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A Necessary Condition For The Existence Of Chaos In Fractional Order Delay Differential Equations

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Abstract—In this paper we propose a necessary condition for the existence of chaos in delay differential equations of fractional order. To explain the proposed theory, we discuss fractional order Liu system and financial system involving delay.

Keywords—Caputo derivative, delay, stability, chaos.

I. Introduction

FRACTIONAL calculus (FC) [1] deals with differentiation and integration of arbitrary real or complex order. Though the history of this subject dates back to 300 years, the applications are rather recent. Fractional calculus has wide range of applications in control theory [2], viscoelasticity [3], diffusion [4], mechanics [5], signal processing [6], biology [7] and social sciences [8]. In the recent article [9], Machado has discussed the applications of FC in the deoxyribonucleic acid (DNA) code, financial market evolution, earthquake dynamics, global warming and musical compositions.

Delay differential equations of fractional order (FDDE) involves non-integer order derivatives as well as time delays. These equations have found many applications in Control Theory [10], Agriculture [11], Chaos [12] and so on. Recently, we have studied delayed fractional Bloch equation [13] arising in NMR.

Since the fractional derivative is non-local, it has ability to model memory and hereditary properties. The time-delay [14] in the model is also having similar properties. Hence, the models containing fractional derivative as well as time-delay are crucial. The stability analysis of fractional delay systems has been studied by many researchers [15], [16], [17], [18], [19], [20].

Though there are some articles dealing with FDDEs, very few of them study the chaos. According to author's knowledge, no any necessary condition for the existence of chaos in FDDEs is available in the literature. We propose such condition in this article. The paper is organized as follows. Section II presents basic definitions in FC, a numerical algorithm for solving FDDEs and the stability analysis. Section III develops the necessary condition for the existence of chaos in FDDEs. Some examples supporting the analysis are presented in Section IV. Finally, Section V outlines the main conclusions.

II. Preliminaries

A. Fractional Calculus

Let us start with some definitions [1], [21].

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Definition 2.1: A real function $f(t)$, $t > 0$ is said to be in space C_α, $\alpha \in \Re$ if there exists a real number $p\ (> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0, \infty)$.

Definition 2.2: A real function $f(t)$, $t > 0$ is said to be in space C_α^m, $m \in I\!N \bigcup \{0\}$ if $f^{(m)} \in C_\alpha$.

Definition 2.3: Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann–Liouville integral of order μ, $\mu > 0$ is given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)\, d\tau, \quad t > 0. \tag{1}$$

Definition 2.4: The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m-1 < \mu < m$, $m \in I\!N \bigcup \{0\}$, is defined as:

$$\begin{aligned} D^\mu f(t) &= \frac{d^m}{dt^m} f(t), \qquad \mu = m \\ &= I^{m-\mu} \frac{d^m f(t)}{dt^m}. \end{aligned} \tag{2}$$

Note that for $m-1 < \mu \leq m,\ m \in I\!N$,

$$I^\mu D^\mu f(t) = f(t) - \sum_{k=0}^{m-1} \frac{d^k f}{dt^k}(0) \frac{t^k}{k!}, \tag{3}$$

$$I^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \tag{4}$$

B. Numerical method for solving fractional differential equations

Bhalekar and Daftardar-Gejji have modified Adams-Bashforth method [22], [23], [24] to solve delay differential equations of fractional order (FDDE) [25]. The method is described below.

Consider the FDDE

$$D^\alpha y(t) = f\left(t, y(t), y(t-\tau)\right), t \in [0, T], \tag{5}$$

$$y(t) = g(t), t \in [-\tau, 0], 0 < \alpha \leq 1. \tag{6}$$

where D^α denotes Caputo fractional derivative of order $\alpha \in (0, 1]$. We use Caputo derivatives because we can define properly the initial conditions as well as because the systems to be solved is a linear one. Applying fractional integration I^α on both sides of (5) and using (6) we obtain

$$y(t) = g(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f\left(\xi, y(\xi), y(\xi-\tau)\right) d\xi. \tag{7}$$

Consider a uniform grid
$\{t_n = nh : n = -k, -k+1, \cdots, -1, 0, 1, \cdots, N\}$
where k and N are integers such that $h = T/N$ and $h = \tau/k$. Let

$$y_h(t_j) = g(t_j), j = -k, -k+1, \cdots, -1, 0 \tag{8}$$

and note that

$$\begin{aligned} y_h(t_j-\tau) &= y_h(jh-kh) \\ &= y_h(t_{j-k}), j=0,1,\cdots,N \end{aligned} \tag{9}$$

Suppose we have already calculated approximations $y_h(t_j) \approx y(t_j)$, $(j=-k,-k+1,\cdots,-1,0,1,\cdots,n)$ and we want to calculate $y_h(t_{n+1})$ using

$$\begin{aligned} & y(t_{n+1}) \\ = \;& g(0)+ \\ & \int_0^{t_{n+1}} \frac{(t_{n+1}-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(\xi, y(\xi), y(\xi-\tau))\, d\xi. \end{aligned} \tag{10}$$

We use approximations $y_h(t_n)$ for $y(t_n)$ in (10). The integral on right hand side of (10) is evaluated using the product trapezoidal quadrature formula. We obtain

$$\begin{aligned} & y_h(t_{n+1}) \\ = \;& g(0)+\frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h(t_{n+1}), y_h(t_{n+1}-\tau)) \\ & +\frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j), y_h(t_j-\tau)) \\ = \;& g(0)+\frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, y_h(t_{n+1}), y_h(t_{n+1-k})) \\ & +\frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \end{aligned} \tag{11}$$

where $a_{j,n+1}$ are given by

$$a_{j,n+1} = \begin{cases} n^{\alpha+1}-(n-\alpha)(n+1)^\alpha, & \text{if } j=0, \\ (n-j+2)^{\alpha+1}+(n-j)^{\alpha+1} & \\ -2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j=n+1. \end{cases} \tag{12}$$

The unknown term $y_h(t_{n+1})$ appears on both sides of (11) and due to nonlinearity of f equation (11) can not be solved explicitly for $y_h(t_{n+1})$. So we replace the term $y_h(t_{n+1})$ on the right hand side by an approximation $y_h^P(t_{n+1})$, called predictor.

Product rectangle rule is used in (10) to evaluate predictor term

$$\begin{aligned} & y_h^P(t_{n+1}) \\ = \;& g(0)+\frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j), y_h(t_j-\tau)) \\ = \;& g(0)+\frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \end{aligned} \tag{13}$$

where $b_{j,n+1}$ is given by

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha-(n-j)^\alpha\right). \tag{14}$$

C. Stability of fractional order linear system with delay

Consider the fractional order linear system of delay differential equations:

$$\begin{aligned} D^\alpha x_1(t) &= \sum_{i=1}^{n} a_{1i} x_i(t-\tau_{1i}) \\ D^\alpha x_2(t) &= \sum_{i=1}^{n} a_{2i} x_i(t-\tau_{2i}) \\ &\;\;\vdots \\ D^\alpha x_n(t) &= \sum_{i=1}^{n} a_{ni} x_i(t-\tau_{ni}) \end{aligned} \tag{15}$$

and the characteristic matrix

$$\Delta(\lambda) = \begin{pmatrix} \lambda^\alpha - a_{11}e^{-\lambda\tau_{11}} & -a_{12}e^{-\lambda\tau_{12}} & \cdots & -a_{1n}e^{-\lambda\tau_{1n}} \\ -a_{21}e^{-\lambda\tau_{21}} & \lambda^\alpha - a_{22}e^{-\lambda\tau_{22}} & \cdots & -a_{2n}e^{-\lambda\tau_{2n}} \\ \vdots & \vdots & & \vdots \\ -a_{n1}e^{-\lambda\tau_{n1}} & -a_{n2}e^{-\lambda\tau_{n2}} & \cdots & \lambda^\alpha - a_{nn}e^{-\lambda\tau_{nn}} \end{pmatrix}$$

Theorem 2.1: [26] If all the roots of characteristic equation $\det(\Delta(\lambda)) = 0$ have negative real parts, then the zero solution of system (15) is asymptotically stable.

III. Theory

Consider a nonlinear system of fractional order delay differential equations

$$D^\alpha x_i(t) = f_i(x_1(t),\cdots,x_n(t),x_1(t-\tau),\cdots,x_n(t-\tau)), \tag{16}$$

$i=1,\cdots,n$, $0<\alpha\le 1$. An equilibrium point $X^* = (x_1^*,\cdots,x_n^*)$ of the system (16) satisfies

$$f_i(x_1^*,\cdots,x_n^*,x_1^*,\cdots,x_n^*) = 0, \quad i=1,\cdots,n. \tag{17}$$

To identify the stability of equilibrium point we perturb the solution around X^* by a time dependent function $\xi(t) = (\xi_1(t),\cdots,\xi_n(t))$. Denoting $x_i = x_i(t)$ and $x_{i\tau} = x_i(t-\tau)$, we have $x_i = \xi_i + x_i^*$, $x_{i\tau} = \xi_{i\tau} + x_i^*$. Then

$$\begin{aligned} D^\alpha \xi_i &= D^\alpha x_i \\ &= f_i(\xi_1+x_1,\cdots,\xi_n+x_n,\xi_{1\tau}+x_{1\tau},\cdots,\xi_{n\tau}+x_{n\tau}). \end{aligned} \tag{18}$$

Using Taylor series expansion, the equation (18) can be linearized about equilibrium point X^* as

$$D^\alpha \xi = J\xi + J_\tau \xi_\tau, \tag{19}$$

where

$$J = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\cdots,n}$$

is the Jacobian matrix with respect to X evaluated at $X = X_\tau = X^*$ and

$$J_\tau = \left(\frac{\partial f_i}{\partial x_{j\tau}}\right)_{i,j=1,\cdots,n}$$

is the Jacobian matrix with respect to X_τ evaluated at $X = X_\tau = X^*$. Thus the characteristic equation of the equilibrium point is

$$|J + e^{-\lambda\tau} J_\tau - \lambda^\alpha I| = 0, \tag{20}$$

where I is the identity matrix.

If all the roots λ of characteristic equation (20) have negative real parts, then the equilibrium point of system (16) is asymptotically stable.

A. Necessary condition for existence of chaos

Consider the fractional order nonlinear system (16) with delay. A necessary condition for the system (16) to exhibit chaos is the instability of the equilibrium points. According to the discussion in Section III, this condition can be written as: "If the system (16) is chaotic then there exist a root of the characteristic equation (20) having positive real part".

IV. Illustrative Examples

In this section, we discuss the examples available in the literature viz. Liu system and financial system.

Example 4.1: Consider the fractional order Liu system with delay [27]

$$\begin{aligned} D^\alpha x(t) &= a\left(y(t) - x(t-\tau)\right), \\ D^\alpha y(t) &= bx(t-\tau) - kx(t)\, z(t), \\ D^\alpha z(t) &= -cz(t-\tau) + hx^2(t), \end{aligned} \tag{21}$$

where $a = 10$, $b = 40$, $k = 1$, $c = 2.5$, $h = 4$.

Now we use our theory and get the stability bounds. The characteristic equation at an equilibrium point $(5, 5, 40)$ turns out to be:

$$det \begin{pmatrix} \lambda^\alpha + 10e^{-\lambda\tau} & -10 & 0 \\ 40 - 40e^{-\lambda\tau} & \lambda^\alpha & 5 \\ -40 & 0 & \lambda^\alpha + 2.5e^{-\lambda\tau} \end{pmatrix} = 0. \tag{22}$$

If we plot $\text{Re}(\lambda)$ for fixed α then the region of curve below x-axis is in the stable region $\text{Re}(\lambda) < 0$. For chaos, the values of τ should be chosen so that the $\text{Re}(\lambda) > 0$. In the Table 1, we have discussed different cases of fixed values for α and the corresponding unstable regions given in the figures. We also have listed the chaotic regions for these values of α which are available in [27].

TABLE I
UNSTABLE REGIONS FOR LIU SYSTEM.

No.	α	Unstable region	Chaos[27]	Fig
1	1	$0 \le \tau \le 0.028$	$0 \le \tau \le 0.005$	1(a)
2	0.97	$0 \le \tau \le 0.021$	$0 \le \tau \le 0.005$	1(b)
3	0.94	$0 \le \tau \le 0.015$	$0 \le \tau \le 0.009$	1(c)
4	0.90	$0 \le \tau \le 0.008$	$0 \le \tau \le 0.007$	1(d)

It is clear from the Table I and Figures 1-4 that the chaotic regions observed in [27] are in the unstable regions obtained by our theory. It can be observed that the unstable region decreases for smaller values of α.

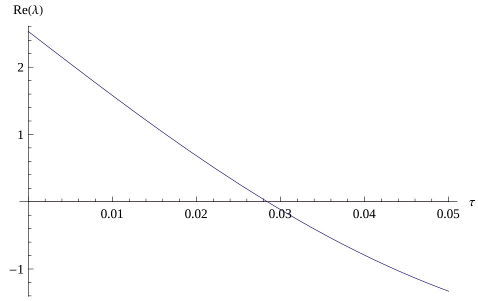

Fig. 1. Liu system, $\alpha = 1$

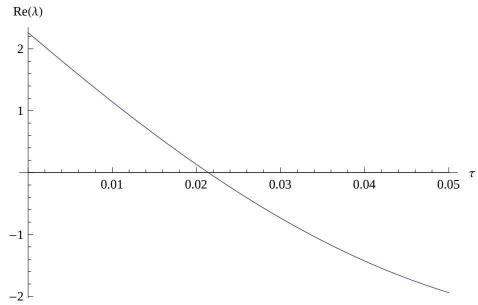

Fig. 2. Liu system, $\alpha = 0.97$

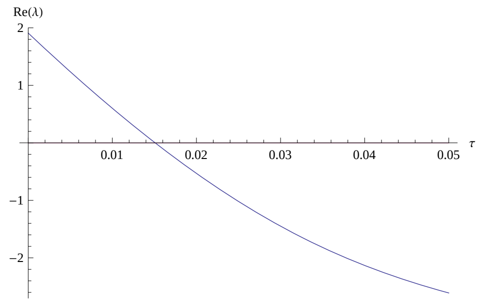

Fig. 3. Liu system, $\alpha = 0.94$

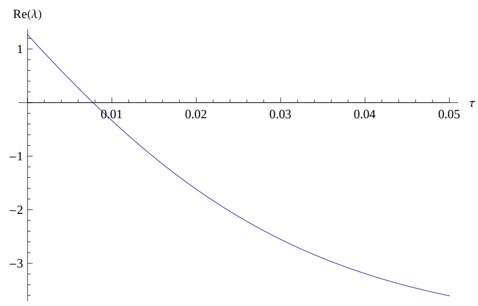

Fig. 4. Liu system, $\alpha = 0.90$

Example 4.2: In [28], Zen *et al.* have studied the fractional order financial system given by

$$\begin{aligned} D^\alpha x(t) &= z(t) + \left(y(t-\tau) - a\right) x(t), \\ D^\alpha y(t) &= 1 - by(t) - x^2(t-\tau), \\ D^\alpha z(t) &= -x(t-\tau) - cz(t), \end{aligned} \tag{23}$$

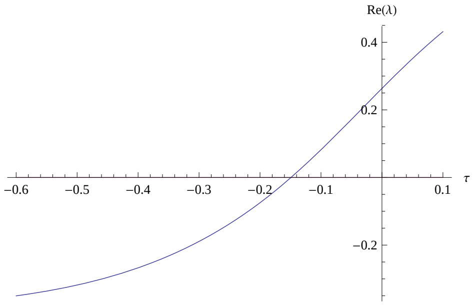

Fig. 5. Financial system, $\alpha = 0.97$

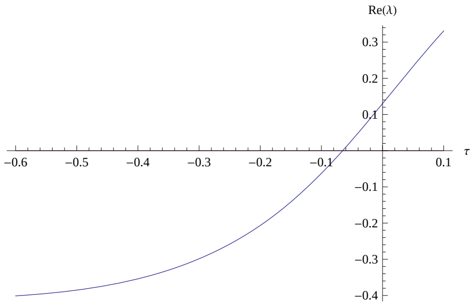

Fig. 6. Financial system, $\alpha = 0.90$

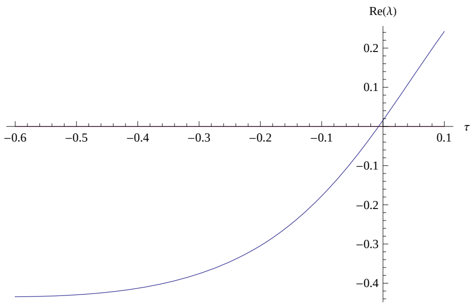

Fig. 7. Financial system, $\alpha = 0.85$

where $a = 3$, $b = 0.1$ and $c = 1$. The characteristic equation at an equilibrium point $(\sqrt{15}/5, 4, -\sqrt{15}/5)$ is

$$det \begin{pmatrix} \lambda^\alpha - 1 & -\frac{\sqrt{15}}{5}e^{-\lambda\tau} & -1 \\ 2\frac{\sqrt{15}}{5}e^{-\lambda\tau} & \lambda^\alpha + 0.1 & 0 \\ e^{-\lambda\tau} & 0 & \lambda^\alpha + 1 \end{pmatrix} = 0. \qquad (24)$$

We have compared the results given in [28] with our theory in Table 2.

TABLE II
UNSTABLE AND CHAOTIC REGIONS FOR FINANCIAL SYSTEM.

No.	α	Unstable region	Chaos [28]	Fig
1	0.97	$\tau \geq -0.15$	$0 < \tau \leq 0.06$	2(a)
2	0.90	$\tau \geq -0.06$	$0 < \tau \leq 0.15$	2(b)
3	0.85	$\tau \geq -0.01$	$0 < \tau \leq 0.05$	2(c)

It can be observed from the Table II and Figures 5-7 that the chaotic regions observed by Zen *et al.* are in the unstable regions obtained from our theory.

V. Conclusion

In this paper, we have used the stability analysis and proposed a necessary condition for the existence of chaos in fractional order delay differential equations (FDDE). Examples from the existing literature are discussed. It is verified that our theory is well in agreement with the results obtained in the literature using numerical simulations. According to author's knowledge, no such criterion for the existence of chaos was proposed before for FDDEs. Also, very few research articles are devoted for the investigation of chaos in FDDEs. We hope that this work will help researchers in this field.

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