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The positive solution for singular eigenvalue problem of one-dimensional p-Laplace operator

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Abstract—In this paper, by constructing a special cone and using fixed point theorem and fixed point index theorem of cone, we get the existence of positive solution for a class of singular eigenvalue value problems with p-Laplace operator, which improved and generalized the result of related paper.

Keywords—Cone, fixed point index, eigenvalue problem, p-Laplace operator, positive solutions.

I. Introduction

THE eigenvalue problems with p-Laplace operator arises in a variety of applied mathematics and physics, and they are widely applied in studying for non-newtonian fluid mechanics, cosmological physics, plasma physics, and theory of elasticity, etc. In recent years, some important results have been obtained by a variety of method(see[1-9]). In paper [10], Wang and Ge study for the following problem

$$\begin{cases} (\phi_p(u'))' + a(t)(t)f(t,u(t)) = 0, t \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

by using fixed point theorem of cone, they get the existence of multiple positive solution. Motivated by paper [4,6,10], we consider the following problems:

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda h(t)f(x(t)), \quad t \in (0,1) \\ \alpha\varphi_p(x(0)) - \beta\varphi_p(x'(0)) = 0, \\ \gamma\varphi_p(x(1)) + \delta\varphi_p(x'(1)) = 0, \end{cases} \tag{1}$$

where $\varphi_p(s) = |s|^{p-2}s, p > 1$, and λ is a positive parameter, $h(t)$ is nonnegative measurable function in $(0,1)$, $h(t)$ may be singular at $t = 0, 1$, $\alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0$ $f(x)$ is nonnegative continuous function in $[0, +\infty)$, f is sup-linear and sub-linear at 0 and ∞.

We first list the following conditions:

$(\mathbf{H_1})$ h(t) is nonnegative function in $(0,1)$, for any closed subinterval of $(0,1)$, $h(t) \neq 0$ and $0 < \int_0^1 h(t)dt < +\infty$;

$(\mathbf{H_2})$ $f \in C([0,+\infty),[0,+\infty))$ and $f(0) = 0$; for $u > 0, f(u) > 0$;

$(\mathbf{H_3})$ $\lim\limits_{x\to 0} \dfrac{f(x)}{x^{p-1}} = a$, where $a \in [0, +\infty]$;

$(\mathbf{H_4})$ $\lim\limits_{x\to +\infty} \dfrac{f(x)}{x^{p-1}} = +\infty$;(f is sup-linear at $x = +\infty$.)

$(\mathbf{H_5})$ $\lim\limits_{x\to 0} \dfrac{f(x)}{x^{p-1}} = 0$;(f is sub-linear at $x = 0$.)

$(\mathbf{H_6})$ $\lim\limits_{x\to +\infty} \dfrac{f(x)}{x^{p-1}} = 0$.($f$ is sub-linear at $x = +\infty$.)

For the sake of convenience, we list the following definitions and lemmas:

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Definition 1.1 If $x \in C[0,1]\bigcap C^1(0,1)$ and satisfy (1), $\varphi_p(x'(t))$ is absolutely continuous in $(0,1)$, $-(\varphi_p(x'(t)))' = \lambda h(t)f(x(t))$ hold almost everywhere in $(0,1)$, we call x is positive solution for problem (1).

Definition 1.2 Let E be a real Banach space, if K is a nonempty convex closed set in E, and satisfy the following conditions:

$(1) x \in K, \lambda \geq 0 \Rightarrow \lambda x \in K; (2) x \in K, -x \in K \Rightarrow x = \theta$, θ is zero element in E; we call K is a cone in E.

Let $E = C[0,1]\bigcap C^1[0,1]$, we induce the order $x < y$: for all $t \in [0,1]$, we have $x(t) < y(t)$. If we denote the norm $\|x\| = max\{\max\limits_{0\leq t\leq 1}|x(t)|, \max\limits_{0\leq t\leq 1}|x'(t)|\}$, then $(E, \|.\|)$ is a Banach space.

Let $K = \{x \in E : x(t) \geq 0, \alpha\varphi_p(x(0)) - \beta\varphi_p(x'(0)) = 0, \gamma\varphi_p(x(1)) + \delta\varphi_p(x'(1)) = 0, x$ is concave function in $[0,1]\}$, then K is a cone in E.

Lemma 1.1 For any $0 < \varepsilon < \frac{1}{2}, x \in K$ has the following properties:
(1) $x(t) \geq \|x\|t(1-t)$, $\forall t \in [0,1]$;
(2) $x(t) \geq \varepsilon^2\|x\|$,$\forall t \in [\varepsilon, 1-\varepsilon]$. (the proof is elementary.)

lemma 1.2 Suppose H_3, H_4 hold, and $a = \infty$, then there exists $R > 0$, such that$\dfrac{f(R)}{R^{p-1}} = \min\limits_{t>0}\dfrac{f(t)}{t^{p-1}}$, suppose H_5, H_6 hold, then there exists $L > 0$, such that $\dfrac{f(L)}{L^{p-1}} = \max\limits_{t>0}\dfrac{f(t)}{t^{p-1}} = C'$.

Lemma 1.3 (see[11]) Let E be Banach space, K is a cone in E, for $r > 0$, we define $K_r = \{x \in K : \|x\| \leq r\}$. Suppose $T : K_r \to K$ is completely continuous, such that $\forall u \in \partial K_r = \{x \in K : \|x\| = r\}$, we have $Tx \neq x$, If $\|x\| \leq \|Tx\|, x \in \partial K_r$, then $i(T, K_r, K) = 0$; if $\|x\| \geq \|Tx\|, x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Lemma 1.4 (see [12]) Let Ω_1, Ω_2 is a bounded open set in E, $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2, A : K\bigcap(\overline{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous. If $\|Ax\| \leq \|x\|, \forall x \in K\bigcap\partial\Omega_1; \|Ax\| \geq \|x\|, \forall x \in K\bigcap\partial\Omega_2$. or $\|Ax\| \leq \|x\|, \forall x \in K\bigcap\partial\Omega_2; \|Ax\| \geq \|x\|, \forall x \in K\bigcap\partial\Omega_1$, then A has fixed point in $K\bigcap(\overline{\Omega}_2 \setminus \Omega_1)$.

II. Conclusion

Theorem 2.1 If conditions $(H_1), (H_2), (H_3), (H_4)$ hold, and $a = +\infty$.

(a) If there exists $\lambda^* > 0$ such that $(\lambda^{*\frac{1}{p-1}} + \max\{(\dfrac{\beta}{\alpha})^{\frac{1}{p-1}}, (\dfrac{\delta}{\gamma})^{\frac{1}{p-1}}\})(\dfrac{f(\bar{R})}{\bar{R}^{p-1}})^{\frac{1}{p-1}}\psi_p(\int_0^1 h(t)dt) \leq 1$

where $\psi_p(t) = |t|^{\frac{1}{p-1}} sgn(t)$ is converse function of φ_p, $\bar{R} \in (0,R]$ is the maximum point of f in $(0,R]$, then for $0 < \lambda < \lambda^*$, Problem (1) has two positive solutions $x_1(t), x_2(t)$, and satisfy $0 < \|x_1\| < R < \|x_2\|$.

(b) There exists λ^{**}, when $\lambda > \lambda^{**}$, the problem (1) has no positive solution.

(a) Proof For any $x \in K$, we have $x'(0) \geq 0, x'(1) \leq 0$, so there exists a constant $\sigma(=\sigma_x)$ such that $x'(\sigma) = 0$, we define $T_\lambda : K \to E$ as follow

$$(T_\lambda x)(t) = \begin{cases} \psi_p(\frac{\beta}{\alpha}\int_0^\sigma \lambda h(r)f(u(r))dr) + \\ \int_0^t \psi_p(\int_s^\sigma \lambda h(r)f(u(r))dr)ds, 0 \leq t \leq \sigma, \\ \psi_p(\frac{\delta}{\gamma}\int_\sigma^1 \lambda h(r)f(u(r))dr) + \\ \int_t^1 \psi_p(\int_\sigma^s \lambda h(r)f(u(r))dr)ds, \sigma \leq t \leq 1, \end{cases}$$

By the definition of T_λ, we know $\forall x \in K, T_\lambda x \in C^1[0,1]$ is nonnegative and satisfy the boundary condition, furthermore,

$$(T_\lambda x)'(t) = \begin{cases} \psi_p \int_t^\sigma \lambda h(r)f(u(r))dr) > 0, \quad 0 \leq t \leq \sigma, \\ -\psi_p(\int_\sigma^t \lambda h(r)f(u(r))dr) < 0, \quad \sigma \leq t \leq 1, \end{cases}$$

is continuous and non-increasing in $[0,1]$, and $(T_\lambda x)'(\sigma) = 0$, so $(T_\lambda x)(\sigma)$ is the maximum value of $T_\lambda x$ in $[0,1]$. Since $(T_\lambda x)'$ is continuous and non-increasing in $[0,1]$, we have $T_\lambda x \in K$, this imply $T_\lambda K \subset K$, furthermore, $-(\varphi_p(T_\lambda x'(t)))' = \lambda h(t)f(x(t))$, so the fixed point of T_λ in K is solution for problem (1).

Similar to the method of [4,5], we know $T_\lambda : K \to K$ is completely continuous.

By (H_1), $\forall \varepsilon > 0$, we have $0 < \int_\varepsilon^{1-\varepsilon} h(t)dt < +\infty$, and when $\varepsilon \leq x \leq 1-\varepsilon$, $y(x) = \int_\varepsilon^x \psi_p(\int_s^x h(r)dr)ds + \int_x^{1-\varepsilon} \psi_p(\int_x^s h(r)dr)ds$ is nonnegative continuous.

Let $P = \min_{\varepsilon \leq x \leq 1-\varepsilon} y(x) > 0$, by (H_3) and $a = \infty$, i.e. $\lim_{x \to 0} \frac{f(x)}{x^{p-1}} = \infty$, we know there exists $0 < r' < R$, such that when $0 \leq x \leq r'$, $f(x) \geq (Mx)^{p-1}$, where $M > 2(\lambda^{\frac{1}{p-1}}\varepsilon^2 P)$, for $x \in \partial K_{r'} = \{x \in K : \|x\| = r'\}$, we have

$$\begin{aligned} 2\|T_\lambda x\| &\geq \int_\varepsilon^\sigma \psi_p(\int_s^\sigma \lambda h(r)f(u(r))dr)ds + \\ &\quad \int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s \lambda h(r)f(u(r))dr)ds \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 r'(\int_\varepsilon^\sigma \psi_p(\int_s^\sigma h(r)dr)ds + \\ &\quad \int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s h(r)dr)ds) \\ &= \lambda^{\frac{1}{p-1}} M\varepsilon^2 r' y(\sigma) \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 r' P \\ &\geq 2r' = 2\|x\|, \quad \sigma \in [\varepsilon, 1-\varepsilon] \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} \lambda h(r)f(u(r))dr)ds \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 r'(\int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} h(r)dr)ds) \\ &= \lambda^{\frac{1}{p-1}} M\varepsilon^2 r' y(1-\varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 r' P \\ &> 2r' > r' = \|x\|, \quad \sigma > 1-\varepsilon, \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s \lambda h(r)f(u(r))dr)ds \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 r' y(\varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 r' P \\ &> 2r' > r' = \|x\|, \quad \sigma < \varepsilon, \end{aligned}$$

so for $x \in \partial K_{r'}$, we have $\|T_\lambda x\| \geq \|x\|$, by lemma 1.3,

$$i(T_\lambda, K_{r'}, K) = 0. \tag{2}$$

By (H_4) $\lim_{x \to +\infty} \frac{f(x)}{x^{p-1}} = +\infty$, there exists $R_1 > 0, \forall x \geq R_1$, we have $f(x) \geq (Mx)^{p-1}$, take $\tilde{R} > max\{R, R_1\}$, for $x \in \partial\tilde{R}, \|x\| = \tilde{R}$, by lemma 1.1, we have

$$\begin{aligned} 2\|T_\lambda x\| &\geq \int_\varepsilon^\sigma \psi_p(\int_s^\sigma \lambda h(r)f(u(r))dr)ds + \\ &\quad \int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s \lambda h(r)f(u(r))dr)ds \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 \tilde{R}(\int_\varepsilon^\sigma \psi_p(\int_s^\sigma h(r)dr)ds + \\ &\quad \int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s h(r)dr)ds) \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 \tilde{R} y(\sigma) \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 \tilde{R} P \\ &\geq 2r = 2\|x\|, \quad \sigma \in [\varepsilon, 1-\varepsilon], \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} \lambda h(r)f(u(r))dr)ds \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 \tilde{R}(\int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} h(r)dr)ds) \\ &= \lambda^{\frac{1}{p-1}} M\varepsilon^2 \tilde{R} y(1-\varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 r P \\ &> 2\tilde{R} > \tilde{R} = \|x\|, \quad \sigma > 1-\varepsilon, \end{aligned}$$

$$\begin{aligned} \|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s \lambda h(r)f(u(r))dr)ds \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 \tilde{R} y(\varepsilon) \\ &\geq \lambda^{\frac{1}{p-1}} M\varepsilon^2 \tilde{R} P \\ &> 2\tilde{R} > \tilde{R} = \|x\|, \quad \sigma < \varepsilon, \end{aligned}$$

so for $x \in \partial K_{\tilde{R}}$, we have $\|T_\lambda x\| \geq \|x\|$, by lemma 1.3,

$$i(T_\lambda, K_{\tilde{R}}, K) = 0. \tag{3}$$

On the other hand, for $x \in \partial K_R$, we have

$$\begin{aligned}\|T_\lambda x\| &\le \psi_p(\int_0^1 \lambda h(r)f(u(r))dr)ds+\\ &\quad max\{\psi_p(\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr),\\ &\quad \psi_p(\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr)\}\\ &\le \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad \psi_p(\int_0^1 h(r)f(\bar{R})dr)\\ &= \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad \psi_p(\tfrac{f(\bar{R})}{\varphi_p(\bar{R})}\varphi_p(\bar{R})\textstyle\int_0^1 h(r)dr)\\ &= \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad (\tfrac{f(\bar{R})}{\bar{R}^{p-1}})^{\frac{1}{p-1}}\psi_p(\textstyle\int_0^1 h(r)dr)\bar{R}\\ &< \bar{R} \le R = \|x\|,\end{aligned}$$

by lemma 1.3,

$$i(T_\lambda, K_R, K) = 1. \tag{4}$$

by (2),(3),(4) and the additivity of fixed point index $i(T_\lambda, K_{\bar{R}} \setminus \dot{K}_R) = -1, i(T_\lambda, K_R \setminus \dot{K}_r) = 1$.

So T_λ has fixed point x_1 in $K_{\bar{R}} \setminus \dot{K}_R$ and x_2 in $K_R \setminus \dot{K}_r$.

Next we show $x_1 \neq x_2$, we only need to show when $x_i \in \partial K_R, i = 1, 2$, $T_\lambda x_i \neq x_i$ hold.

If it is not true, when $x_i \in \partial K_R, i = 1, 2$, $T_\lambda x_i = x_i$, so $\|T_\lambda x_i\| = \|x_i\|$. Since x_i satisfy (1), we have

$$\begin{aligned}\|T_\lambda x_i\| &\le \psi_p(\int_0^1 \lambda h(r)f(u(r))dr)ds+\\ &\quad max\{\psi_p(\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr),\\ &\quad \psi_p(\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr)\}\\ &\le \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad \psi_p(\int_0^1 h(r)f(\bar{R})dr)\\ &= \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad \psi_p(\tfrac{f(\bar{R})}{\varphi_p(\bar{R})}\varphi_p(\bar{R})\textstyle\int_0^1 h(r)dr)\\ &= \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad (\tfrac{f(\bar{R})}{R^{p-1}})^{\frac{1}{p-1}}\psi_p(\textstyle\int_0^1 h(r)dr)\bar{R}\\ &\le \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad (\tfrac{f(\bar{R})}{R^{p-1}})^{\frac{1}{p-1}}\psi_p(\textstyle\int_0^1 h(r)dr)\|x_i\|\end{aligned}$$

this imply

$$\begin{aligned}1 &\le \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad (\tfrac{f(\bar{R})}{R^{p-1}})^{\frac{1}{p-1}}\psi_p(\textstyle\int_0^1 h(r)dr).\end{aligned} \tag{5}$$

this is a contradiction, so $x_1 \neq x_2$.

Last, obviously $0 < \|x_1\| < R < \|x_2\|$.

(b) Proof Suppose there exists a subsequence $\{\lambda_n\}$, and $\lambda_n > n$ such that for any n, problem (1) has a positive solution $x_n \in K$, by H_3, $\forall x > 0$, we have $f(x) \ge \bar{C}x^{p-1}$, where $\bar{C} = \frac{f(R)}{R^{p-1}}$, when $\sigma < \varepsilon$, by lemma 1.1, we have

$$\begin{aligned}\|x_n\| &\ge \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s \lambda_n h(r)f(u(r))dr)ds\\ &\ge \lambda_n^{\frac{1}{p-1}}\int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s h(r)\bar{C}(u_n)^{\frac{1}{p-1}}dr)ds\\ &\ge (\lambda_n\bar{C})^{\frac{1}{p-1}}\varepsilon^2\|x_n\|\int_\varepsilon^{1-\varepsilon}\psi_p(\int_\varepsilon^s h(r)dr)ds\\ &= (\lambda_n\bar{C})^{\frac{1}{p-1}}\varepsilon^2\|x_n\|y(\varepsilon)\\ &\ge (\lambda_n\bar{C})^{\frac{1}{p-1}}\varepsilon^2\|x_n\|P,\end{aligned}$$

so

$$1 \ge n\bar{C}\varepsilon^{2(p-1)}P^{p-1}. \tag{6}$$

Since n is sufficient large, so we get a contradiction.

When $\sigma > 1-\varepsilon$ and $\sigma \in [\varepsilon, 1-\varepsilon]$, we can get the similar result.

So there exists λ^{**}, when $\lambda > \lambda^{**}$, problem (1) has no positive solution, the proof is finished.

Theorem 2.2 If $(H_1),(H_3),(H_4)$ hold , and $0 < a < +\infty$, if there exists $\lambda^{***} > 0$ and $\lambda^{***\frac{1}{p-1}}(1+\max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\psi_p(a\int_0^1 h(t)dt) \le 1$, where $\psi_p(t) = |t|^{\frac{1}{p-1}}sgn(t)$ is converse function of φ_p, so for $0 < \lambda < \lambda^{***}$, problem (1) has a positive solution.

Proof Take $\epsilon > 0$, such that $(\lambda^{\frac{1}{p-1}} + max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})(a+\epsilon)^{\frac{1}{p-1}}\psi_p(\int_0^1 h(t)dt) < 1$. by H_3, there exists $\eta > 0$ such that when $0 \le x \le \eta$, $f(x) \le x^{p-1}(a+\epsilon)$. so for $x \in \partial K_\eta$, we have

$$\begin{aligned}\|T_\lambda x\| &\le \psi_p(\int_0^1 \lambda h(r)f(u(r))dr)ds+\\ &\quad max\{\psi_p(\frac{\beta}{\alpha}\int_0^1 \lambda h(r)f(u(r))dr),\\ &\quad \psi_p(\frac{\delta}{\gamma}\int_0^1 \lambda h(r)f(u(r))dr)\}\\ &\le \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad \psi_p(\int_0^1 h(r)x^{p-1}(r)(a+\epsilon)dr)\\ &\le \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad (a+\epsilon)^{\frac{1}{p-1}}\psi_p(\textstyle\int_0^1 h(r)dr)\|x\|\\ &\le \lambda^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times\\ &\quad (a+\epsilon)^{\frac{1}{p-1}}\psi_p(\textstyle\int_0^1 h(r)dr)\eta\\ &< \eta = \|x\|.\end{aligned}$$

By H_4, there exists $\varrho > 0$, such that when $x \ge \varrho, f(x) \ge (Mx)^{p-1}$, choose $\mu > max\{\varrho, \eta\}$, by the similar method with theorem 2.1, we can show when $x \in \partial K_\mu$, $\|T_\lambda x\| \ge \|x\|$, so if we define

$\Omega_1 = \{x \in K : \|x\| < \eta\}$, $\Omega_2 = \{x \in K : \|x\| < \mu\}$, by lemma 1.4, T_λ has at least one fixed point $x \in K$, and $\mu > \|x\| > \eta$, the proof is finished.

Corollary In condition H_3, let $a = 0$, then $\forall\lambda > 0$, problem (1) has at least one positive solution.

Theorem 2.3 If H_1, H_2, H_5, H_6 hold, then
(a)$\forall\varepsilon \in (0, \frac{1}{2})$, there exists $\lambda_* = \lambda_*(\varepsilon) > 0$, such that for all $\lambda > \lambda_*$, problem (1) has at least two x_1, x_2 and $0 < \|x_1\| < L < \|x_2\|$.

(b) If there exist $\lambda_{**} > 0$ such that $\lambda_{**}{}^{\frac{1}{p-1}}(1 + \max\{(\dfrac{\beta}{\alpha})^{\frac{1}{p-1}}, (\dfrac{\delta}{\gamma})^{\frac{1}{p-1}}\})(C'^{\frac{1}{p-1}})\psi_p\displaystyle\int_0^1 h(t)dt) \leq 1$, then for all $\lambda < \lambda_{**}$, problem (1) has no positive solution, where $C' = \frac{f(L)}{L^{p-1}}$.

(a) Proof For any $0 < \varepsilon < \frac{1}{2}, \forall x \in K$ and $\|x\| = L$, Let $v = \min\limits_{\varepsilon \leq t \leq 1-\varepsilon} \dfrac{f(u(t))}{u(t)^{p-1}}$, by (H_2) and lemma 1.1, $v > 0$, let $\lambda_* = \dfrac{2^{p-1}}{(\varepsilon^2 Q)^{p-1} v}$, where $Q = \min\limits_{\varepsilon \leq x \leq 1-\varepsilon} y(x) > 0$, then for $\lambda > \lambda_*$ we have

$$\begin{aligned}
2\|T_\lambda x\| &\geq \int_\varepsilon^\sigma \psi_p(\int_s^\sigma \lambda h(r) f(u(r))dr)ds + \\
&\int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s \lambda h(r) f(u(r))dr)ds \\
&\geq (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 (\int_\varepsilon^\sigma \psi_p(\int_s^\sigma h(r)dr)ds + \\
&\int_\sigma^{1-\varepsilon} \psi_p(\int_\sigma^s h(r)dr)ds) \\
&= (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 Q L \\
&> 2L = 2\|x\|, \sigma \in [\varepsilon, 1-\varepsilon],
\end{aligned}$$

$$\begin{aligned}
\|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_s^{1-\varepsilon} \lambda h(r) f(u(r))dr)ds \\
&\geq (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 Q L \\
&> \|x\|, \sigma > 1-\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\|T_\lambda x\| &\geq \int_\varepsilon^{1-\varepsilon} \psi_p(\int_\varepsilon^s \lambda h(r) f(u(r))dr)ds \\
&\geq (\lambda v)^{\frac{1}{p-1}} \varepsilon^2 Q L \\
&\geq \|x\|, \sigma < \varepsilon,
\end{aligned}$$

so for $x \in \partial K_L$, we have $\|T_\lambda x\| > \|x\|$.

For the same λ, choose $\epsilon' > 0$ such that $\epsilon'(\lambda^{\frac{1}{p-1}} + max\{(\dfrac{\beta}{\alpha})^{\frac{1}{p-1}}, (\dfrac{\delta}{\gamma})^{\frac{1}{p-1}}\})\psi_p(\displaystyle\int_0^1 h(r)dr) < 1$, by (H_5), there exists $0 < l < L$, such that when $0 \leq x \leq l$, $f(x) \leq (\epsilon' x)^{p-1}$, so for $x \in \partial K_l$, we have

$$\begin{aligned}
\|T_\lambda x\| &\leq \psi_p(\int_0^1 \lambda h(r) f(u(r))dr)ds + \\
&\quad max\{\psi_p(\frac{\beta}{\alpha}\int_0^1 \lambda h(r) f(u(r))dr), \\
&\quad \psi_p(\frac{\delta}{\gamma}\int_0^1 \lambda h(r) f(u(r))dr)\} \\
&\leq \lambda^{\frac{1}{p-1}}(1 + max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\
&\quad \psi_p(\int_0^1 h(r) f(x(r))dr) \\
&\leq \lambda^{\frac{1}{p-1}}(1 + max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\
&\quad \psi_p \textstyle\int_0^1 (\epsilon' x)^{p-1} h(r)dr) \\
&\leq \lambda^{\frac{1}{p-1}}(1 + max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\
&\quad \epsilon' \psi_p(\textstyle\int_0^1 h(r)dr) l \\
&< l = \|x\|.
\end{aligned}$$

We define a new function $\bar{f}(x) = \max\limits_{0 \leq s \leq x} f(s)$, so $\bar{f}(x)$ is nondecreasing monotonously, by (H_6) $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x^{p-1}} = 0$, we can get $\lim\limits_{x \to +\infty} \dfrac{\bar{f}(x)}{x^{p-1}} = 0$, for the same $\epsilon' > 0$, there exists $S > 0$ such that when $x \leq S$, $\bar{f}(x) \leq (\epsilon' x)^{p-1}$, choose $L' = max\{L, S\}$, so for $x \in K_{L'}$, we have

$$\begin{aligned}
\|T_\lambda x\| &\leq \psi_p(\int_0^1 \lambda h(r) f(u(r))dr)ds + \\
&\quad max\{\psi_p(\frac{\beta}{\alpha}\int_0^1 \lambda h(r) f(u(r))dr), \\
&\quad \psi_p(\frac{\delta}{\gamma}\int_0^1 \lambda h(r) f(u(r))dr)\} \\
&\leq \lambda^{\frac{1}{p-1}}(1 + max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\
&\quad \psi_p(\int_0^1 h(r)\bar{f}(L')dr) \\
&\leq \lambda^{\frac{1}{p-1}}(1 + max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}}, (\frac{\delta}{\gamma})^{\frac{1}{p-1}}\}) \times \\
&\quad \epsilon' \psi_p \textstyle\int_0^1 h(r)dr) L' \\
&< L' = \|x\|.
\end{aligned}$$

we define $\Omega_1 = \{x \in K : \|x\| < L\}$,$\Omega_2 = \{x \in K : \|x\| < L'\}$, by lemma 1.4, T_λ has at least two fixed points $x_1(t), x_2(t)$ in K, and satisfy $l \leq \|x_1\| \leq L \leq \|x_2\| \leq L'$.

Similarly to the proof of theorem 2.1, T_λ has no fixed point in ∂K_L, so $x_1(t) \neq x_2(t)$, the proof is finished.

(b) Proof Suppose there exists a subsequence $\lambda_n < \lambda_{**}$ and $\lambda_n \in (0, \frac{1}{n})$ such that for $\forall n$ problem (1) has a positive solution $x_n \in K$. since $x > 0$, $f(x) \leq (C'x)^{p-1}$, where

$C' = \frac{f(L)}{L^{p-1}}$, we have

$$\begin{aligned}\|x_{\lambda_n}\| &\leq \psi_p(\int_0^1 \lambda_n h(r) f(x_{\lambda_n}(r))dr)ds+ \\ & max\{\psi_p(\frac{\beta}{\alpha}\int_0^1 \lambda h(r) f(x_{\lambda_n}(r))dr), \\ & \psi_p(\frac{\delta}{\gamma}\int_0^1 \lambda h(r) f(x_{\lambda_n}(r))dr)\} \\ &\leq \lambda_n^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times \\ & \psi_p(\int_0^1 h(r) C' x_{\lambda_n}^{p-1}(r)dr) \\ &\leq \lambda_n^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})\times \\ & C'^{\frac{1}{p-1}}\psi_p(\int_0^1 h(r)dr)\|x_{\lambda_n}\|,\end{aligned}$$

i.e.

$$1 \leq \lambda_n^{\frac{1}{p-1}}(1+max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})C'^{\frac{1}{p-1}}\psi_p(\int_0^1 h(r)dr). \tag{7}$$

since $\lambda_n < \lambda_{**}$, so $\lambda_n^{\frac{1}{p-1}}(1 + max\{(\frac{\beta}{\alpha})^{\frac{1}{p-1}},(\frac{\delta}{\gamma})^{\frac{1}{p-1}}\})C'^{\frac{1}{p-1}}\psi_p(\int_0^1 h(r)dr) < 1$, this is contradiction, the proof is finished.

Example 1

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda(1-t)^{p_1}t^{p_2}(cx^{q_1}(t)+x^{q_2}(t)), & t\in(0,1) \\ x(0)=x(1)=0, \end{cases}$$

where λ is a positive parameter, $c \in R^+ \bigcup\{0\}, -1 < p_1 < 0, -1 < p_2 < 0, 0 < q_1 \leq p-1 < q_2$.

we cinsider the following two cases:

(1) when $0 < q_1 < p-1 < q_2$ and $c > 0$, $R = \bar{R} = (c\frac{p-1-q_1}{q_2-p+1})^{\frac{1}{q_2-q_1}}$, and $\lambda^* = \frac{((q_2-p+1)^{q_2-p+1}(p-1-q_1)^{p-1-q_1})^{\frac{1}{q_2-q_1}}}{q_2-q_1} \times (\beta(p_1+1,p_2+1))^{-1} \times c^{\frac{q_2-p+1}{q_1-q_2}}$.

By theorem 2.1, if $\lambda \in (0,\lambda^*)$, then problem (1) has at least two positive solutions x_1, x_2 satisfy $0 < \|x_1\| < R < \|x_2\|$, there exists λ^{**} sufficient large, when $\lambda > \lambda^{**}$, problem (1) has no positive solution.

(2) $q_1 = p-1$, and $c \geq 0$, if $c > 0$, then $h(t) = (1-t)^{p_1}t^{p_2}$, and $f(x) = cx^{q_1}(t)+x^{q_2}(t)$ satisfy all the conditions of theorem 2, and $\beta = 0, \delta = 0$. Let $\lambda^{***} = (c\beta(p_1+1,p_2+1))^{-1}$, where β is β function, for $0 < \lambda < \lambda^{***}$, problem (1) has at least one positive solution.

If $c = 0$, by corollary, for each $\lambda > 0$, problem (1) has at least one positive solution.

Example 2

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda h(t)(e^{x(t)}-1), & t\in(0,1) \\ x(0)=x(1)=0 \end{cases}$$

where λ is positive parameter, $h(t)$ is same as above, we consider three cases:

case 1 $p > 2$. let $\lambda^* = (\frac{f(\bar{R})}{\bar{R}^{p-1}}\beta(p_1+1,p_2+1))^{-1}$, where $R = \bar{R} \in (p-2,p-1)$ is the only zero point of function $\chi(x) = e^x(x-p+1)+p-1$, by theorem 2.1, when $\lambda \in (0,\lambda^*)$, problem (1) has at least two solutions, and $0 < \|x_1\| < R < \|x_2\|$. there exists λ^{**} sufficient large, when $\lambda > \lambda^{**}$, problem (1) has no solution.

case 2 $p = 2$, let $\lambda^{***} = (\beta(p_1+1,p_2+1))^{-1}$, where β is β function. by theorem 2.2, for $0 < \lambda < \lambda^{***}$ problem (1) has at least one positive solution.

case 3 $1 < p < 2$, in this case $a = 0$, by corollary, for each $\lambda > 0$, problem (1) has at least one positive solution.

Example 3

$$\begin{cases} -(\varphi_p(x'(t)))' = \lambda t^{-\alpha}x(t)^q e^{-x(t)}, & t\in(0,1), \\ x(0)=x(1)=0 \end{cases}$$

where $0 < \alpha < 1, p-1 < q$.

By theorem 2.3, for $\varepsilon \in (0,\frac{1}{2})$, let $\lambda_*(\varepsilon) = (\frac{2}{v\varepsilon^2 Q})^{p-1}$, for $0 < \lambda > \lambda_*$, problem (1) has at least two positive solutions, and $0 < \|x_1\| < q-p+1 < \|x_2\|$. there exists λ_{**} sufficient small, when $\lambda < \lambda_{**}$ problem (1) has no solution. Specially, $p = 2$, let $\lambda_*(\varepsilon) = (\frac{2}{v\varepsilon^2 Q})^{p-1}, v = \varepsilon^{2(q-1)}(q-1)e^{\varepsilon^2(1-q)}$, and $Q = \frac{\varepsilon^{2-\alpha}+(1-\varepsilon)^{2-\alpha}-2^{\alpha-1}}{(1-\alpha)(2-\alpha)}$, we can get for $0 < \lambda > \lambda_*$, problem (1) has at least two positive solutions, and $0 < \|x_1\| < q-1 < \|x_2\|$.

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