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A family of improved secant-like method with super-linear convergence

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Abstract—A family of improved secant-like method is proposed in this paper. Further, the analysis of the convergence shows that this method has super-linear convergence. Efficiency are demonstrated by numerical experiments when the choice of α is correct.

Keywords—Nonlinear equations, Secant method, Convergence order, Secant-like method.

I. Introduction

As we all know, Newton's method is an important and basic approach for solving nonlinear equations and its formulation is given by [1]–[4]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

which converges quadratically.

To improve the local order of convergence, a number of modified methods have been studied [6]–[12], but a potential problem in these method is the evaluation of the derivative. In many cases, it is expensive to compute the derivative, and the above methods are still restrict in practical applications. Thus it is useful to have another method, which does not require evaluation of derivative. Such a method is the Secant method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \tag{2}$$

which closely resembles with the Regula-falsi method. Also the Secant method is a variation of the Newton's method due to Newton's himself [5]. For a good review of these algorithms, some excellent textbooks are available (see [1], [2]). To improve this method, many modified methods called secant-like method have been proposed in [13]–[17].

In the present work, we propose a new type of hybrid technique by (6) proposed in [17], and then we present a family of improved iterative method for solving nonlinear equations without derivatives. Analysis of the convergence shows that the asymptotic convergence order of this method is $(1+\sqrt{5})/2$. The practical utility is demonstrated by numerical results.

II. Notation

Let $f(x)$ be a real function with a simple root x^* and let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers that converges to x^*.

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We say that the *order of convergence* is q if there exists a $q \in \mathbb{R}^+$ such that

$$\lim_{n\to+\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^q} = C \neq 0, \infty.$$

Let $e_n = x_n - x^*$ be the n-th iterate error. We call

$$e_{n+1} = Ce_n^q + \cdots, \tag{3}$$

the *error equation*. If we can obtain the error equation for the method, then the value of q is its order of convergence.

III. New method

In [17], Kanwar et al. drew a parabola with vertex at $(x_n, 0)$ and axis parallel to y-axis

$$y = \alpha(x - x_n)^2, \tag{4}$$

where α is the scaling parameter, such that this parabola intersects the approximated line to function $y = f(x)$ at the point $(x_{n+1}, f(x_{n+1}))$, which passing through $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$. The function of the approximated line is given by

$$y = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_n). \tag{5}$$

From (4) and (5), they obtained

$$\alpha(x_{n+1} - x_n)^2 = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x_{n+1} - x_n). \tag{6}$$

Solving (6) for x_{n+1} with quadratic formula yields the following iteration formula called Kanwar's method in [17]

$$\begin{aligned} x_{n+1} =& x_n - 2(x_n - x_{n-1})f(x_n) \div \{(f(x_n) - f(x_{n-1})) \\ & \pm\sqrt{(f(x_n) - f(x_{n-1}))^2 + 4\alpha(x_n - x_{n-1})^2 f(x_n)}\Big\} \end{aligned} \tag{7}$$

This method is also a family of secant-like method with super-linear convergence.

However, we find that the formula (7) includes the computation of square roots. It is likely to be difficult in some cases. In order to construct an improved method, we rewrite (6) by the following form

$$\alpha(x_{n+1} - x_n)^2 - \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x_{n+1} - x_n) - f(x_n) = 0. \tag{8}$$

To solve the equation (8) for x_{n+1}, we factor $x_{n+1} - x_n$ from the first two terms to obtain

$$\left[\alpha(x_{n+1} - x_n) - \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right](x_{n+1} - x_n) - f(x_n) = 0,$$

from which it follows that

$$x_{n+1} - x_n = -\frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}} - \alpha(x_{n+1} - x_n)}.$$

Approximating the difference x_{n+1} remaining on the right-hand side of above equation by secant method given by (2), we obtain a family of new secant-like method

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})(f(x_n) - f(x_{n-1}))}{[f(x_n) - f(x_{n-1})]^2 - \alpha f(x_n)(x_n - x_{n-1})^2}. \tag{9}$$

where α is the scaling parameter.

If we let $\alpha \to 0$, then (9) reduces to secant method. Relation (9) defined the secant-like method for solving nonlinear equations.

Now, we can call the Kanwar's method defined by (7) the irrational formula and the improved method defined by (9) the rational formula.

IV. Convergence analysis

For the the family of improved method defined by (9), we have the following analysis of convergence.

Theorem 1. *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $x^* \in D$. Let $f(x)$ have first, second and third derivatives in the interval D, then the asymptotic convergence of the method defined by (9) is $(1 + \sqrt{5})/2$.*

Proof: Let $e_n = x_n - x^*$. Using Taylor expansion and taking into account $f(x^*) = 0$, we have

$$f(x_n) = f'(x^*)\left[e_n + C_2 e_n^2 + C_3 e_n^3 + \cdots\right], \tag{10}$$

where $C_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}$, $k = 2, 3, \cdots$. Furthermore, we get

$$f(x_{n-1}) = f'(x^*)\left[e_{n-1} + C_2 e_{n-1}^2 + C_3 e_{n-1}^3 + \cdots\right], \tag{11}$$

$$\begin{aligned} &f(x_n) - f(x_{n-1}) \\ &= f'(x^*)\left[(e_n - e_{n-1}) + C_2(e_n^2 - e_{n-1}^2) + \cdots\right] \\ &= f'(x^*)(e_n - e_{n-1})\left[1 + C_2(e_n + e_{n-1}) + \cdots\right] \end{aligned} \tag{12}$$

From (9), (10), (11) and (12) , we obtain

$$\begin{aligned} e_{n+1} =& e_n - \left(e_n + C_2 e_n^2 + \cdots\right)\left[1 + C_2(e_n + e_{n-1}) + \cdots\right] \\ &\div \Big\{\left[1 + C_2(e_n + e_{n-1}) + \cdots\right]^2 \\ &- \frac{\alpha}{f'(x^*)}\left(e_n + C_2 e_n^2 + \cdots\right)\Big\} \\ =& e_n - e_n + 2C_2 e_n^2 + C_2 e_n e_{n-1} + \cdots \\ &\div \Big\{1 + \left(2C_2 - \frac{\alpha}{f'(x^*)}\right) e_n + 2C_2 e_{n-1} \\ &+ C_2^2(e_n + e_{n-1})^2 - \frac{\alpha}{f'(x^*)} C_2 e_n^2 + \cdots\Big\} \\ =& e_n - \left[e_n + 2C_2 e_n^2 + C_2 e_n e_{n-1} + \cdots\right] \\ &\times \Big\{1 - \left[\left(2C_2 - \frac{\alpha}{f'(x^*)}\right) e_n + 2C_2 e_{n-1}\right] \\ &+ \left[\left(2C_2 - \frac{\alpha}{f'(x^*)}\right) e_n + 2C_2 e_{n-1}\right]^2 + \cdots\Big\} \\ =& \left(4C_2 - \frac{\alpha}{f'(x^*)}\right) e_n^2 + 3C_2 e_n e_{n-1} + \cdots \\ =& 3C_2 e_n e_{n-1} + \cdots . \end{aligned} \tag{13}$$

Then (13) becomes

$$e_{n+1} = 3C_2 e_n e_{n-1} + \cdots . \tag{14}$$

From (3) we have

$$e_n = C e_{n-1}^q + \cdots \tag{15}$$

and

$$e_{n+1} = C e_n^q + \cdots = C^{q+1} e_{n-1}^{q^2} + \cdots . \tag{16}$$

Substituting (15) and (16) into (14) gives

$$C^{q+1} e_{n-1}^{q^2} = 3C_2 C e_{n-1}^{q+1} + \cdots \tag{17}$$

which implies that

$$q^2 - q - 1 = 0 \tag{18}$$

It is obtained from (18) that the asymptotic convergence order $(1 + \sqrt{5})/2$. ■

V. Numerical Example

Now, we employ the Kanwar's method defined by (7) proposed by Kanwar et al. in [17] and the present methods called Improved method defined by (9) to solve some nonlinear equations. All the examples in the current work are computed using MATLAB with double precision, the iterative method is stopped when $|f(x)| < 1e - 14$ or $|x_n - x_{n-1}| < 1e - 14$.

Example 1. Consider a nonlinear equation $f(x) = x^3 - e^{-x} = 0$ with initial guess $x_{-1} = 1$, $x_0 = 1.5$ and the parameter $\alpha = 1/4$. The roots of this problem is $x^* = -1.20764782713092$.

Example 2. Consider a nonlinear equation $f(x) = \cos x - x = 0$ with initial guess $x_{-1} = 0$, $x_0 = 1.5$ and the parameter $\alpha = 1/2$. The roots of this problem is $x^* = 0.739085133215161$.

Example 3. Consider a nonlinear equation $f(x) = \sin^2 x - x^2 + 1 = 0$ with initial guess $x_{-1} = 2$, $x_0 = 1.5$ and the parameter $\alpha = 0.1$. The roots of this problem is $x^* = 1.40449164821534$.

From the numerical results shown in Table I, II and III, it can be seen that this new method has super linear convergence and is superior to Kanwar's method. An interesting observation for Kanwar's method defined by (7) is that an additional conditions should be discussed during the iteration process. We should limit the iteration directions of Kanwar's method when doing iteration, if $f(x_n) - f(x_{n-1}) > 0$, we should using the positive sign in Kanwar's method (7), otherwise ,we should using the negative sign.

The numerical results also shows that the choice of α can affect the number of iterations of the Kanwar's method (7) and

TABLE I
COMPARISON OF KANWAR'S METHOD AND IMPROVED METHOD FOR EXAMPLE 1

	Kanwar's method		Improved method	
n	x_n	$f(x_n)$	x_n	$f(x_n)$
-1	1	0.632120558828558	1	0.632120558828558
0	1.5	3.15186983985157	1.5	3.15186983985157
1	0.879344204524759	0.264894577398489	0.85454038286769	0.198540238461398
2	0.822439406786893	0.116944852708724	0.811045660452703	0.0891086927319675
3	0.777508311247124	0.0104690415383335	0.775503388809559	0.00592008176193187
4	0.773091057163286	0.000469085983007156	0.772973354788399	0.000203743914093668
5	0.772883850230285	2.00825277235639e-006	0.772883177442087	4.91972071248181e-007
6	0.772882959320998	3.87163356929676e-010	0.772882959167484	4.11846667880411e-011
7	0.77288295914921	2.77555756156289e-016	0.77288295914921	0
8	0.77288295914921	0	0.77288295914921	0

TABLE II
COMPARISON OF KANWAR'S METHOD AND IMPROVED METHOD FOR EXAMPLE 2

	Kanwar's method		Improved method	
n	x_n	$f(x_n)$	x_n	$f(x_n)$
-1	0	1	0	1
0	1.5	-1.4292627983323	1.5	-1.4292627983323
1	0.547436329668064	0.306425387098374	0.806443324072783	-0.11437320284828
2	0.713709162514262	0.042229744908763	0.747060401562036	-0.013370952686709
3	0.740231323136458	-0.00191876255794388	0.739217253949664	-0.000221125301004599
4	0.739078529136091	1.10526500561869e-005	0.739085370076688	-3.96414321923722e-007
5	0.739085131541512	2.80103884531258e-009	0.739085133222087	-1.15918386001113e-011
6	0.739085133215163	-4.21884749357559e-015	0.739085133215161	0
7	0.739085133215161	1.11022302462516e-016	0.739085133215161	0

TABLE III
COMPARISON OF KANWAR'S METHOD AND IMPROVED METHOD FOR EXAMPLE 3

	Kanwar's method		Improved method	
n	x_n	$f(x_n)$	x_n	$f(x_n)$
-1	2	-2.17317818956819	2	-2.17317818956819
0	1.5	-0.255003751699777	1.5	-0.255003751699777
1	1.43350075457049	-0.0736563430257404	1.43364454266792	-0.0740296356595003
2	1.40648462754061	-0.00495524495120003	1.40652803323954	-0.00506333874252474
3	1.40453598195165	-0.000110061118280935	1.40453736051422	-0.000113483603850995
4	1.40449171733435	-1.71586070241148e-007	1.40449172113812	-1.81028830592567e-007
5	1.40449164821774	-5.96012128539769e-012	1.40449164821795	-6.48414655302076e-012
6	1.40449164821534	3.33066907387547e-016	1.40449164821534	-4.44089209850063e-016
7	1.40449164821534	-4.44089209850063e-016	1.40449164821534	3.33066907387547e-016

TABLE IV
THE NUMBERS OF KANWAR'S ITERATIONS (7) BY THE VALUES OF α.

			α					
Equation	x_0	x_1	10	2	1	0.1	0.01	Roots
$f_1 = \tan^{-1} x$	2	3	21	7	7	8	8	1.5707963267949
$f_2 = xe^{x^2} - \sin^2 x + 3\cos x + 5$	-2	-1	9	9	9	9	9	-1.20764782713092
$f_3 = x^3 - 2x - 5$	2	3	8	7	7	7	7	2.09455148154233
$f_4 = xe^x - 1$	1	2	10	9	9	9	9	0.567143290409784
$f_5 = xe^x - \cos x$	2	3	12	12	12	12	12	0.517757363682458

TABLE V
THE NUMBERS OF IMPROVED METHOD ITERATIONS (9) BY THE VALUES OF α.

			α					
Equation	x_0	x_1	10	2	1	0.1	0.01	Roots
$f_1 = \tan^{-1} x$	2	3	17	10	9	8	8	1.5707963267949
$f_2 = xe^{x^2} - \sin^2 x + 3\cos x + 5$	-2	-1	9	9	9	9	9	-1.20764782713092
$f_3 = x^3 - 2x - 5$	2	3	11	8	8	7	7	2.09455148154233
$f_4 = xe^x - 1$	1	2	fail	7	9	9	9	0.567143290409784
$f_5 = xe^x - \cos x$	2	3	10	11	11	11	12	0.517757363682458

the improved method (9). From the numerical results shown in Table IV and V, it can be seen that when the values of α greater, the number of iteration steps is greater; as α close to 0, the number of iteration consistent with the secant method.

VI. CONCLUSION

We present a family of iterative method for solving non-linear equations. Theorem 1 shows that the asymptotic convergence order of this method is $(1+\sqrt{5})/2$. The numerical results states that this family of iterative method has super linear convergence, and the choice of α can affect the number of iterations. This method requires no derivatives, so it is especially efficient when the computational cost of the derivative is expensive.

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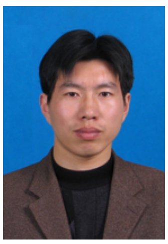

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