-periodic. Furthermore, if $n = 1$ and u is wave periodic, then ψ_u is also wave periodic and $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in (\rho_u(T^+), \rho_u(T))$.*

For any positive integer k, define $\varphi^*_{u,k} \in L^\infty([0, \rho_u(T)], \mathbb{R}^m)$ as

$$\varphi^*_{u,k}(\varrho) = \varphi^*_u(\rho_u(T)k + \varrho), \forall \varrho \in [0, \rho_u(T)].$$

Definition 3. *The operator $\mathcal{H}$ is said to be strongly consistent with respect to input u and initial condition ξ^0 if and only if it is consistent with respect to u and ξ^0, and the sequence of functions $\varphi^*_{u,k}$ converges in $L^\infty([0, \rho_u(T)], \mathbb{R}^m)$ as $k \to \infty$.*

If the operator $\mathcal{H}$ is strongly consistent with respect to input u and initial condition ξ^0, then the graph $\{(\varphi^\circ_u(\varrho), \psi_u(\varrho)), \varrho \in [0, \rho_u(T)]\}$ represents the so-called hysteresis loop, where $\varphi^\circ_u = \lim_{k\to\infty} \varphi^*_{u,k}$.

III. Problem Statement

The LuGre model is given by [3]:

$$\begin{aligned}
\dot{x}(t) &= -\sigma_0 \frac{|\dot{u}(t)|}{g(\dot{u}(t))} x(t) + \dot{u}(t), && (1)\\
x(0) &= x_0, && (2)\\
F(t) &= \sigma_0 x(t) + \sigma_1 \dot{x}(t) + f(\dot{u}(t)). && (3)
\end{aligned}$$

where $t \geq 0$ denotes time; the parameters $\sigma_0 > 0$ and $\sigma_1 > 0$ are respectively the stiffness and the microscopic damping friction coefficients; the function $g \in C^0(\mathbb{R}, \mathbb{R})$[2] represents

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[1] $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ is the space of locally integrable functions $\mathbb{R}_+ \to \mathbb{R}^n$.

[2] $C^0(\mathbb{R}, \mathbb{R})$ is the Banach space of continuous functions defined from $\mathbb{R}$ to $\mathbb{R}$, endowed with the norm $\|\cdot\|_\infty$.

the macrodamping friction with $g(\vartheta) > 0, \forall\vartheta \in \mathbb{R}$; $x(t) \in \mathbb{R}$ is the average deflection of the bristles; $x_0 \in \mathbb{R}$ is the initial state; $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ is the relative displacement and is the input of the system; $F(t)$ is the friction force and is the output of the system; and $f \in C^0(\mathbb{R},\mathbb{R})$ is a memoryless function.

In Equation (1), the function $g(\dot{u})$ is measurable [23, Theorem 1.12(d)]. Thus, the differential equation (1) can be seen as a linear time-varying system that satisfies all assumptions of [10, Theorem 3]. This implies that a unique absolutely continuous solution of (1) exists on $\mathbb{R}_+$.

In equations (1)-(3), consider the following operators:

- The operator $\mathcal{H}_s : W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R} \to L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $\mathcal{H}_s(u, x_0) = x$
- The operator $\mathcal{H}_o : W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R} \to L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $\mathcal{H}_o(u, x_0) = F$

Now consider the following system.

$$\begin{aligned} \dot{x}(t) &= -\sigma_0 \frac{|v(t)|}{g(v(t))} x(t) + v(t), && (4) \\ x(0) &= x_0, && (5) \\ F(t) &= \sigma_0 x(t) + \sigma_1 \dot{x}(t) + f(v(t)). && (6) \end{aligned}$$

in which $v \in L^\infty(\mathbb{R}_+,\mathbb{R})$. In equations (4)-(6), consider the following operators:

- The operator $\mathcal{H}_s' : L^\infty(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R} \to L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $\mathcal{H}_s'(v, x_0) = x$
- The operator $\mathcal{H}_o' : L^\infty(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R} \to L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $\mathcal{H}_o'(v, x_0) = F$

Observe that the operators $\mathcal{H}_s'$ and $\mathcal{H}_o'$ are causal due to the uniqueness of the solutions of Equation (1).
Consider the left-derivative operator Δ_- defined on $W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ by

$$[\Delta_-(u)](t) = \lim_{\tau \uparrow t} \frac{u(\tau) - u(t)}{\tau - t}$$

The operator Δ_- is causal as $[\Delta_-(u)](t)$ depends only on values of $u(\tau)$ for $\tau \leq t$, and we have $\Delta_-(u) = \dot{u}$ a.e. as $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ so that $\Delta_-(u) \in L^\infty(\mathbb{R}_+,\mathbb{R})$.
Note that $\mathcal{H}_s = \mathcal{H}_s' \circ \Delta_-$ and $\mathcal{H}_o = \mathcal{H}_o' \circ \Delta_-$ so that the operators $\mathcal{H}_s$ and $\mathcal{H}_o$ are causal. Observe also that $\mathcal{H}_s$ and $\mathcal{H}_o$ satisfy Assumption 1.

Proposition 1. *Let $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$. There exists a unique function $v_u \in L^\infty(I_u,\mathbb{R})$ that is defined by $v_u \circ \rho_u = \dot{u}$. Moreover, $\|v_u\|_{\infty,I_u} \leq \|\dot{u}\|_\infty$. Assume that $\dot{u}$ is nonzero on a set $A \subseteq \mathbb{R}$ that satisfies $\mu(\rho_u(\mathbb{R}\backslash A)) = 0$. Then, v_u is nonzero almost everywhere.*

Proof. The operator $\Delta_- : W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \to L^\infty(\mathbb{R}_+,\mathbb{R})$ is causal and satisfies Assumption 1. The first part of Proposition 1 follows immediately from Lemma 3. Now, let $B = \{\varrho \in I_u / v_u(\varrho) = 0\}$, then $B \subseteq \rho_u(\mathbb{R}\backslash A)$ which implies that $\mu(B) = 0$. □

Remark 1. *Observe that if $\dot{u}$ is nonzero almost everywhere, then $\mu(\mathbb{R}\backslash A) = 0$ so that by [28] we have $\mu(\rho_u(\mathbb{R}\backslash A)) = 0$ as ρ_u is absolutely continuous. An example in which $\dot{u}$ does not need to be nonzero almost everywhere, is when u is constant on some interval, or on a finite number of intervals, or an infinite number of intervals such that this infinite number has measure zero (for example countable).*

In the rest of the paper, we consider that the input u satisfies the conditions of Proposition 1. Consider the time scale change $s_\gamma(t) = t/\gamma, \gamma > 0, t \geq 0$. When the input $u \circ s_\gamma$ is used instead of u, system (1)-(3) becomes

$$\begin{aligned} \dot{x}_\gamma(t) &= -\sigma_0 \frac{\left|\frac{\dot{u} \circ s_\gamma(t)}{\gamma}\right|}{g\left(\frac{\dot{u} \circ s_\gamma(t)}{\gamma}\right)} x_\gamma(t) + \frac{\dot{u} \circ s_\gamma(t)}{\gamma}, && (7) \\ x_\gamma(0) &= x_0, && (8) \\ F_\gamma(t) &= \sigma_0 x_\gamma(t) + \sigma_1 \dot{x}_\gamma(t) + f\left(\frac{\dot{u} \circ s_\gamma(t)}{\gamma}\right). && (9) \end{aligned}$$

When $\gamma = 1$, system (7)-(9) reduces to (1)-(3).

Lemma 3 shows that for any $\gamma > 0$, there exists a unique function $x_{u \circ s_\gamma} \in L^\infty(I_u,\mathbb{R})$ such that $x_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma} = x_\gamma$, and a unique function $\varphi_{u \circ s_\gamma} \in L^\infty(I_u,\mathbb{R})$ such that $\varphi_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma} = F_\gamma$. Using the change of variables $\varrho = \rho_{u \circ s_\gamma}(t)$, it follows from Equations (7)-(9), Lemma 2 and Proposition 1 that

$$\begin{aligned} \dot{x}_{u \circ s_\gamma}(\varrho) &= -\frac{\sigma_0}{g\left(\frac{v_u(\varrho)}{\gamma}\right)} x_{u \circ s_\gamma}(\varrho) + \dot{\psi}_u(\varrho), && (10) \\ x_{u \circ s_\gamma}(0) &= x_0, && (11) \\ \varphi_{u \circ s_\gamma}(\varrho) &= \sigma_0 x_{u \circ s_\gamma}(\varrho) + \frac{\sigma_1}{\gamma} |v_u(\varrho)| \dot{x}_{u \circ s_\gamma}(\varrho) \\ &\quad + f\left(\frac{v_u(\varrho)}{\gamma}\right), && (12) \end{aligned}$$

for all $\gamma > 0$ and for almost all $\varrho \in I_u$.

Problem statement: The aim of this paper is to analyze the convergence properties of the sequence of functions $\varphi_{u \circ s_\gamma}$ in order to study the consistency and strong consistency of the operator $\mathcal{H}_o$.

IV. MAIN RESULTS

This section presents the main result of the paper, which is Lemma 6.

The following lemma generalizes Theorem 4.18 in [13, p.172]. Indeed, in [13], continuous differentiability is needed, while in Lemma 5, we only need absolute continuity. Also, in [13], the inequality on the derivative of the Lyapunov function is needed everywhere, while in Lemma 5 it is needed only almost everywhere.

Lemma 5. *Consider a function $z : [0,\omega) \subseteq \mathbb{R}_+ \to \mathbb{R}_+$, where $\omega > 0$ is finite or infinite. Assume the following*

1) *The function z is absolutely continuous on each compact interval of $[0,\omega)$.*

2) *There exist* $z_1 \geq 0$ *and* $z_2 > 0$ *such that* $z_1 < z_2$, $z(0) < z_2$ *and*

$$\begin{cases} \dot{z}(t) \leq 0 \ \text{for almost all } t \in [0,\omega) \\ \text{that satisfy } z_1 < z(t) < z_2. \end{cases} \tag{13}$$

Then, $z(t) \leq \max(z(0), z_1)$, $\forall t \in [0,\omega)$.

Example 1. We want to study the stability of the following system

$$\begin{aligned} \dot{x}(t) &= -x^3(t) + u(t), & (14) \\ x(0) &= x_0, & (15) \end{aligned}$$

where x_0 and state x take values in $\mathbb{R}$, and input $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$. System (14)-(15) has an absolutely continuous solution that is defined on an interval of the form $[0,\omega)$ [10, p.4].

Let $z : [0,\omega) \to \mathbb{R}_+$ be such that $z(t) = x^2(t), \forall t \in [0,\omega)$. The function z is absolutely continuous on each compact subset of $[0,\omega)$ because x is absolutely continuous. Thus, Condition 1 in Lemma 5 is satisfied.

We have for almost all $t \in [0,\omega)$ that

$$\begin{cases} \dot{z}(t) = 2x(t) \cdot \dot{x}(t) = 2x(t)\left(-x^3(t) + u(t)\right) \\ \leq -2z^2(t) + 2\|u\|_\infty \sqrt{z(t)}. \end{cases}$$

Thus,

$$\dot{z}(t) \leq 0 \ \text{ for almost all } t \in [0,\omega) \ \text{ that satisfy } \|u\|_\infty^{2/3} < z(t).$$

Therefore, Condition 2 in Lemma 5 is satisfied with $z_1 = \|u\|_\infty^{2/3}$ and z_2 can be any positive real number such that $z_2 > \max(z(0), z_1) = \max\left(x_0^2, \|u\|_\infty^{2/3}\right)$. Thus, we deduce from Lemma 5 that $z(t) \leq \max\left(z(0), \|u\|_\infty^{2/3}\right) = \max\left(x_0^2, \|u\|_\infty^{2/3}\right)$, $\forall t \in [0,\omega)$, and hence $|x(t)| \leq \max\left(|x_0|, \sqrt[3]{\|u\|_\infty}\right)$, $\forall t \in [0,\omega)$.

Lemma 6. *Let* $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ *be such that* $\dot{u}$ *is nonzero on a set* $A \subseteq \mathbb{R}$ *that satisfies* $\mu(\rho_u(\mathbb{R}\backslash A)) = 0$. *Then the following holds:*

- *There exist* $E, \gamma_1 > 0$ *such that* $\|F_\gamma\|_\infty \leq E$, $\forall \gamma > \gamma_1$. [3]
- *The operator* $\mathcal{H}_o$ *is consistent with respect to input* u *and initial condition* x_0, *that is there exists a unique function* $\varphi_u^* \in W^{1,\infty}(I_u,\mathbb{R})$ *such that*

  $$\lim_{\gamma\to\infty} \left\|\varphi_{u \circ s_\gamma} - \varphi_u^*\right\|_{\infty,I_u} = 0,$$

  where

  $$\begin{aligned} \varphi_u^*(\varrho) &= \sigma_0 e^{\frac{-\sigma_0 \varrho}{g(0)}} \left[x_0 + \int_0^\varrho e^{\sigma_0\tau/g(0)} \dot{\psi}_u(\tau)\, d\tau\right] \\ &\quad + f(0), \forall \varrho \in I_u. \end{aligned} \tag{16}$$

Moreover, if u *is* T*-periodic, then the operator* $\mathcal{H}_o$ *is strongly consistent with respect to input* u *and initial condition* x_0. *That is, there exists a unique function* $\varphi_u^\circ \in W^{1,\infty}([0,\rho_u(T)],\mathbb{R})$ *such that*

$$\lim_{\gamma\to\infty} \left\|\varphi_{u,k}^* - \varphi_u^\circ\right\|_{\infty,[0,\rho_u(T)]} = 0,$$

where

$$\varphi_u^\circ(\varrho) = \sigma_0\, h_\infty(\varrho) + f(0), \forall \varrho \in [0,\rho_u(T)],$$

$$\dot{h}_\infty(\varrho) = \frac{-\sigma_0}{g(0)} h_\infty(\varrho) + \dot{\psi}_u(\varrho), \ \text{for almost all } \varrho \in [0,\rho_u(T)].$$

In this case, $h_\infty(0)$ *may be different than* x_0.
Additionally, if the input u *is wave periodic (see Definition 2), then we have*

$$\varphi_u^\circ(0) = \frac{g(0)}{e^{\frac{\sigma_0 \rho_u(T)}{g(0)}} - 1} \cdot \left(2e^{\frac{\sigma_0 \rho_u\left(T^+\right)}{g(0)}} - 1 - e^{\frac{\sigma_0 \rho_u(T)}{g(0)}}\right) + f(0), \tag{17}$$

and

$$\varphi_u^\circ(\varrho) = \begin{cases} Q_1 e^{\frac{-\sigma_0 \varrho}{g(0)}} + g(0) + Q(\varrho) & \forall \varrho \in [0,\rho_u(T^+)] \\ \\ Q_2 e^{\frac{-\sigma_0 \varrho}{g(0)}} - g(0) + Q(\varrho) & \forall \varrho \in [\rho_u(T^+),\rho_u(T)] \end{cases}$$

where

$$\begin{aligned} Q_1 &= \varphi_u^\circ(0) - g(0), \\ Q_2 &= \varphi_u^\circ(0) + g(0)\left(2e^{\frac{\sigma_0 \rho_u\left(T^+\right)}{g(0)}} - 1\right), \\ Q(\varrho) &= f(0)\left(1 - e^{\frac{-\sigma_0 \varrho}{g(0)}}\right), \forall \varrho \in [0,\rho_u(T)]. \end{aligned}$$

Example 2. Consider the LuGre model (1)-(3) with $f(\vartheta)$ taking the form $f(\vartheta) = \sigma_2 \vartheta, \forall \vartheta \in \mathbb{R}$, where the parameter σ_2 is the viscous friction coefficient. A possible choice for $g(\vartheta)$ that leads to a reasonable approximation of the Stribeck effect is [5]:

$$g(\vartheta) = F_C + (F_S - F_C)\, e^{-|\vartheta/v_s|^\alpha}, \forall \vartheta \in \mathbb{R}, \tag{18}$$

where $F_C > 0$ is the Coulomb friction force, $F_S > 0$ is the stiction force, $v_s > 0$ is the Stribeck velocity, and α is a positive constant.

Take $\sigma_0 = 4$ N/m, $v_S = 0.001$ m/s, $F_S = 3$ N, $F_C = 1$ N, $\sigma_1 = 1$ Ns/m, $\sigma_2 = 1$ Ns/m, and $x(0) = 0$ m. Let $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ be the wave periodic function of period $T = 2$ s and with $T^+ = 1$ s, such that $u(t) = t$ (in meters), $\forall t \in [0,1]$ s, and $u(t) = 2 - t$, $\forall t \in [1,2]$ s. Then ρ_u is the identity mapping and hence $I_u = \mathbb{R}_+$, $\psi_u = u$ and $v_u = \dot{u}$ a.e. Note that $T = \rho_u(T) = 2$ and $T^+ = \rho_u(T^+) = 1$. The functions u and g are plotted respectively in Fig. 1a and Fig. 1b.

Lemma 6 implies that the operator $\mathcal{H}_o$ is consistent with respect to input u and initial condition x_0; that is $\lim_{\gamma\to\infty} \left\|\varphi_{u \circ s_\gamma} - \varphi_u^*\right\|_{\infty,I_u} = 0$, where $\varphi_u^* \in W^{1,\infty}(I_u,\mathbb{R})$ is defined as

$$\varphi_u^*(\varrho) = 4e^{-\frac{4}{3}\varrho} \int_0^\varrho e^{4\tau/3} \dot{\psi}_u(\tau)\, d\tau, \forall \varrho \in I_u = \mathbb{R}_+.$$

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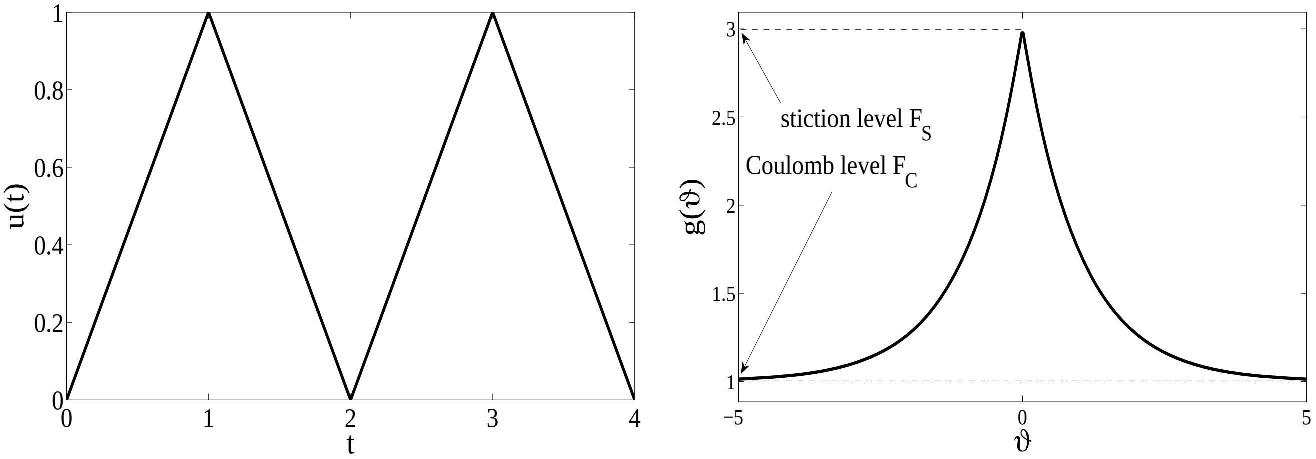
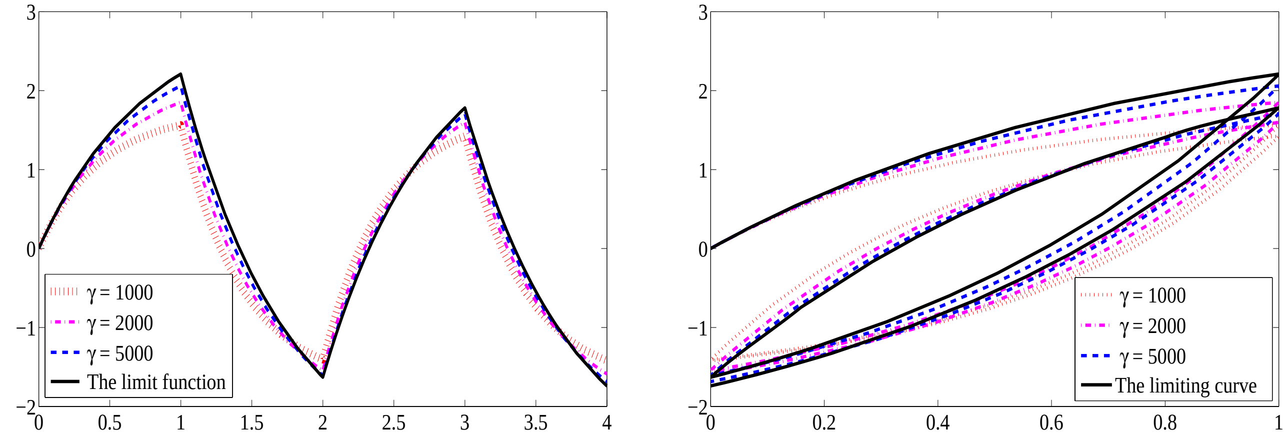
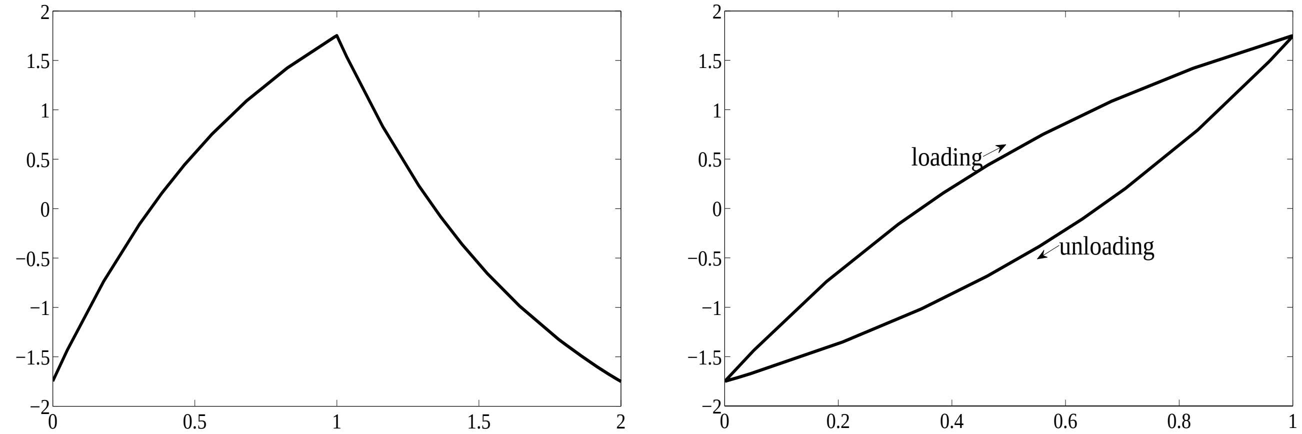
[3] F_γ is given in (9).



(a) $u(t)$ versus t. (b) $g(\vartheta)$ versus ϑ.



(c) $\varphi_{u\circ s_\gamma}(\varrho)$ versus ϱ for different values of γ. (d) $\varphi_{u\circ s_\gamma}(\varrho)$ versus $\psi_u(\varrho)$ for different values of γ.



(e) $\varphi_u^\circ(\varrho)$ versus ϱ. (f) $\varphi_u^\circ(\varrho)$ versus $\psi_u(\varrho)$ (the hysteresis loop).

Fig. 1: Simulations of Example 2.

Moreover, the operator $\mathcal{H}_o$ is strongly consistent with respect to input u and initial condition x_0; that is

$$\lim_{\gamma\to\infty} \left\| \varphi^*_{u,k} - \varphi^\circ_u \right\|_{\infty,[0,2]} = 0,$$

where $\varphi^\circ(0) = \frac{3}{e^{\frac{8}{3}}-1}\left(2e^{\frac{4}{3}} - 1 - e^{\frac{8}{3}}\right) \approx -1.7483488$, and

$$\varphi^\circ(\varrho) = \begin{cases} e^{\frac{-4\varrho}{3}}\left[\varphi^\circ(0) - 3\right] + 3 & \varrho \in [0,1] \\ e^{\frac{-4\varrho}{3}}\left[\varphi^\circ(0) + 6e^{\frac{4}{3}} - 3\right] - 3 & \varrho \in [1,2] \end{cases}$$

Fig. 1c shows the uniform convergence of $\varphi_{u\circ s_\gamma}$ to φ^*_u as $\gamma \to \infty$. Fig. 1d shows that the graphs $\left\{\left(\varphi_{u\circ s_\gamma}(\varrho), \psi_u(\varrho)\right), \varrho \in I_u = \mathbb{R}_+\right\}$ converge to the set $\left\{\left(\varphi^*_u(\varrho), \psi_u(\varrho)\right), \varrho \in I_u = \mathbb{R}_+\right\}$ as $\gamma \to \infty$. The hysteresis loop $\left\{\left(\varphi^\circ_u(\varrho), \psi_u(\varrho)\right), \varrho \in [0, \rho_u(T)] = [0,2]\right\}$ in presented in Fig. 1f. Fig. 1e shows the function $\varphi^\circ_u(\varrho)$ versus ϱ. Observe that $\varphi^\circ_u(0) \approx -1.7483488$ is different than $\varphi^*_u(0) = 0$.

V. CONCLUSION

In this paper, the LuGre model is seen as an operator $\mathcal{H}$ that associates to an input u and initial condition x_0 an output $\mathcal{H}(u, x_0)$, all belonging to some appropriate spaces. Following the research carried out in [12], the consistency and strong consistency of the operator are analyzed. The main result of the paper is given in Lemma 6. To illustrate this result, numerical simulations are carried out in Example 2.

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