World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:7, No:2, 2013

Preconditioned Jacobi method for fuzzy linear systems

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Abstract—A preconditioned Jacobi (PJ) method is provided for solving fuzzy linear systems whose coefficient matrices are crisp M-matrices and the right-hand side columns are arbitrary fuzzy number vectors. The iterative algorithm is given for the preconditioned Jacobi method. The convergence is analyzed with convergence theorems. Numerical examples are given to illustrate the procedure and show the effectiveness and efficiency of the method.

Keywords—preconditioning, M-matrix, Jacobi method, fuzzy linear system (FLS).

I. Introduction

FUZZY linear systems (FLSs) are linear systems whose parameters are all or partially represented by fuzzy numbers. FLSs have many applications in control problems, information, physics, statistics, engineering, economics, finance and even social sciences. Therefore, it is important to establish mathematical models and numerical methods for solving FLSs.

Friedman et al. [11] suggested a general model for solving a class of $n \times n$ FLSs

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n, \end{cases} \tag{1}$$

where the coefficient matrix $A = (a_{ij})$ is a crisp matrix and y_i is a fuzzy number, $1 \le i,\ j \le n$. Many authors study numerical methods for solving FLS (1), such as Abbasbandy, Ezzati and Jafarian [1], [2], [3], [9], Allahviranloo [4], [5], [6], Dehghan and Hashemi [8], Fariborzi Araghi and Fallahzadeh [10], Liu [12], Miao, Wang and Zheng [13], [14], [17], [18], [19], Nasser, Matinfar and Sohrabi [15], Zhu, Joutsensalo and Hämäläinen [20].

In this paper, a preconditioned Jacobi method named PJ is provided for solving FLS (1) whose coefficient matrix is an M-matrix. Also, we compare the numerical results with Jacobi method given in [4].

The remainder of the paper is organized as follows. In Section 2, we give some basic definitions and results about fuzzy number and FLS. In Section 3, we propose the PJ method with the convergence theorems. Numerical examples are given in Section 4 to illustrate the method and the conclusion is drawn in Section 5.

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II. Preliminaries

Following [11], a fuzzy number is defined as $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, which satisfies,

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$,
- $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$,
- $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

To define a solution to the system (1) we should recall the arithmetic operations of arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$, $0 \le r \le 1$, and real number k,

(1) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,
(2) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$, and
(3) $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \ge 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 2.1: A fuzzy number vector $X = (x_1, x_2, \cdots, x_n)^T$ given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), \quad 1 \le i \le n, 0 \le r \le 1, \tag{2}$$

is called a solution of the fuzzy linear system (1) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i. \end{cases} \tag{3}$$

With (3), we can extend FLS (1) to a $2n \times 2n$ crisp linear system

$$SX = Y \tag{4}$$

where $S = (s_{kl})$, s_{kl} are determined as follows

$$\begin{array}{llll} a_{ij} \ge 0 & \Rightarrow & s_{ij} = a_{ij}, & s_{n+i,n+j} = a_{ij}, \\ a_{ij} < 0 & \Rightarrow & s_{i,n+j} = a_{ij}, & s_{n+i,j} = a_{ij}, \end{array} \quad 1 \le i,j \le n, \tag{5}$$

and any s_{kl} which is not determined by the above items is zero, $1 \le k,\ l \le 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}, Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ \overline{y}_1 \\ \vdots \\ \overline{y}_n \end{bmatrix}. \tag{6}$$

What's more, the matrix S has the structure $\begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$, $A = S_1 + S_2$, and (4) can be rewritten as follows

$$\begin{cases} S_1\underline{X} + S_2\overline{X} = \underline{Y}, \\ S_2\underline{X} + S_1\overline{X} = \overline{Y}, \end{cases} \tag{7}$$

where

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix}, \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix}, \underline{Y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_n \end{bmatrix}, \overline{Y} = \begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \vdots \\ \overline{y}_n \end{bmatrix}. \tag{8}$$

The following theorem indicates when FLS (1) has a unique solution.

Theorem 2.2: The matrix S is nonsingular if and only if the matrices $A = S_1 + S_2$ and $S_1 - S_2$ are both nonsingular. See [11].

Under the conditions of Theorem 2.2, the solution of (1) is thus unique but may still not be an appropriate fuzzy vector. Thus, we have the following definition.

Definition 2.3: Let $X = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \le i \le n\}$ denote the unique solution of (1) from (4). If $(\underline{x}_i(r), \overline{x}_i(r))$, $1 \le i \le n$ are all fuzzy numbers then X is called a strong solution; otherwise, X is called a weak solution.

To develop the preconditioned Jacobi method for FLS (1) whose coefficient matrix is an M-matrix, we first give the definition of M-matrix and some results about FLS (1) with M-matrix.

Definition 2.4 ([7]): Any matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is called a nonsingular M-matrix if $a_{ii} > 0$, $a_{ij} \le 0$ $(i \ne j)$, $1 \le i, j \le n$, and A^{-1} is nonnegative.

Theorem 2.5 ([18]): The matrix S in (4) is symmetric positive definite if and only if the matrices $A = S_1 + S_2$ and $S_1 - S_2$ are both symmetric positive definite.

In the next section, we consider a fuzzy linear system with $A = (a_{ij})$ which is an arbitrary $n\times n$ symmetric matrix having negative off-diagonal elements and positive row sums (positive column sums), i.e.,

$$\begin{cases} a_{ii} > 0 & i = 1, 2, \cdots, n, \\ a_{ij} = a_{ji} < 0 & i \ne j, 1 \le i, j \le n, \\ \sum_{k=1}^{n} a_{ik} > 0 & i = 1, 2, \cdots, n. \end{cases} \tag{9}$$

It is easy to be verified that such kind of matrix is a positive definite M-matrix, and we have the following result.

Theorem 2.6: Suppose the coefficient matrix A of a fuzzy linear system (1) satisfies (9), then the coefficient matrix S of the extended system (4) is also a positive definite M-matrix.

Proof: As A satisfying (9) is a positive definite M-matrix and $S_1 - S_2$ is positive definite, by Theorem 2.5, S is positive definite. According to the relation between A and S by (4), we can get

$$\begin{cases} s_{ii} > 0 & i = 1, 2, \cdots, 2n, \\ s_{ij} = s_{ji} \le 0 & i \ne j, 1 \le i, j \le 2n, \\ \sum_{k=1}^{2n} s_{ik} > 0 & i = 1, 2, \cdots, 2n. \end{cases} \tag{10}$$

Thus, we can easily verify that S is an M-matrix. The proof is completed. ■

III. The Preconditioned Jacobi Method

Suppose the coefficient matrix A of (1) satisfies (9), then the coefficient matrix $S = (s_{ij})$ of the extended system (4) meets (10). Let $P = (p_{ij})$ be the $2n \times 2n$ symmetric positive definite matrix as in [16] with the entries given by

$$p_{ij} = \frac{\delta_{ij}}{s_{ij}} + \frac{1}{\widetilde{s}}, \tag{11}$$

where δ_{ij} is the Kronecker delta function and

$$\widetilde{s} = \sum_{j=1}^{2n} \sum_{j=1}^{2n} s_{ij}. \tag{12}$$

In this way, the matrix P is a good approximating inverse of S, and

$$P = \frac{1}{\widetilde{s}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [1, 1, \cdots, 1] + \mathrm{diag}(1/\mathrm{s}_{11}, 1/\mathrm{s}_{22}, \cdots, 1/\mathrm{s}_{2\mathrm{n},2\mathrm{n}}). \tag{13}$$

Thus, it could be a good preconditioner for S.

We consider the preconditioned system $PSX = PY$ with the Jacobi method, referred to PJ method. Let $PS = D - L - U$, where

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}, L = \begin{bmatrix} L_1 & 0 \\ -S_2 & L_1 \end{bmatrix}, U = \begin{bmatrix} U_1 & -S_2 \\ 0 & U_1 \end{bmatrix}, \tag{14}$$

$D_1 = \mathrm{diag}(s_{ii})$, $i = 1, 2, \cdots n$, $D_1 - L_1 - U_1 = S_1$, and L_1 and U_1 are strictly lower and upper triangular matrices. The algorithm is as followed.

Algorithm 3.1: The PJ method can be implemented as follows:

Step 1. Calculate preconditioner P;

Step 2. Choose an initial vector X, calculate $R = PY - PSX$ and set $k = 0$;

Step 3. While $\|R\|_2 > \varepsilon \|Y\|_2$ and $k < k_{\max}$, do

$$\begin{cases} \underline{X}_{k+1} = D_1^{-1}(L_1 + U_1)\underline{X}_k - D_1^{-1} S_2 \overline{X}_k + D_1^{-1}\underline{Y}, \\ \overline{X}_{k+1} = D_1^{-1}(L_1 + U_1)\overline{X}_k - D_1^{-1} S_2 \underline{X}_k + D_1^{-1}\overline{Y}. \end{cases}$$

For Jacobi method, we have the following convergence theorem.

Theorem 3.2: If A satisfies (9), then the Jacobi method for the extended system (4) is convergent.

Proof: By theorem 2.6, S is symmetric positive definite and strictly diagonally dominant. Suppose D_S is the diagonal matrix of S, then $2D_S - S$ is strictly diagonally dominant and symmetric and its diagonal entries are positive, thus $2D_S - S$ is positive definite. Therefore, the Jacobi method for the extended system (4) is convergent. ■

For preconditioned Jacobi method, we have the following results.

Theorem 3.3: If PS is strictly diagonally dominant, then the PJ method for the extended system (4) is convergent.

Proof: For Jacobi method, if the coefficient matrix is strictly diagonally dominant, then the iterative scheme is convergent. ■

The numerical examples in the next section show that PJ method has a much faster convergence rate than Jacobi method.

TABLE 1
ITERATIONS (IT), CPU TIME (CPU) AND RELATIVE ERROR (ERR) FOR EXAMPLE 4.1

n	Jacobi						PJ					
	x_a			x_b			x_a			x_b		
	IT	CPU	ERR	IT	CPU	ERR	IT	CPU	ERR	IT	CPU	ERR
16	140	0.03	9.6e-7	140	0.03	9.6e-7	72	0.01	9.8e-7	92	0.02	9.6e-7
32	183	0.06	9.9e-7	183	0.06	9.9e-7	105	0.02	9.8e-7	137	0.03	9.2e-7
64	227	0.46	9.7e-7	227	0.45	9.7e-7	140	0.25	9.7e-7	185	0.34	9.4e-7
128	271	2.62	9.9e-7	271	2.61	9.9e-7	174	1.73	9.8e-7	232	2.32	9.8e-7
256	316	18.88	1.0e-6	316	18.51	1.0e-6	206	10.75	9.7e-7	278	14.88	9.6e-7
512	363	139.07	9.7e-7	363	138.83	9.7e-7	236	87.52	9.6e-7	321	119.87	9.7e-7

TABLE 2
ITERATIONS (IT), CPU TIME (CPU) AND RELATIVE ERROR (ERR) FOR EXAMPLE 4.2

n	Jacobi						PJ					
	x_a			x_b			x_a			x_b		
	IT	CPU	ERR	IT	CPU	ERR	IT	CPU	ERR	IT	CPU	ERR
16	98	0.03	9.6e-7	99	0.03	9.3e-7	46	0.01	8.3e-7	67	0.01	8.3e-7
32	196	0.09	9.4e-7	197	0.06	9.9e-7	80	0.03	9.5e-7	134	0.06	9.7e-7
64	390	0.73	9.9e-7	394	0.69	9.8e-7	135	0.21	9.8e-7	269	0.53	9.9e-7
128	779	7.22	1.0e-6	787	7.11	9.9e-7	218	2.11	9.8e-7	539	4.93	9.9e-7
256	1557	83.30	1.0e-6	1573	83.26	1.0e-6	329	19.47	1.0e-6	1079	48.77	9.9e-7
512	3114	1108.6	1.0e-6	3146	1118.9	1.0e-6	443	146.75	1.0e-6	2159	748.51	9.9e-7

IV. NUMERICAL EXAMPLES

We use MATLAB 7.12 to solve the preconditioned systems. In our experiments, the initial guess is zero and the stopping criterion is

$$\frac{\left\|R^{(k)}\right\|_2}{\left\|R^{(0)}\right\|_2} < 10^{-6}, \tag{15}$$

where $R^{(k)}$ is the residual vector after k iterations. In the tables, x_a and x_b mean that we solve $SX = Y$ as two numeric systems

$$S\begin{bmatrix} x_{a1} \\ x_{a2} \\ \vdots \\ x_{a,2n} \end{bmatrix} = \begin{bmatrix} y_{a1} \\ y_{a2} \\ \vdots \\ y_{a,2n} \end{bmatrix} and S\begin{bmatrix} x_{b1} \\ x_{b2} \\ \vdots \\ x_{b,2n} \end{bmatrix} = \begin{bmatrix} y_{b1} \\ y_{b2} \\ \vdots \\ y_{b,2n} \end{bmatrix} \tag{16}$$

not one symbolic system

$$S\begin{bmatrix} x_{a1} + x_{b1}r \\ x_{a2} + x_{b2}r \\ \vdots \\ x_{a,2n} + x_{b,2n}r \end{bmatrix} = \begin{bmatrix} y_{a1} + y_{b1}r \\ y_{a2} + y_{b2}r \\ \vdots \\ y_{a,2n} + y_{b,2n}r \end{bmatrix} \tag{17}$$

in the actual calculations.

Example 4.1: Consider $n \times n$ fuzzy linear system $Ax = y$ with

$$\begin{cases} a_{ij} = -\frac{1}{i}, & i < j, \\ a_{ij} = a_{ji}, & i > j, \quad 1 \le i, j \le n, \\ a_{ij} = -\sum_{k \ne i} a_{ik} + \frac{i}{n}, & i = j, \end{cases} \tag{18}$$

and

$$y = \begin{bmatrix} (1+r, 3-r) \\ (1+r, 3-r) \\ \vdots \\ (1+r, 3-r) \end{bmatrix}. \tag{19}$$

We can see that A is a positive definite M-matrix. Thus, the extended system $SX = Y$ also has a positive definite coefficient matrix S which is a strictly diagonally dominant M-matrix. By PJ and Jacobi methods, we have the results in Table 1.

Example 4.2: Consider $n \times n$ fuzzy linear system $Ax = y$ with

$$\begin{cases} a_{ij} = -\frac{1}{2} - \frac{i}{2n}, & i < j, \\ a_{ij} = a_{ji}, & i > j, \quad 1 \le i, j \le n, \\ a_{ij} = -\sum_{k \ne i} a_{ik} + 1 + \frac{i}{n}, & i = j, \end{cases} \tag{20}$$

and

$$y = \begin{bmatrix} (1+r, 3-r) \\ (2+r, 4-r) \\ \vdots \\ (n+r, (n+2)-r) \end{bmatrix}. \tag{21}$$

We can see that A is a positive definite M-matrix. Thus, the extended system $SX = Y$ also has a positive definite coefficient matrix S which is a strictly diagonally dominant M-matrix. By PJ and Jacobi methods, we have the results in Table 2.

Tables 1 and 2 give the number of iterations required for convergence, CPU time and relative error, which show that the PJ method is much better than Jacobi method.

V. CONCLUSION

We present a PJ (Preconditioned Jacobi) method for $n \times n$ fuzzy linear system. If the proposed matrix S by Friedman et al. [11] is a positive definite M-matrix and the preconditioned coefficient matrix PS is strictly diagonally dominant, then for any initial vector X_0, the PJ iteration will converge to the unique solution of $SX = Y$. The numerical results manifest that the method is effective and faster than Jacobi method.

ACKNOWLEDGMENT

K. Wang was supported by Shanghai College Teachers Visiting abroad for Advanced Study Program (B.60-A101-12-010), Shanghai Leading Academic Discipline Project (J50101) and a grant of "The First-class Discipline of Universities in Shanghai".

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