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Stability of fractional differential equation

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Abstract—We study a Dirichlet boundary value problem for Lane-Emden equation involving two fractional orders. Lane-Emden equation has been widely used to describe a variety of phenomena in physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres,and thermionic currents. However, ordinary Lane-Emden equation does not provide the correct description of the dynamics for systems in complex media. In order to overcome this problem and describe dynamical processes in a fractalmedium, numerous generalizations of Lane-Emden equation have been proposed. One such generalization replaces the ordinary derivative by a fractional derivative in the Lane-Emden equation. This gives rise to the fractional Lane-Emden equation with a single index. Recently, a new type of Lane-Emden equation with two different fractional orders has been introduced which provides a more flexible model for fractal processes as compared with the usual one characterized by a single index. The contraction mapping principle and Krasnoselskiis fixed point theorem are applied to prove the existence of solutions of the problem in a Banach space. Ulam-Hyers stability for iterative Cauchy fractional differential equation is defined and studied.

Keywords—Fractional calculus; fractional differential equation; Lane-Emden equation; Riemann-Liouville fractional operators; Volterra integral equation.

I. INTRODUCTION

The theory of singular boundary value problems has become an important area of investigation in the past three decades (see [1-5]). One of the equations describing this type is the Lane-Emden equation. Lane-Emden type equations, first published by Jonathan Homer Lane in 1870 [6], and further explored in detail by Emden [7], represents such phenomena and having significant applications, is a second- order ordinary differential equation with an arbitrary index, known as the polytropic index, involved in one of its terms. The Lane-Emden equation describes a variety of phenomena in physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres,and thermionic currents [8].

The solution of the Lane-Emden problem, as well as other various linear and nonlinear singular initial value problems in quantum mechanics and astrophysics, is numerically challenging because of the singularity behavior at the origin. The approximate solutions to the Lane-Emden equation were given by homotopy perturbation method [9], variational iteration method [10], and Sinc-Collocation method [11], an implicit series solution [12]. Recently, Parand et. al [13] proposed an approximation algorithm for the solution of the nonlinear Lane-Emden type equation using Hermite functions collocation method. Moreover, Adibi and Rismani [14] introduced a modified Legendre-spectral method. Finally, Bhrawy and Alofi [15] imposed a Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations.

Lane-Emden equations have the following form

$$u''(t) + \frac{a}{t}u'(t) + f(t,u) = g(t), \quad 0 < t \leq 1,\, a \geq 0 \quad (1)$$

with the initial condition

$$u(0) = A, \quad u'(0) = B,$$

where A, B are constants, $f(t,u)$ is a continuous real valued function and $g(t) \in C[0,1]$.

II. FRACTIONAL CALCULUS

Fractional calculus and its applications (that is the theory of derivatives and integrals of any arbitrary real or complex order) has importance in several widely diverse areas of mathematical physical and engineering sciences. It generalized the ideas of integer order differentiation and n-fold integration. Fractional derivatives introduce an excellent instrument for the description of general properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids and rocks, and in many other fields (see [16,17]).

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [18], Erdlyi-Kober operators [19], Weyl-Riesz operators [20], Caputo operators [21] and Grnwald-Letnikov operators [22], have appeared during the past three decades. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [23]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed in [24,25].

One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators (see[22]). The Riemann-Liouville fractional derivative could

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hardly pose the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. Moreover, this operator possesses advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms (see[26]).

Definition 2.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denoted the convolution product (see [22]), $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0,\ t \leq 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. The fractional (arbitrary) order derivative of the function f of order $0 \leq \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 2.1. From Definition 2.1 and Definition 2.2, we have

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha},\ \ \mu > -1;\ 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha},\ \ \mu > -1;\ \alpha > 0.$$

In this note, we consider the fractional Lane-Emden equations of the following form

$$D^\beta\Big(D^\alpha + \frac{a}{t}\Big)u(t) + f(t,u) = g(t), \tag{2}$$

$$(0 < t \leq 1,\ a \geq 0,\ 0 < \alpha, \beta \leq 1)$$

with the boundary condition

$$u(0) = \mu, \quad u(1) = \nu,$$

where $f(t,u)$ is a continuous real valued function and $g(t) \in C[0,1]$.

Now, we state a known result due to Krasnoselskii [27] which is needed to prove the existence of solution.

Theorem 2.1 Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

III. Existence of solutions

In this section, we establish the solution of Eq. (2). Assume that $\mathcal{B}$ is a Banach space X of all continues bounded functions endow with the supremum norm. We introduce the following linear problem for Lane-Emden equation involving two fractional orders:

$$D^\beta\Big(D^\alpha + \frac{a}{t}\Big)u(t) = g(t), \tag{3}$$

$$(0 < t \leq 1,\ a \geq 0,\ 0 < \alpha, \beta \leq 1)$$

with the boundary condition

$$u(0) = \mu, \quad u(1) = \nu.$$

Lemma 3.1 A unique solution of the linear two-point boundary value problem for Lane-Emden equation (3) is given by

$$\begin{aligned} u(t) =& \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds - \frac{a}{\tau}u(\tau)\Big)d\tau \\ &- t^\alpha\Big[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds - \frac{a}{\tau}u(\tau)\Big)d\tau\Big] \\ &+ (\nu-\mu)t^\alpha + \mu, \end{aligned} \tag{4}$$

where $g \in C[0,1]$.

Proof. The general solution of

$$D^\beta\Big(D^\alpha + \frac{a}{t}\Big)u(t) = g(t)$$

can be written as

$$\begin{aligned} u(t) =& \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds - \frac{a}{\tau}u(\tau)\Big)d\tau \\ &- \frac{c_0}{\Gamma(\alpha+1)}t^\alpha - c_1. \end{aligned} \tag{5}$$

Using the boundary conditions for (3) we obtain

$$c_1 = -\mu,$$

$$\frac{c_0}{\Gamma(\alpha+1)} = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds - \frac{a}{\tau}u(\tau)\Big)d\tau + \mu - \nu.$$

Substituting the last assertion in (5), we obtain the solution given by (4). This completes the proof.

Theorem 3.1 Let $f : [0,1] \times X \longrightarrow X$ be a jointly continuous function satisfying the condition

$$|f(t,u) - f(t,v)| \leq L|u-v|, \quad \forall\, t \in [0,1],\ u, v \in X.$$

Moreover, assume that $\sup_{t\in[0,1]} |g(t)| = \gamma$. Then the boundary value problem (2) has a unique solution provided

$$\ell := \frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{2a\Gamma(\alpha)}{\Gamma(2\alpha)} < 1, \quad a \geq 0.$$

Proof. Define $P : X \to X$ by

$$\begin{aligned}
&(Pu)(t)\\
&= \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big(g(s)-f(s,u(s))\big)ds\\
&- \frac{a}{\tau}u(\tau)\Big)d\tau\\
&- t^\alpha\Big[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big(g(s)-f(s,u(s))\big)ds\\
&- \frac{a}{\tau}u(\tau)\Big)d\tau\Big] + (\nu-\mu)t^\alpha + \mu.
\end{aligned} \tag{6}$$

Setting $\sup_{t\in[0,1]}|f(t,u)| = M$ and we chose

$$r \ge \frac{1}{1-\epsilon}\Big(\frac{2(\gamma+M)}{\Gamma(\alpha+\beta+1)} + 2\mu + \nu\Big), \tag{7}$$

where ϵ such that $\ell \le \epsilon < 1$. Now we show that $PB_r \subset B_r$, where $B_r := \{u \in X : \|u\| \le r\}$. For $u \in X$, we have

$$\begin{aligned}
&\|(Pu)(t)\|\\
&= \sup_{t\in[0,1]} |\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big(g(s)-f(s,u(s))\big)ds\\
&- \frac{a}{\tau}u(\tau)\Big)d\tau - t^\alpha\Big[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\\
&\times \big(g(s)-f(s,u(s))\big)ds - \frac{a}{\tau}u(\tau)\Big)d\tau\Big] + (\nu-\mu)t^\alpha + \mu|\\
&\le \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big|g(s)\\
&- f(s,u(s)) - f(s,0) + f(s,0)\big|ds + |\frac{a}{\tau}u(\tau)|\Big)d\tau\\
&+ \sup_{t\in[0,1]} t^\alpha\Big[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big|g(s)\\
&- f(s,u(s)) - f(s,0) + f(s,0)\big|ds + |\frac{a}{\tau}u(\tau)|\Big)d\tau\Big]\\
&+ \sup_{t\in[0,1]} |(\nu-\mu)|t^\alpha + |\mu|\\
&\le \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau\\
&\times [\gamma + M + L\|u\|]\\
&+ \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\tau}d\tau a\|u\|\\
&+ \sup_{t\in[0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau\\
&\times [\gamma + M + L\|u\|]\\
&+ \sup_{t\in[0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\tau}d\tau a\|u\|\\
&+ \sup_{t\in[0,1]} |(\nu-\mu)|t^\alpha + |\mu|
\end{aligned}$$

since $\tau \in [0,1]$ and $\alpha \in (0,1]$ yields

$$\begin{aligned}
\|(Pu)(t)\| &\le 2[\gamma + M\\
&+ L\|u\|]\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau\\
&+ 2a\|u\|\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\tau^{\alpha-1}d\tau + |\nu| + 2|\mu|\\
&\le \frac{2[\gamma+M+Lr]}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^1 (1-\tau)^{\alpha-1}\tau^\beta d\tau\\
&+ \frac{2ar}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-1}d\tau + |\nu| + 2|\mu|
\end{aligned}$$

Using (7), and the relations for the Beta function

$$B(z,w) = \int_0^1 (1-\tau)^{z-1}\tau^{w-1}d\tau = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

and

$$B(z,w+1) = \int_0^1 (1-\tau)^{z-1}\tau^{w}d\tau = \frac{\Gamma(z)\Gamma(w+1)}{\Gamma(z+w+1)}$$

we find that

$$\begin{aligned}
\|(Pu)(t)\| &\le \frac{2[\gamma+M+Lr]}{\Gamma(\alpha)\Gamma(\beta+1)}B(\alpha,\beta+1)\\
&+ \frac{2ar}{\Gamma(\alpha)}B(\alpha,\alpha) + |\nu| + 2|\mu|\\
&= \frac{2[\gamma+M+Lr]}{\Gamma(\alpha+\beta+1)} + \frac{2ar\Gamma(\alpha)}{\Gamma(2\alpha)} + |\nu| + 2|\mu|\\
&\le (\ell + 1 - \epsilon)r \le r.
\end{aligned}$$

We proceed to prove that P is a contraction mapping. For $u, v \in X$ and for all $t \in [0,1]$ we pose

$$\begin{aligned}
&\|(Pu)(t) - (Pv)(t)\| = \sup_{t\in[0,1]} |(Pu)(t) - (Pv)(t)|\\
&\le \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\\
&\times \big|f(s,u(s)) - f(s,v(s))\big|ds + \frac{a}{\tau}|u(\tau)-v(\tau)|\Big)d\tau\\
&+ \sup_{t\in[0,1]} t^\alpha\Big[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big|f(s,u(s))\\
&- f(s,v(s))\big|ds + \frac{a}{\tau}|u(\tau)-v(\tau)|\Big)d\tau\Big]\\
&\le \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau[L\|u-v\|]\\
&+ \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\tau}d\tau a\|u-v\|\\
&+ \sup_{t\in[0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau[L\|u-v\|]\\
&+ \sup_{t\in[0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\tau}d\tau a\|u-v\|\\
&\le 2[L\|u-v\|]\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau\\
&+ 2a\|u-v\|\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\tau^{\alpha-1}d\tau\\
&= \ell\|u-v\|,
\end{aligned}$$

then P is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof.

Theorem 3.2 Let $f : [0,1]\times X \longrightarrow X$ be a jointly continuous function and maps bounded subsets of $[0,1]\times X$ into relatively compact subsets of X. Furthermore, assume that

$$|f(t,u) - f(t,v)| \le L|u-v|, \quad \forall\, t\in[0,1],\ u,v\in X,$$
$$|f(t,u)| \le \sigma(t), \quad \sigma \in (L^1[0,1], R^+)$$

and $\sup_{t\in[0,1]} |g(t)| = \gamma$. If

$$\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{a\Gamma(\alpha)}{\Gamma(2\alpha)} < 1$$

then the boundary value problem (2) has at least one solution on [0, 1].

Proof. Define two operators A and B on B_r where

$$r \ge \frac{\frac{2(\gamma+\|\sigma\|_{L^1})}{\Gamma(\alpha+\beta+1)} + 2|\mu| + |\nu|}{1 - \frac{2a\Gamma(\alpha)}{\Gamma(2\alpha)}}$$

as follows:

$$(Au)(t) := \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big(g(s) - f(s,u(s))\big)ds - \frac{a}{\tau}u(\tau)\Big)d\tau$$

and

$$(Bu)(t) := -t^\alpha\Big[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big(g(s) - f(s,u(s))\big)ds - \frac{a}{\tau}u(\tau)\Big)d\tau\Big] + (\nu-\mu)t^\alpha + \mu.$$

For $u, v \in B_r$ we obtain

$$\|Au + Bv\| \le \Big(\frac{2(\gamma+\|\sigma\|_{L^1})}{\Gamma(\alpha+\beta+1)} + \frac{2a\Gamma(\alpha)r}{\Gamma(2\alpha)} + 2|\mu| + |\nu|\Big) \le r.$$

Thus, $A, B \in B_r$. Furthermore, for $u, v \in B_r$,

$$\begin{aligned}
\|(Bu)(t) - (Bv)(t)\| &\le \sup_{t\in[0,1]} t^\alpha\Big[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big|f(s,u(s)) - f(s,v(s))\big|ds + \frac{a}{\tau}|u(\tau)-v(\tau)|\Big)d\tau\Big] \\
&\le \sup_{t\in[0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau \times [L\|u-v\|] + \sup_{t\in[0,1]} t^\alpha \int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\tau}d\tau a\|u-v\| \\
&\le [L\|u-v\|]\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau + a\|u-v\|\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\tau^{\alpha-1}d\tau \\
&= [\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{a\Gamma(\alpha)}{\Gamma(2\alpha)}]\|u-v\|,
\end{aligned}$$

it follows that B is a contraction mapping in B_r. The continuity of f implies that the operator A is continuous. Also, A is uniformly bounded on B_r as

$$\begin{aligned}
&\|(Au)(t)\| \\
&= \sup_{t\in[0,1]} |\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big(g(s) - f(s,u(s))\big)ds - \frac{a}{\tau}u(\tau)\Big)d\tau| \\
&\le \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\big|g(s) + f(s,u(s))\big|ds + |\frac{a}{\tau}u(\tau)|\Big)d\tau \\
&\le \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau[\gamma + \|\sigma\|_{L^1}] + \sup_{t\in[0,1]} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{\tau^\alpha}{\tau}d\tau a\|u\| \\
&\le [\gamma + \|\sigma\|_{L^1}]\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)d\tau + a\|u\|\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\tau^{\alpha-1}d\tau \\
&\le \frac{[\gamma + \|\sigma\|_{L^1}]}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\alpha)ar}{\Gamma(2\alpha)}.
\end{aligned}$$

Next we show that A is a compact operator in B_r. Let $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned}
&\|(Au)(t_1) - (Au)(t_2)\| \\
&= \|\int_0^{t_1} \frac{(t_1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}(g(s) - f(s,u(s)))ds + \frac{a}{\tau}u(\tau)\Big)d\tau \\
&- \int_0^{t_2} \frac{(t_2-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^\tau \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}(g(s) - f(s,u(s)))ds + \frac{a}{\tau}u(\tau)\Big)d\tau\| \\
&\le \frac{(\gamma + \|f\|)}{\Gamma(\alpha+\beta+1)}|t_1^{\alpha+\beta} - t_2^{\alpha+\beta}| + \frac{a\Gamma(\alpha)}{\Gamma(2\alpha)}|t_1^{2\alpha-1} - t_2^{2\alpha-1}|
\end{aligned}$$

which is independent of u, where

$$\|f\| = sup_{t\in[0,1]}|f(t,u(t))|.$$

Thus, A is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets. Therefore, A is relatively compact on B_r. Hence, by the Arzela- Ascoli theorem, A is compact on B_r. Thus all the assumptions of Theorem 2.1 are satisfied and the conclusion of Theorem 2.1 implies that the boundary value problem (2) has at least one solution on [0, 1]. This completes the proof.

IV. Applications

In this section, we illustrate some applications of the theory established in this paper.

Example 4.1 Consider the following (BVP)

$$D^{1/2}\Big(D^{1/2} + \frac{0.1}{t}\Big)u(t) + \frac{1}{(t+2)^2}\frac{|u|}{1+|u|} = t, \quad t\in(0,1) \tag{8}$$

$$u(0) = \mu, \quad u(1) = \nu.$$

It is clear that

$$|f(t,u) - f(t,v)| \le \frac{1}{4}|u - v|,$$

and $\sup_{t\in(0,1)} g(t) = 1$. Moreover, we have

$$\ell = \frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{2a\Gamma(\alpha)}{\Gamma(2\alpha)} = 0.5 + 0.3544 = 0.8544 < 1$$

Thus, by Theorem 3.1, the boundary value problem (8) has a unique solution on [0, 1].

Example 4.2 Consider the following (BVP)

$$D^{1/4}\Big(D^{1/2} + \frac{0.1}{t}\Big)u(t) + \frac{1}{(t+2)^2}\frac{|u|}{1+|u|} = t, \quad t \in (0,1) \tag{9}$$

$$u(0) = \mu, \quad u(1) = \nu.$$

It is clear that $|f(t,u)| \le \frac{1}{(t+2)^2} := \sigma(t)$,

$$|f(t,u) - f(t,v)| \le \frac{1}{4}|u - v|,$$

and $\sup_{t\in(0,1)} g(t) = 1$. Moreover, we have

$$\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{a\Gamma(\alpha)}{\Gamma(2\alpha)}$$

$$= \frac{1}{3\Gamma(\frac{3}{4})} + 0.1772 = 0.2721 + 0.1772 = 0.4493 < 1.$$

Thus, by Theorem 3.2, the boundary value problem (9) has at least one solution on [0, 1].

Note that when $a \ge 1$ the (BVPS) (8) and (9) have not solutions in [0,1] :

Example 4.3 Consider the following (BVP)

$$D^{\beta}\Big(D^{\alpha} + \frac{1}{t}\Big)u(t) + \frac{1}{(t+2)^2}\frac{|u|}{1+|u|} = t, \quad t \in (0,1) \tag{10}$$

$$u(0) = \mu, \quad u(1) = \nu.$$

It is clear that $|f(t,u)| \le \frac{1}{(t+2)^2} = \sigma(t)$,

$$|f(t,u) - f(t,v)| \le \frac{1}{4}|u - v|,$$

and $\sup_{t\in(0,1)} g(t) = 1$. Moreover, since $\Gamma(\alpha) > \Gamma(2\alpha)$ for $0 < \alpha \le 1$, we have

$$\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{a\Gamma(\alpha)}{\Gamma(2\alpha)} > 1, \quad \forall\, 0 < \alpha, \beta \le 1.$$

V. Ulam-Hyers stability

The analysis on stability of fractional differential equations is more complicated than the classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels. Recently, Li and Zhang [28] provided an overview on the stability results of the fractional differential equations. Particular, Li et al. [29] devoted to study the MittagLeffler stability and the Lyapunovs methods; Deng [30] derived sufficient conditions for the local asymptotical stability of nonlinear fractional differential equations and Li et all studied the stability of fractional-order nonlinear dynamic systems using the Lyapunov direct method and generalized Mittag-Leffler stability [31]. Furthermore, there are few works on the Ulam stability of fractional differential equations, which maybe provide a new way for the researchers to investigate the stability of fractional differential equations from different perspectives. First the Ulam stability and data dependence for fractional differential equations with Caputo derivative has been posed by Wang et al. [32] while Ibrahim [33] with Riemann-Liouville derivative. Finally, the author generalized the Ulam-Hyers stability for fractional differential equation including infinite power series [34-36].

In this section, we continuo our study by imposing the Ulam-Hyers stability for fractional differential equation.

Definition 4.1. Problem (2) has the Ulam-Hyers stability if there exists a positive constant K with the following property: For every $\epsilon > 0$, $u \in C_{L,\alpha}$, if

$$\Big|D^{\beta}\Big(D^{\alpha} + \frac{a}{t}\Big)u(t) + f(t,u) - g(t)\Big| < \epsilon$$

then there exists some $v \in X$ satisfying

$$D^{\beta}\Big(D^{\alpha} + \frac{a}{t}\Big)v(t) + f(t,v) = g(t)$$

with $v(0) = \mu$, $v(1) = \nu$ such that

$$|u(t) - v(t)| < K\epsilon.$$

Theorem 4.1. Let the assumptions of Theorem 3.1 hold. If

$$\sup|D^{\beta}\Big(D^{\alpha}+\frac{a}{t}\Big)u(t)| \ge \frac{2[\gamma+M+Lr]}{\Gamma(\alpha+\beta+1)} + \frac{2ar\Gamma(\alpha)}{\Gamma(2\alpha)} + |\nu| + 2|\mu|$$

then problem (2) has the Ulam-Hyers stability in X.

Proof. For every $\epsilon > 0, u \in C_{L,\alpha}$, we let

$$\Big|D^{\beta}\Big(D^{\alpha} + \frac{a}{t}\Big)u(t) + f(t,u) - g(t)\Big| < \epsilon.$$

Under the assumptions of Theorem 3.1, Eq.(2) has a unique solution in X. Since

$$|u(t)| \le \frac{2[\gamma+M+Lr]}{\Gamma(\alpha+\beta+1)} + \frac{2ar\Gamma(\alpha)}{\Gamma(2\alpha)} + |\nu| + 2|\mu|$$

yields

$$\sup|u| \le \sup|D^{\beta}\Big(D^{\alpha} + \frac{a}{t}\Big)u(t)|.$$

Thus we obtain

$$\sup|u(t)-v(t)| \le \sup|D^{\beta}\Big(D^{\alpha}+\frac{a}{t}\Big)\big(u(t)-v(t)\big)|$$
$$\le \sup|D^{\beta}\Big(D^{\alpha}+\frac{a}{t}\Big)u(t)-D^{\beta}\Big(D^{\alpha}+\frac{a}{t}\Big)v(t)$$
$$-f(t,u)+f(t,v)+g(t)-g(t)|+\sup|f(t,u)-f(t,v)|$$
$$\le \epsilon + L\sup|u(t)-v(t)|;$$

hence we receive

$$\sup|u(t)-v(t)| \le \frac{\epsilon}{1-L} := K\epsilon.$$

In view of (4), we observe that $v(0)=\mu, v(1)=\nu$ yields

$$|u(t)-v(t)| \le K\epsilon.$$

Thus (2) has the Ulam-Hyers stability.

VI. Conclusion

The existence of solutions for a Dirichlet boundary value problem involving Lane-Emden equation with two different fractional orders has been discussed. We applied the concepts of fractional calculus together with fixed point theorems to establish the existence results. First of all, we found the unique solution for a linear Dirichlet boundary value problem involving Lane-Emden equation with two different fractional orders, which in fact provides the platform to prove the existence of solutions for the associated nonlinear fractional Lane-Emden equation with two different orders. Our approach is applicable to a variety of real world problems. Moreover, the Ulam-Hyers stability is discussed for the iterative Cauchy fractional differential equation by using the lower bound of the fractional differential derivative.

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