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A new seed projection method for solving shifted systems with multiple right-hand sides

Chao Li,Hao Liu

Abstract—In this paper, we propose a new seed projection method for solving shifted systems with multiple right-hand sides. This seed projection method uses a seed selection strategy. Numerical experiments are presented to show the efficiency of the newly method.

Keywords—shifted systems, multiple right-hand sides, seed projection.

I. Introduction

THROUGHOUT this paper, we consider techniques for the solutions of several shifted systems of equations

$$(A - \sigma_i I)\, x_i = b_i, \quad i = 1, \cdots, p, \tag{1}$$

with $\sigma_i \in R$, $b_i \in R^n$, and $A \in R^{n\times n}$. I is the identity matrix in $R^{n\times n}$, set $A_i = A - \sigma_i I$, A_i is an $n \times n$ nonsingular and nonsymmetric real matrix.

Such shifted systems arise in a variety of practical applications such as control theory, structural dynamics, and quantum chromodynamics; see [1,2] and references therein. In many practical applications, the right-hand sides are not arbitrary, and often there is information sharable among the right-hands.

When $A_i \equiv A_j = A$, $\forall i, j = 1, \cdots, p$, problem (1) reduces to linear systems with multiple right-hand sides

$$AX = B = [b_1, b_2, \cdots, b_p]\,. \tag{2}$$

Two methods for solving (2) have been discussed in the literature are block iterative method [3], and seed projection method [4]. When $b_i \equiv b_j = b$, $\forall i, j = 1, \cdots, p$, equation (1) can be written as shifted systems

$$(A - \sigma_i I)\, x_i = b, \quad i = 1, \cdots, p. \tag{3}$$

Because of the invariance of Krylov subspace under shifting, i.e.,$K_m(A, v) = K_m(A - \sigma I, v)$, a nature solution to problem (3) is the krylov subspace method such as FOM method [2], QMR method [5], GMRES method [6], etc. In generally, $\sigma_i \neq \sigma_j$, $b_i \neq b_j$ for $i \neq j$. Based on the seed projection method for the CG method proposed in [4], Gu and Zhu propose a seed projection method using the GMRES method for solving (1), see [7]. This method is effective especially when the right-hand sides are close. This method (referred as the GS-shift method) generates a Krylov subspace from a set of vectors obtained by solving one of the system (1), called a seed system, by GMRES method and then projects the residuals of other nonseed systems, called nonseed systems, orthogonally onto the generated Krylov subspace to get the approximate solutions. The whole process is repeated with another unsolved system as a seed until all the systems are solved.

C. Li and H. Liu are with the Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, P. R. China. e-mail: littlemaths@126.com(C. Li), hliu@nuaa.edu.cn(H. Liu).

From [8], we know that seed selection plays an important role in the seed projection method. It is interesting to give a good approach to the seed selection that increase the convergence rate of the GS-shift method. In this paper, we propose a new seed projection method which uses a seed selection strategy for solving (1). Numerical experiments are given in this paper will show that our method is more effective than the GS-shift method.

In section 2, we give a quick overview of GS-shift method. In section 3, we present the new seed projection method with a seed selection strategy for solving shifted systems with multiple right-hand sides. In section 4, some numerical experiments are presented to show the efficiency of the method. Finally, we make some concluding remarks in section 5.

II. GS-shift Method

In this section, we recall the GS-shift algorithm for solving (1). This will allow us to simplify the presentation of the new method in section 3. Details of the algorithm can be found in [7]. We summarize it in the following algorithm.

Algorithm 1. GS-Shift

1) $x_i = x_i^{(0)}$, $r_i = b_i - A_i x_i$, $i = 1, \cdots, p$.
2) For $l = 1, 2 \cdots, p$, until all the systems are solved
3) Set $s = l$, select the lth system as seed system.
4) For $k = 1, 2, \cdots$, until seed system convergence
5) $[V_{m+1}, H]$=Arnoldi(A_s, r_s).
6) $d_s = \arg\min_{d\in R^m} \left\|\beta e_1 - \bar{H}_m d\right\|_2$, where $\beta = \|r_s\|_2$.
7) $\bar{x}_s = x_s + V_m d_s$.
8) $\bar{r}_s = b_s - A_s \bar{x}_s$.
9) For $j = l + 1, \cdots, p$, compute all the approximate projection solutions of the nonseed systems
10) $d_j = \arg\min_{d\in R^m} \left\|V_{m+1}^{\mathrm{T}} r_j - \bar{H}_j d\right\|_2$, where $\bar{H}_j = \bar{H}_m + (\sigma_s - \sigma_j)\begin{pmatrix} I_m \\ 0 \end{pmatrix}$.
11) $\bar{x}_j = x_j + V_m d_j$.
12) $\bar{r}_j = b_j - A_j \bar{x}_j$.
13) If $\|\bar{r}_j\|_2 \leq \varepsilon$, delete the jth system.
14) End(j).
15) If $\|\bar{r}_s\|_2 \leq \varepsilon$, go to 17).
16) End(k).
17) End(l).

We now make a few description about the algorithm. We apply restarted GMRES method to solve the seed system, function Arnoldi applies the Arnoldi procedure to build an orthogonal basis $V_{m+1} = [v_1, ..., v_m]$ for the Krylov subspace $\mathcal{K}_{m+1}(A_s, r_s)$. The well-known relation $A_s V_m = V_{m+1}\bar{H}_m$ holds, where $\bar{H}_m$ is the upper Hessenberg matrix, then we can deduce the relation $A_j V_m = V_{m+1}\bar{H}_j$, where $\bar{H}_j = \bar{H}_m + (\sigma_s - \sigma_j)\begin{pmatrix} I_m \\ 0 \end{pmatrix}$. The solution of seed system is given by $\bar{x}_s = x_s + V_m d_s$, where $d_s = \arg\min_{d\in R^m} \left\|\beta e_1 - \bar{H}_m d\right\|_2$, and $\beta = \|r_s\|_2$. The solutions of the nonseed systems are approximated by projecting the residual $\bar{r}_j = b_j - A_j\bar{x}_j$ onto $\mathcal{K}_{m+1}(A_s, r_s)$, and solving the least squares problem $d_j = \arg\min_{d\in R^m} \left\|V_{m+1}^{\mathrm{T}} r_j - \bar{H}_j d\right\|_2$. Thus the approximate solution of the nonseed systems can be derived by $\bar{x}_j = x_j + V_m d_j$. After the current seed system is solved to desired accuracy, the whole process is repeated with another unsolved system as a seed until all the systems are solved.

III. GS-SHIFT METHOD WITH A SEED SELECTION STRATEGY

In this section, we propose a new seed projection method with a seed selection strategy to solve (1). As reported in [8,9], the seed plays an important role in the reduction of the norm of nonseed residuals. A good seed system should allow more information shareable among the nonseed systems. In order to let a seed system include more content of each system, the seed system can be chosen as a linear combination of current residuals. However, this will cause extra work to solve an artificial system. So we usually restrict the seed to one of the current residuals. One possible way is to choose the seed such that

$$\|r_s\|_2 \geq \|r_j\|_2, \quad j = 1, \cdots, p, \tag{4}$$

i.e.,

$$\|r_s\|_2 = \max_{1\leq j\leq p} \|r_j\|_2. \tag{5}$$

In our algorithm, referred to as GSS-Shift, we select the seed system according to (5). In generally, a new seed is selected when the current seed system is convergent. However, our algorithm select a new seed once the GMRES method, which we solve the solution of the seed system, restart. This alternative of seeds is a primary move for restarting the GMRES phase instead of restarting the GMRES phase till the seed system convergence. Details of the GSS-Shift algorithm can be described as follows.

Algorithm 2. GSS-Shift

1) $x_i = x_i^{(0)}$, $r_i = b_i - A_i x_i$, $i = 1, \cdots, p$.
2) $[s, r_s]$=SEED(R), where $R = [r_1, r_2, \cdots, r_p]$.
3) $\beta = \|r_s\|_2$.
4) While $\beta > \varepsilon$
5) $[V_{m+1}, H]$=Arnoldi(A_s, r_s).
6) $d_s = \arg\min_{d\in R^m} \left\|\beta e_1 - \bar{H}_m d\right\|_2$.
7) $\bar{x}_s = x_s + V_m d_s$.
8) $\bar{r}_s = b_s - A_s\bar{x}_s$.
9) For $j = 1, \cdots, p$, and $j \neq s$, compute all the approximate projection solutions of the nonseed systems
10) If $\|r_j\|_2 > \varepsilon$
11) $d_j = \arg\min_{d\in R^m} \left\|V_{m+1}^{\mathrm{T}} r_j - \bar{H}_j d\right\|_2$, where $\bar{H}_j = \bar{H}_m + (\sigma_s - \sigma_j)\begin{pmatrix} I_m \\ 0 \end{pmatrix}$.
12) $\bar{x}_j = x_j + V_m d_j$.
13) $\bar{r}_j = b_j - A_j\bar{x}_j$.
14) End(If).
15) End(j).
16) Set $x_i := \bar{x}_i$, $r_i := \bar{r}_i$, $i = 1, \cdots, p$.
17) $[s, r_s]$=SEED(R), $\beta = \|r_s\|_2$.
18) End(While).

We note that the function SEED is used to obtain the seed system by (5). When R is applied on the function SEED, returns s and r_s , where s is the index of column of R having the maximum norm.

The properties of residuals about GS-Shift algorithm also apply to our algorithm, details about the properties can be found in [7].

IV. NUMERICAL EXPERIMENTS

In this section, we give some experimental results of using the GSS-Shift algorithms to solve (1) and compare its performances with the GS-Shift method and the restarted GMRES(m) algorithm that applied on each system separately, denoted by G-Shift.

For all the methods, we use a restarted strategy every m iterations, i.e., we apply m steps of Arnoldi process every restarted. The initial guess is $X^{(0)} = 0$ and the stopping criterion is $\|r_j\|_2 < 10^{-6}$ for $j = 1, \cdots, p$. All codes are written in Matlab.

The experiments are based on the following matrix A and shift array $\Sigma = [\sigma_1, \cdots, \sigma_p]$, hence $A_i = A - \sigma_i I$. The matrix A in these problems is a bidiagonal matrix with 0.1 in each superdiagonal position. Set $A = (D; 0.1)$, where D is a diagonal matrix. In addition, $p = 5$ is set.
$A1 = (D1; 0.1)$, $D1 = diag(1, 2, \cdots, 1000)$;
$A2 = (D2; 0.1)$, $D2 = diag(-1, -2, -3, -4, 5, 6, \cdots, 1000)$.
$\Sigma(1) = [0.000, -0.001, -0.002, -0.003, -0.004]$;
$\Sigma(2) = [0.0, -0.1, -0.2, -0.3, -0.4]$;
$\Sigma(3) = [-0.072, -0.036, -0.018, -0.009, -0.009]$.
Two kinds of right-hand sides are chosen.
$B1 = [b_1, b_2, \cdots, b_p]$,
$(b_j)_i = -\cos(5\cos(t_i - 2(j-1)\pi/128))$, $j = 1, \cdots, p$, where $t_i = 1 + 0.1(i-1)$, $i = 1, \cdots, n$;
$B2 = [b_1, b_2, \cdots, b_p]$,
$(b_j)_i = j \cdot \cos((2j+i) \times 10^6) \cdot \sin((3(4-j)+i) \times 10^6)$, $i = 1, \cdots, n$; $j = 1, \cdots, p$.

TABLE I
NUMBER OF ITERATIONS TO CONVERGENCE OF EACH SYSTEM FOR $A = A1$ WITH $m = 15$

RHS	Σ	GSS-Shift	GS-Shift	G-Shift
$B1$	$\Sigma(1)$	(72)21,15,9,11,16	(96)34,25,17,10,10	(170)34,34,34,34,34
	$\Sigma(2)$	(71)32,9,10,7,13	(90)34,22,15,12,7	(148)34,32,29,27,26
	$\Sigma(3)$	(73)16,14,12,14,17	(97)32,24,17,12,12	(167)32,33,34,34,34
$B2$	$\Sigma(1)$	(79)4,15,21,7,32	(111)32,27,30,10,12	(168)32,32,37,36,31
	$\Sigma(2)$	(79)17,15,22,11,14	(117)32,25,26,17,17	(147)32,30,32,29,24
	$\Sigma(3)$	(83)7,14,17,11,34	(118)30,26,29,17,16	(163)30,31,36,35,31

TABLE II
NUMBER OF ITERATIONS TO CONVERGENCE OF EACH SYSTEM FOR $A = A2$ WITH $m = 25$

RHS	Σ	GSS-Shift	GS-Shift	G-Shift
$B1$	$\Sigma(1)$	(64)27,9,9,7,12	(128)46,33,24,15,10	(255)46,52,52,52,53
	$\Sigma(2)$	(92)10,5,9,9,59	(150)46,35,24,22,23	(295)46,56,58,66,69
	$\Sigma(3)$	(67)32,6,10,8,11	(130)48,35,17,17,13	(259)48,53,53,52,53
$B2$	$\Sigma(1)$	(68)2,9,21,5,31	(155)45,41,43,7,19	(260)45,50,58,54,53
	$\Sigma(2)$	(89)3,9,11,14,52	(198)45,43,45,28,37	(299)45,53,65,64,72
	$\Sigma(3)$	(73)6,9,27,5,26	(176)46,41,47,21,21	(262)46,51,58,54,53

In Tables 1 and 2, we list the number of iterations to convergence for each seed system to be refined. In the parentheses, we give the total number of iterations. The experiment result show that our method is more efficient than other two methods.

V. CONCLUSION

In this paper, we have proposed a new GMRES seed projection method for solving shifted systems with multiple right-hand sides. To define the new method, we use a seed selection strategy. Experimental results show that our method is effective than the GS-Shift method.

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