The symmetric solutions for boundary value problems of second-order singular differential equation

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Abstract—In this paper, by constructing a special operator and using fixed point index theorem of cone, we get the sufficient conditions for symmetric positive solution of a class of nonlinear singular boundary value problems with p-Laplace operator, which improved and generalized the result of related paper.

Keywords—Banach space, cone, fixed point index, singular differential equation, p-Laplace operator, symmetric solutions.

I. INTRODUCTION

The boundary value problems with p-Laplace operator arises in a variety of applied mathematics and physics, and they are widely applied in studying for non-newtonian fluid mechanics, cosmological physics, plasma physics, and theory of elasticity, etc. In recent years, some important results have been obtained by a variety of method(see[1-4]). On the other hand, the study for the symmetric and multiple solutions to this problem is more and more active (see[5-6]). In paper [5], Sun study for the problem

\[
\begin{aligned}
& (u'' + a(t)f(t, u(t))) = 0, \quad t \in (0, 1) \\
& u(0) = cu(\eta) = u(1),
\end{aligned}
\]

where \( \alpha \in (0, 1), \eta \in (0, \frac{1}{2}) \), by using spectrum theory, Sun get the existence of symmetric and multiple solution. But when \( p \neq 2 \), \( \phi_p(u) \) is nonlinear, so the method of the paper [5] is not suitable to p-Laplace operator. In paper [6], Tian and Liu study for the problem

\[
\begin{aligned}
& (\phi_p(u'))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1) \\
& u(0) = \alpha u(\eta) = u(1),
\end{aligned}
\]

where \( \phi(s) \) is p-Laplace operator. Motivated by paper [5,6], we consider the existence of solution for the following problems:

\[
\begin{aligned}
& (\phi_p(u'))' + h_1(t)f(u, v) = 0, \\
& (\phi_p(v'))' + h_2(t)g(u, v) = 0, \\
& u(0) = \gamma u(\eta) = u(1), \\
& v(0) = \gamma v(\eta) = v(1),
\end{aligned}
\]

where \( t \in (0, 1), \gamma \in (0, 1), \eta \in (0, \frac{1}{2}) \), \( \phi(s) \) is a p-Laplace operator, i.e. \( \phi_p(s) = |s|^{p-2}s, p > 1 \). Obviously, if \( \frac{p}{2} + \frac{1}{q} = 1 \), then \( (\phi_p)^{-1} = \phi_q \).

Compare with above paper, our method is different. By constructing a new operator, and using fixed point index theorem, we get the sufficient condition of the existence of symmetric solution, which improved and generalized the result of paper [5,6,7].

In this paper, we always suppose that the following conditions hold:

\[
\begin{aligned}
& (H_1) \quad f \in C([0, +\infty) \times [0, +\infty), [0, +\infty)), \quad g \in C([0, +\infty), [0, +\infty)). \\
& (H_2) \quad h_1 \in C([0, 1], [0, +\infty)), \quad h_1(t) = h_2(1 - t), t \in (0, 1), \text{ for any subinterval of } (0, 1), \quad h_1(t) \neq 0, \quad \text{and} \\
& \int_0^1 h_1(t)dt < +\infty (i = 1, 2). \\
& (H_3) \quad \text{There exists } \alpha \in (0, 1), \text{ such that } \lim_{u \to +\infty} \frac{g(u)}{u^\alpha} = +\infty \text{ and } \lim_{u \to +\infty} \frac{f(u, v)}{u^\alpha} \text{ hold uniformly to } u \in R^+. \\
& (H_4) \quad \text{There exists } \beta \in (0, +\infty), \text{ such that } \lim_{u \to 0^+} \frac{g(u)}{u^\beta} = 0 \text{ and } \lim_{v \to 0^+} \frac{f(u, v)}{v^\beta} < +\infty \text{ hold uniformly to } u \in R^+. \\
& (H_5) \quad \text{There exists } n \in (0, 1), \text{ such that } \lim_{u \to 0^+} \frac{g(u)}{u^\beta} = +\infty \text{ and } \lim_{v \to 0^+} \frac{f(u, v)}{v^\beta} \text{ hold uniformly to } u \in R^+. \\
& (H_6) \quad f(u, v) \text{ and } g(u) \text{ are nondecreasing with respect to } u \text{ and } v, \text{ and there exists } R > 0, \text{ such that} \\
& \frac{\gamma}{\gamma - 1} \int_0^\frac{1}{\gamma} \phi_p(k_1(s))dsf(R, \gamma - 1) \int_0^\frac{1}{\gamma} \phi_q(k_1(s))ds \times g(R) < R, \text{ where } k_1(s) = \int_0^s h_1(\tau)d\tau, \gamma \in (1, 2). \\
\end{aligned}
\]

For convenience, we list the following definitions and lemmas:

Definition 1.1 If \( u(t) = u(1 - t), t \in [0, 1] \), we call \( u(t) \) is symmetric in \([0, 1]\).

Definition 1.2 If \( (u, v) \) is a positive solution of problem (1), and \( u, v \) is symmetric in \([0, 1]\), we call \( (u, v) \) is symmetric positive solution of problem (1).

Definition 1.3 If \( u(\lambda t_1 + (1 - \lambda) t_2) \geq \lambda u(t_1) + (1 - \lambda) u(t_2) \), we call \( u(t) \) is concave in \([0, 1]\).

Let \( E = C[0, 1], \text{ define the norm } ||u|| = \max_{t \in [0, 1]} |u(t)|, \text{ obviously } (E, ||||) \text{ is a Banach space.} \)

Let \( K = \{u \in E | u(t) > 0, u(t) \text{ is a symmetric concave function, } t \in [0, 1]\}, \text{ then } K \text{ is a cone in } E. \text{ By } (H_1), (H_2), \text{ the solution of problem (1) is equivalent to the solution of system of equation (2).} \)
We define $$T, t \in [0, 1]$$, by
\[
\begin{align*}
(Tu)(t) = & \begin{cases}
\int_0^t \phi_q \left( \int_0^{s/2} h_1(\tau) f(u(\tau), v(\tau))d\tau \right) ds + \\
\int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds, \quad 0 \leq t \leq \frac{1}{2} \\
\int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds, \quad \frac{1}{2} \leq t \leq 1.
\end{cases}
\end{align*}
\]
where
\[
\begin{align*}
v(t) = & \begin{cases}
\int_0^t \phi_q \left( \int_0^{s/2} h_2(\tau)g(u(\tau))d\tau \right) ds + \\
\int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds, \quad 0 \leq t \leq \frac{1}{2} \\
\int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds, \quad \frac{1}{2} \leq t \leq 1.
\end{cases}
\end{align*}
\]
Obviously $$Tu \in E$$, it is easy to show if $$T$$ has fixed point $$u$$, then by (4), problem (1) has a solution $$(u, v)$$.

**Lemma 1.1** Let $$(H_1), (H_2)$$, then $$T : K \to K$$ is completely continuous. 

**Proof** \(\forall u \in K\), by $$(H_1), (H_2)$$, we can get $$(Tu)(t) \geq 0, t \in [0, 1]$$.

\[
v'(t) = \begin{cases}
\phi_q \left( \int_0^{s/2} h_2(\tau)g(u(\tau))d\tau \right), 0 \leq t \leq \frac{1}{2} \\
-\phi_q \left( \int_0^{s/2} h_2(\tau)g(u(\tau))d\tau \right), \quad \frac{1}{2} \leq t \leq 1,
\end{cases}
\]
correspondingly $$(\phi_q(v'))' = -h_2(t)g(u) \leq 0, 0 < t < 1$$, so $$v$$ is concave in $$[0, 1]$$.

Next we show $$v$$ is symmetric in $$[0, 1]$$.

When $$t \in [0, \frac{1}{2}], 1 - t \in [\frac{1}{2}, 1]$$, so
\[
v(1 - t) = \int_0^t \phi_q \left( \int_0^{s/2} h_2(\tau)g(u(\tau))d\tau \right) ds + \\
\int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds = v(t).
\]

Similarly, we have $$v(1 - t) = v(t), t \in [\frac{1}{2}, 1]$$. So $$v$$ is a symmetric concave function in $$[0, 1]$$.

Next we show $$Tu$$ is symmetric in $$[0, 1]$$, when $$t \in [0, \frac{1}{2}], 1 - t \in [\frac{1}{2}, 1]$$, so
\[
(Tu)(1 - t) = \int_0^t \phi_q \left( \int_0^{s/2} h_2(\tau)g(u(\tau))d\tau \right) ds + \\
\int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds = (Tu)(t).
\]

Similarly, we have $$(Tu)(1 - t) = (Tu)(t), t \in [\frac{1}{2}, 1]$$, so $$Tu$$ is concave in $$[0, 1]$$, so $$TK \subset K$$. On the other hand, let $$D$$ be a arbitrary bounded set of $$K$$, then there exist constant $$c > 0$$, such that $$D \subset \{u \in K||u|| \leq c\}$$. Let $$b = \max_{u \in [a, c]} g(u)$$, so $$\forall u \in D$$, we have

\[
||v|| = \int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds + \\
\int_0^{s/2} \phi_q \left( \int_0^t h_2(\tau)g(u(\tau))d\tau \right) ds \leq c.
\]

Let $$L = \max_{u \in [a, c]} f(u, v)$$, so $$\forall u \in D$$, we have

\[
||Tu|| = \int_0^{s/2} \phi_q \left( \int_0^t h_1(\tau)f(\tau), v(\tau))d\tau \right) ds + \\
\int_0^{s/2} \phi_q \left( \int_0^t h_1(\tau)f(\tau), v(\tau))d\tau \right) ds \leq c.
\]
\[
\left\| (Tu) \right\| = \max \left\{ |\phi_q(\int_{0}^{\frac{1}{2}} h_1(\tau) f(u(\tau), v(\tau))d\tau)|, \right. \\
\left. |\phi_q(\int_{\frac{1}{2}}^{1} h_1(\tau) f(u(\tau), v(\tau))d\tau)| \right\} \\
\leq L^{{r-1}} \phi_q(\int_{0}^{1} h_1(\tau)d\tau).
\]

By Arzela-Ascoli theorem, we know \( TD \) is compact set. By Lebesgue dominated convergence theorem, it is easy to show \( T \) is continuous in \( K \), so \( T : K \rightarrow K \) is completely continuous.

**Lemma 1.2** For any \( 0 < \varepsilon < \frac{1}{4}, u \in K \), we have

1. \( u(t) \geq \|u\|(1 - t), \forall t \in [0, 1] \),
2. \( u(t) \geq \varepsilon^2 \|u\|, \forall t \in [\varepsilon, 1 - \varepsilon] \).

(1) If \( \|u\| \geq x, \forall x \in K \),
(2) \( \|Tu\| \geq \|u\|, \forall x \in K \). By Lemma 1.3, \( K \) is a cone of \( E \) in Banach space, \( \Omega_1 \) and \( \Omega_2 \) are open subsets in \( E \), \( \theta \in \Omega_1, \Omega_2 \subset \Omega_2 \), and satisfy one of the following conditions:

**Lemma 1.3** (see [8]) Let \( K \) be a cone of \( E \) in Banach space, \( \Omega_1 \) and \( \Omega_2 \) are open subsets in \( E \), \( \theta \in \Omega_1, \Omega_2 \subset \Omega_2 \), and satisfy one of the following conditions:

1. \( \|Tu\| \geq \|u\|, \forall x \in K \),
2. \( \|Tu\| \geq \|u\|, \forall x \in K \). By Lemma 1.3, \( K \) is a complete discontinuous operator, and satisfy one of the conditions:

**Lemma 1.4** (see [9]) Let \( K \) be a cone of \( E \) in Banach space, \( K_r = \{ x \in K \ | \ x \leq r \} \), suppose \( A : K_r \rightarrow K \) is a completely continuous, and satisfy \( T \) is continuous, and satisfy \( \|T\| \geq 1, \forall x \in K \).

1. If \( \|T\| \leq x, \forall x \in K \), then \( i(T, K_r, K) = 1 \),
2. If \( \|T\| \geq x, \forall x \in K \), then \( i(T, K_r, K) = 0 \).

II. Conclusion

**Theorem 2.1** Suppose \( (H_1) - (H_4) \) hold, then problem (1) has at least one positive solution.

**Proof** By \( (H_3) \), there exist \( \nu \) and a sufficient large number \( M > 0 \), such that

\[
f(u, v) \geq \nu^{p-1} v^{(p-1)\alpha}, \forall u \in R^+, v > M,
\]

\[
g(u) \geq C_0^{p-1} u^{\frac{1}{\alpha}}, \forall u > M,
\]

where \( C_0 = \max \left\{ \left( \frac{2}{\alpha} \right)^{1 - \alpha} \int_{\frac{1}{2}}^{1} \phi_q(k_2(s))ds \right\}^{-1}, \)

\[
(\nu^{(p-1)\alpha} \left( \frac{1}{2} \right)^{1 - \alpha} \int_{\frac{1}{2}}^{1} \phi_q(k_1(s))ds)^{p-1} + 1.
\]

Let \( N = (M + 1)e^{-2} \), if \( u \in K \cap \partial K_N \), by Lemma 2, \( \min_{0 \leq t \leq 1 - \varepsilon^2} u(t) \geq \varepsilon^2 \|u\| = \varepsilon^2 N = M + 1, \) by (3)-(6) and the symmetric property, for any \( t \in [\varepsilon, 1 - \varepsilon]\)

\[
v(t) = \int_{0}^{t} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds + \int_{0}^{1} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds \\
\geq \int_{0}^{1} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds \\
\geq \int_{0}^{1} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds \\
\geq \int_{0}^{1} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds \\
\geq \int_{0}^{1} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds \\
\geq \int_{0}^{1} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds \\
\geq \int_{0}^{1} \phi_q(\int_{s}^{1} h_2(\tau)g(u(\tau))d\tau)ds \\
\geq 2 \|u\|. \]

so \( \|Tu\| > \|u\|, \forall u \in K \cap K_N \), by Lemma 1.4, we can get

\[ i(T, K \cap K_N, K) = 0. \]

On the other hand, by the second limit of \( H_1 \), there exists a sufficient small number \( r_1 \in (0, 1) \) such that

\[
C_1^{p-1} = \sup_{v \in (r_1, 1]} \frac{f(u, v)}{v^{(p-1)\alpha}} < +\infty.
\]

Let \( \varepsilon = \min\left\{ \frac{r_1(1 - \gamma)}{\gamma}, \frac{C_1}{\gamma(1 - \gamma)} \right\} \), by the first limit of \( H_4 \), there exist a sufficient small number \( r_2 \in (0, 1) \) such that

\[
g(u) \leq \varepsilon^{p-1} u^{\frac{1}{\alpha}}, \forall u \in [0, r_2]. \]
Take $r = \min\{r_1, r_2\}$, by (9), we can get

$$v(t) = \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds + \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds + \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds$$

By (8), we can get

$$||Tu|| \leq \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds + \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds + \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds$$

By using Jensen inequality, $0 < q \leq 1$, and (11)-(13), we can get

$$(Tu)(t) \leq \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds$$

By Theorem 2.2, we have

Proof (H2), there exists $\mu > 0$ and a sufficient small number $\xi \in (0, 1)$, such that

$$f(u, v) \geq \mu^{p-1}v^{n-1}, \forall u \in R^+, 0 \leq v \leq \xi,$$

and

$$g(u) \geq (C_{2u})^{r-1}, \forall 0 \leq u \leq \xi,$$

where

$$C_2 = 2\left(\frac{\mu^2}{1-\gamma}\right)^{n}\int_0^n \phi_q(k_1(s))ds \int_0^n \phi_q(k_2(s))^{n}ds^{-1}$$

since $g \in C(R^+, R^+)$, $g(0) \equiv 0$, so there exists $\sigma \in (0, \xi)$ such that $\forall u \in [0, \sigma]$, we have

$$g(u) \leq \left(\int_0^t \phi_q\left(\int_s^t h_1(\tau)d\tau\right)ds\right)^{-1},$$

this imply

$$v(t) \leq \int_0^t \phi_q\left(\int_s^t h_2(\tau)g(u(\tau))d\tau\right)ds \leq \xi, \forall u \in K \cap \partial K_\sigma.$$


