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A new robust stability criterion for dynamical neural networks with mixed time delays

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Abstract—In this paper, we investigate the problem of the existence, uniqueness and global asymptotic stability of the equilibrium point for a class of neural networks, the neutral system has mixed time delays and parameter uncertainties. Under the assumption that the activation functions are globally Lipschitz continuous, we drive a new criterion for the robust stability of a class of neural networks with time delays by utilizing the Lyapunov stability theorems and the Homomorphic mapping theorem. Numerical examples are given to illustrate the effectiveness and the advantage of the proposed main results.

Keywords—Neural networks, Delayed systems, Lyapunov function, Stability analysis.

I. INTRODUCTION

IN recent years, neural networks have been widely used in solving various classes of engineering problems such as control systems, optimization, image processing, associative memory design and signal processing. The key feature of the designed neural network, in such applications, is to be convergent. When a neural network is designed to function as an associative memory, it is desired that the neural network has multiple equilibrium points. Therefore, there has been a great deal of interest to the stability properties of neural networks in the past literature. When a neural network is employed to solve optimization problems, then the neural network must have unique equilibrium point which is globally asymptotically stable. But, in hardware implementation of neural networks, some parameters associated with the dynamical behavior of neural network may be subjected to some changes. Therefore, in order to be able to completely characterize equilibrium and stability properties of the neural network, we must take into account the delay parameters and uncertainties in the mathematical model of the neural network. In the recent literature, many papers have studied the existence, uniqueness and global robust asymptotic stability of the equilibrium point for different classes of delayed neural networks and presented various robust stability conditions [1]-[15]. In [1], the author have applied many methods to study the existence, uniqueness and global asymptotic stability of the equilibrium point for the class of neural networks with multiple time delays and parameter uncertainties, and got a new robust stability criterion. But the paper have conservation. In the current paper, we will present a new alternative sufficient condition for global robust stability of delayed neural networks with multiple and distributed time delays. At the end of this paper we will give two numerical examples to clarify the problem which we study.

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II. PROBLEM STATEMENT

The dynamical behavior of the neural network we consider is assumed to be governed by the following system of ordinary differential equations:

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) \\ &\quad + \sum_{j=1}^{n} d_{ij} \int_{t-\sigma}^{t} f_j(x_j(s))ds + u_i, i = 1,2,\cdots,n.\end{aligned} \tag{1}$$

Where n is the number of the neurons, $x_i(t)$ denotes the state of the neuron i at time t, $f_i(\cdot)$ denotes activation functions, a_{ij}, b_{ij} and d_{ij} denote the strengths of connectivity between neurons j and i; τ_{ij} and σ represent the time delay required in transmitting a signal from the neuron i, u_i is the constant input to the neuron i, c_i is the charging rate for the neuron i.

In order to complete characterize the equilibrium and stability properties of the neural network defined by (1), it will be assume that the parameters a_{ij} and b_{ij} and c_i and d_{ij} of neural system (1) are uncertain and have bounded norms with being defined in the following intervals:

$$\begin{aligned} C_I &:= \{C = diag(c_i) : 0 < \underline{C} \leq C \leq \overline{C}, \\ &\quad i.e., 0 < \underline{c_i} \leq c_i \leq \overline{c_i}, i = 1,2,\cdots,n\} \\ A_I &:= \{A = (a_{ij})_{n\times n} : 0 < \underline{A} \leq A \leq \overline{A}, \\ &\quad i.e., 0 < \underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}, i,j = 1,2,\cdots,n\} \\ B_I &:= \{B = (b_{ij})_{n\times n} : 0 < \underline{B} \leq B \leq \overline{B}, \\ &\quad i.e., 0 < \underline{b_{ij}} \leq b_{ij} \leq \overline{b}_{ij}, i,j = 1,2,\cdots,n\} \\ D_I &:= \{D = (d_{ij})_{n\times n} : 0 < \underline{D} \leq D \leq \overline{D}, \\ &\quad i.e., 0 < \underline{d_{ij}} \leq d_{ij} \leq \overline{d}_{ij}, i,j = 1,2,\cdots,n\} \end{aligned} \tag{2}$$

In order to achieve the task of finding conditions that ensure robust stability of neural network (1), we need to prove that the conditions to be obtained must guarantee that the unique equilibrium point of system (1) is globally asymptotically stable for all $C \in C_I, A \in A_I, B \in B_I, D \in D_I$. Therefor, our main goal will be studying the dynamical analysis of neural network (1) under the parameter uncertainties defined by (2).

We will assume that the functions f_i are Lipschitz conditions satisfying

$$| f_i(x)-f_i(y) |\leq \ell_i | x-y |, i=1,2,\cdots,n, \forall x,y \in R, x \neq y.$$

Where $\ell_i > 0$ denotes a Lipschitz constant. This class of function is denote by $f \in \mathcal{L}$.

In order to obtain our robust stability result, we will need to make use of some commonly used vector and matrix norms. Let $v = (v_1, v_2, \cdots, v_n)^T$ be a vector of dimension n and Q=$(q_{ij})_{n\times n}$ be a real $n \times n$ matrix. For $v = (v_1, v_2, \cdots, v_n)^T$ and Q=$(q_{ij})_{n\times n}$, $| v |$ will be denote $| v |= (| v_1 |, | v_2 |, \cdots, | v_n |)^T$ and $| S |$ will be denote $| Q |= (| q_{ij} |)_{n\times n}$. Then, we consider the following norms:

$$\begin{aligned}
&\| v \|_1 = \textstyle\sum_{i=1}^n | v_i |,\\
&\| v \|_2 = \{\textstyle\sum_{i=1}^n | v_i |^2\}^{\frac{1}{2}},\\
&\| v \|_\infty = \max_{1\leq i\leq n} | v_i |,\\
&\| Q \|_1 = \max_{1\leq i\leq n} \textstyle\sum_{j=1}^n | q_{ji} |,\\
&\| Q \|_2 = [\lambda_{max}(Q^T Q)]^{\frac{1}{2}},\\
&\| Q \|_\infty = \max_{1\leq i\leq n} \textstyle\sum_{j=1}^n | q_{ij} | .
\end{aligned}$$

Lemma 1 [2]. If the map $H(x) \in C^0$ satisfies the following conditions:

$$\begin{aligned}
&(i) H(x) \neq H(y) for all x \neq y,\\
&(ii) \| H(x) \| \to \infty as \| H(x) \| \to \infty,
\end{aligned}$$

then, $H(x)$ is homeomorphism of R^n.

Lemma 2 [3]. Let A be any real matrix such that$A \in A_I$ with A_I being defined as

$$\begin{aligned}
A_I := \{&A = (a_{ij})_{n\times n} : 0 < \underline{A} \leq A \leq \overline{A},\\
&i.e., 0 < \underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}, i,j = 1,2,\cdots,n\}
\end{aligned}$$

Then, the following inequality holds:

$$\| A \|_2 \leq \| \hat{A} \|_2$$

where $\hat{A} = (\hat{a_{ij}})_{n\times n}$ with $\hat{a_{ij}} = \max\{| \underline{a_{ij}} |, | \overline{a_{ij}} |\}$.

Lemma 3 [4]. Let A be any real matrix such that $A \in A_I$ with A_I being defined as

$$\begin{aligned}
A_I := \{&A = (a_{ij})_{n\times n} : 0 < \underline{A} \leq A \leq \overline{A},\\
&i.e., 0 < \underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}, i,j = 1,2,\cdots,n\}
\end{aligned}$$

Then, the following inequality holds:

$$\| A \|_2 \leq \sqrt{\| A^* \|_2^2 + \| A_* \|_2^2 + 2 \| A_*^T | A^* | \|_2}$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A}), A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

Lemma 4 [4]. Let A be any real matrix such that $A \in A_i$ with A_i being defined as

$$\begin{aligned}
A_I := \{&A = (a_{ij})_{n\times n} : 0 < \underline{A} \leq A \leq \overline{A},\\
&i.e., 0 < \underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}}, i,j = 1,2,\cdots,n\}.
\end{aligned}$$

Then, the following inequality holds:

$$\| A \|_2 \leq \| A^* \|_2 + \| A_* \|_2$$

where$A^* = \frac{1}{2}(\overline{A} + \underline{A}), A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

Lemma 1 is very important for the prove of the existence and uniqueness of equilibrium point of the neural network defined by (1). And others define different upper bound norms for the interval matrices, which will play key roles in the following proofs.

III. Main result

Now, we can studying the robust stability of the equilibrium point defined by (1).

1. Existence and uniqueness of equilibrium point.

Theorem 1. For the neural network defined by (1), assume that the network parameters satisfy (2) and $f \in \mathcal{L}$. Then, the neural network model (1) has a unique equilibrium point for every input vector $u = [u_1, u_2, \cdots, u_n]^T$, if the following condition holds:

$$\begin{aligned}
\varepsilon = c_m - \ell_M \| Q \|_2 - \ell_M \sqrt{\| \hat{B} \|_1 \| \hat{B} \|_\infty}\\
-\lambda_{max}(D^*) > 0
\end{aligned}$$

where $c_m = \min(\underline{c_i})$, $\ell_M = \max(\ell_i)$, $\| Q \|_2 = \min\{\| A^* \|_2 + \| A_* \|_2, \sqrt{\| A^* \|_2^2 + \| A_* \|_2^2 + 2 \| A_*^T | A^* | \|_2}, \| \hat{A} \|_2\}$, with $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $\hat{A} = (\hat{a}_{ij})_{n\times n}$, $\hat{a}_{ij} = \max\{| \underline{a}_{ij} |, | \overline{a}_{ij} |\}$, $\hat{B} = (\hat{b}_{ij})_{n\times n}$, with $\hat{b}_{ij} = max(| \underline{b}_{ij} |, | \overline{b}_{ij} |)$, $D^* = (d_{ij}^*)_{n\times n}$, $d_{ij}^* = \frac{|d_{ij}|\sigma\ell_j + |d_{ji}|\sigma\ell_i}{2}$, λ_i is eigenvalue of D^*, and $\lambda_{max} = \max_{1\leq i\leq n}\{\lambda_i\}, i = 1,2,\cdots,n$.

Proof: In order to proceed with proof of the existence and uniqueness,we consider the following mapping associated with system (1).

$$H(x) = -Cx + Af(x) + Bf(x) + D\int_{t-\sigma}^t f(x(s))ds + u \quad (3)$$

Let x^* be an equilibrium point of Eq.(1),then we have

$$H(x^*) = -Cx^* + Af(x^*) + Bf(x^*) + D\sigma f(x^*) + u = 0$$

Note that $H(x) = 0$ is an equilibrium point of (1). Therefor, it follows from Lemma 1 that, for the system defined by (1), there exist a unique equilibrium point for every input vector u if $H(x)$ is homeomorphism of R^n. We now prove that $H(x)$ is homeomorphism of R^n. we choose two vectors $x, y \in R^n$ such that $x \neq y$. For $H(x)$ defined by (3), we can write

$$\begin{aligned}
H(x) - H(y) = &-C(x-y) + A(f(x) - f(y))\\
&+B(f(x) - f(y)) + D\sigma(f(x) - f(y))
\end{aligned}$$

Next, we multiply both sides by $(x-y)^T$, we get

$$\begin{aligned}
(x-y)^T&(H(x) - H(y)) = -(x-y)^T C(x-y)\\
&+(x-y)^T A(f(x) - f(y)) + (x-y)^T B(f(x) - f(y))\\
&+(x-y)^T D\sigma(f(x) - f(y))\\
&= -\textstyle\sum_{i=1}^n c_i(x_i - y_i)^2 + (x-y)^T A(f(x) - f(y))\\
&+\textstyle\sum_{i=1}^n \sum_{j=1}^n b_{ij}(x_i - y_i)(f_j(x_j) - f_j(y_j))\\
&+(x-y)^T D\sigma(f(x) - f(y)).
\end{aligned} \quad (4)$$

We can write the following inequalities

$$\begin{aligned}
-\textstyle\sum_{i=1}^n c_i(x_i - y_i)^2 &\leq -\textstyle\sum_{i=1}^n \underline{c_i}(x_i - y_i)^2\\
&\leq -c_m \textstyle\sum_{i=1}^n (x_i - y_i)^2 = -c_m \| x - y \|_2^2
\end{aligned} \quad (5)$$

We also get that

$$\begin{aligned}
(x-y)^T &A(f(x) - f(y))\\
&\leq \| A \|_2 \| x - y \|_2 \| f(x) - f(y) \|_2\\
&\leq \ell_M \| A \|_2 \| x - y \|_2^2\\
&\leq \ell_M \| Q \|_2 \| x - y \|_2^2
\end{aligned} \quad (6)$$

and

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^n b_{ij}(x_i - y_i)(f_j(x_j) - f_j(y_j)) \\
&\quad \le \sum_{i=1}^n \sum_{j=1}^n \mid b_{ij} \mid\mid x_i - y_i \mid\mid f_j(x_j) - f_j(y_j) \mid \\
&\quad \le \sum_{i=1}^n \sum_{j=1}^n \mid b_{ij} \mid \ell_j \mid x_i - y_i \mid\mid x_j - y_j \mid \\
&\quad \le \ell_M \sum_{i=1}^n \sum_{j=1}^n \mid b_{ij} \mid \tfrac{1}{2}\Big(\sqrt{\tfrac{\|B\|_1}{\|B\|_\infty}}(x_i - y_i)^2 \\
&\quad + \sqrt{\tfrac{\|B\|_\infty}{\|B\|_1}}(x_j - y_j)^2\Big) \\
&\quad = \tfrac{1}{2}\ell_M \sum_{i=1}^n \sum_{j=1}^n \Big(\mid b_{ij} \mid \sqrt{\tfrac{\|B\|_1}{\|B\|_\infty}} \\
&\quad + \mid b_{ji} \mid \sqrt{\tfrac{\|B\|_\infty}{\|B\|_1}}\Big)(x_i - y_i)^2 \\
&\quad \le \tfrac{1}{2}\ell_M\Big(\| B \|_\infty \sqrt{\tfrac{\|B\|_1}{\|B\|_\infty}} + \| B \|_1 \sqrt{\tfrac{\|B\|_\infty}{\|B\|_1}}\Big) \| x - y \|_2^2 \\
&\quad = \ell_M \sqrt{\| B \|_1 \| B \|_\infty} \| x - y \|_2^2
\end{aligned}$$

From (2), it is easy to get $\|B\|_1 \le \|\hat{B}\|_1$ and $\|B\|_\infty \le \|\hat{B}\|_\infty$. So,we obtain the following inequalities

$$\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^n b_{ij}(x_i - y_i)(f_j(x_j) - f_j(y_j)) \\
&\quad \le \ell_M \sqrt{\|\hat{B}\|_1 \|\hat{B}\|_\infty} \|x - y\|_2^2
\end{aligned} \tag{7}$$

$$\begin{aligned}
&(x-y)^T D\sigma(f(x) - f(y)) \\
&= \sum_{i=1}^n \sum_{j=1}^n (x_i - y_i) d_{ij} \sigma(f_j(x_j) - f_j(y_j)) \\
&\le \sum_{i=1}^n \sum_{j=1}^n \mid x_i - y_i \mid\mid d_{ij} \mid \sigma \mid f_j(x_j) - f_j(y_j) \mid \\
&\le \sum_{i=1}^n \sum_{j=1}^n \mid x_i - y_i \mid\mid d_{ij} \mid \sigma \ell_j \mid x_i - y_i \mid \\
&= \begin{pmatrix} \mid x_1 - y_1 \mid \\ \mid x_2 - y_2 \mid \\ \vdots \\ \mid x_n - y_n \mid \end{pmatrix}^T D^* \begin{pmatrix} \mid x_1 - y_1 \mid \\ \mid x_2 - y_2 \mid \\ \vdots \\ \mid x_n - y_n \mid \end{pmatrix} \\
&\le \lambda_{max}(D^*) \| x - y \|_2^2
\end{aligned} \tag{8}$$

Then, using (5)-(8) in (4) yields

$$\begin{aligned}
&(x-y)^T(H(x) - H(y)) \le -c_m \|x - y\|_2^2 \\
&\quad + \ell_M \|Q\|_2 \|x - y\|_2^2 + \ell_M \sqrt{\|\hat{B}\|_1 \|\hat{B}\|_\infty} \|x - y\|_2^2 \\
&\quad + \lambda_{max}(D^*) \|x - y\|_2^2 \\
&\quad = -(c_m - \ell_M \|Q\|_2 - \ell_M \sqrt{\|\hat{B}\|_1 \|\hat{B}\|_\infty} \\
&\quad - \lambda_{max}(D^*)) \|x - y\|_2^2 \\
&\quad = -\varepsilon \|x - y\|_2^2.
\end{aligned}$$

For $x \neq y$ and $\varepsilon > 0$, we got

$$(x-y)^T(H(x) - H(y)) < 0 \tag{9}$$

Thus, from which we can conclude that if $x \neq y$, then $H(x) \neq H(y)$.

Let $y = 0$ in (9), we have

$$x^T(H(x) - H(0)) \le -\varepsilon \|x\|_2^2.$$

Taking the absolute value of both sides, we have

$$\mid x^T(H(x) - H(0)) \mid \ge \varepsilon \|x\|_2^2.$$

Since,

$$\mid x^T(H(x) - H(0)) \mid \le \|x\|_\infty \|H(x) - H(0)\|_1.$$

$$\|x\|_\infty \|H(x) - H(0)\|_1 \ge \varepsilon \|x\|_2^2.$$

And,$\|x\|_\infty \le \|x\|_2$, so

$$\|H(x) - H(0)\|_1 \ge \varepsilon \|x\|_2$$

Then, we have

$$\|H(x)\|_1 + \|H(0)\|_1 \ge \varepsilon \|x\|_2$$

$$\|H(x)\|_1 \ge \varepsilon \|x\|_2 - \|H(0)\|_1.$$

Since,$\|H(0)\|_1$ is finite, we can get the conclusion that if $\|x\| \to \infty$, that $\|H(x)\| \to \infty$. By Lemma 1, it is easy to know that $H(x)$ is homeomorphism of R^n. So, we have completed the proof of the existence and uniqueness of the equilibrium point for the neural networks defined by (1). ■

2. Stability of equilibrium point.

In the section 2, we have completed the proof of the existence and uniqueness of the equilibrium point for the networks defined by (1). Next, we begin to proof stability of equilibrium point for the system (1). We will first simplify system (1) as follows: we let $z_i(\cdot) = x_i(\cdot) - x_i^*$, $i = 1, 2, \cdots, n$, and note that the $z_i(\cdot)$ are governed by:

$$\begin{aligned}
\dot{z}_i(t) &= -c_i z_i(t) + \sum_{j=1}^n a_{ij} g_j(z_j(t)) \\
&\quad + \sum_{j=1}^n b_{ij} g_j(z_j(t - \tau_{ij})) \\
&\quad + \sum_{j=1}^n d_{ij} \int_{t-\sigma}^t g_j(z_j(s)) ds \\
&\quad i = 1, 2, \cdots, n.
\end{aligned} \tag{10}$$

where $g_i(z_i(\cdot)) = f_i(z_i(\cdot) + x_i^*) - f_i(x_i^*), i = 1, 2, \cdots, n$, and $g_i \in \mathcal{L}$. We also note that

$$\mid g_i(z) \mid \le \ell_i \mid z \mid, g_i(0) = 0, i = 1, 2, \cdots, n.$$

Here, note that $z \to 0$ implies that $x \to x^*$. It is sufficient to prove the stability of the transformed system (10) instead of considering the stability of x^* of system (1).

Theorem 2. For the neural network defined by (1), assume that the network parameters satisfy (2) and $f \in \mathcal{L}$. Then, the neural network model (10) is globally asymptotically stably if the following condition holds:

$$\begin{aligned}
\varepsilon = c_m - \ell_M \| Q \|_2 - \ell_M \sqrt{\| \hat{B} \|_1 \| \hat{B} \|_\infty} \\
- \lambda_{max}(D^*) > 0
\end{aligned}$$

where $c_m = \min(\underline{c_i})$, $\ell_M = \max(\ell_i)$, $\| Q \|_2 = \min\{\| A^* \|_2 + \| A_* \|_2, \sqrt{\| A^* \|_2^2 + \| A_* \|_2^2 + 2 \| A_*^T \mid A^* \| \|_2}, \| \hat{A} \|_2\}$, with $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $\hat{A} = (\hat{a}_{ij})_{n\times n}$, $\hat{a}_{ij} = \max\{\mid \underline{a}_{ij} \mid, \mid \overline{a}_{ij} \mid\}$, $\hat{B} = (\hat{b}_{ij})_{n\times n}$, with $\hat{b}_{ij} = max(\mid \underline{b}_{ij} \mid, \mid \overline{b}_{ij} \mid)$, $D^* = (d_{ij}^*)_{n\times n}$, $d_{ij}^* = \frac{|d_{ij}|\sigma\ell_j + |d_{ji}|\sigma\ell_i}{2}$, λ_i is eigenvalue of D^*, and $\lambda_{max} = \max_{1\le i\le n}\{\lambda_i\}, i = 1, 2, \cdots, n$.

Proof: We construct the follow positive definite Lyapunov functional:

$$\begin{aligned}
V_1(z(t)) &= \tfrac{1}{2} \sum_{i=1}^n z_i^2(t) \\
&\quad + \tfrac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \ell_M \mid b_{ij} \mid \beta \int_{t-\tau_{ij}}^t z_j^2(s) ds \\
V_2(z(t)) &= \tfrac{1}{2\sigma} \int_{t-\sigma}^t \int_s^t \mid z(l) \mid^T D^* \mid z(l) \mid dl ds \\
V(z(t)) &= V_1(z(t)) + V_2(z(t))
\end{aligned}$$

where β are positive constants to be determined later. The time derivative of the functional along the trajectories of system (10) is obtained as follows:

$$\begin{aligned}\dot{V}_1(z(t)) = &-\sum_{i=1}^n c_i z_i^2(t)\\ &+\sum_{i=1}^n\sum_{j=1}^n a_{ij} z_i(t) g_j(z_j(t))\\ &+\sum_{i=1}^n\sum_{j=1}^n b_{ij} z_i(t) g_j(z_j(t-\tau_{ij}))\\ &+\sum_{i=1}^n\sum_{j=1}^n d_{ij} z_i(t)\int_{t-\sigma}^t g_j(z_j(s))ds\\ &+\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \ell_M \mid b_{ij}\mid \beta(z_j^2(t)-z_j^2(t-\tau_{ij}))\end{aligned} \tag{11}$$

Then, we can write the following inequalities:

$$\begin{aligned}-\sum_{i=1}^n c_i z_i^2(t) &\le -\sum_{i=1}^n \underline{c}_i z_i^2(t)\\ &\le -\sum_{i=1}^n c_m z_i^2(t)\\ &= -c_m \parallel z(t)\parallel_2^2\end{aligned} \tag{12}$$

and

$$\begin{aligned}\sum_{i=1}^n\sum_{j=1}^n a_{ij} z_i(t) g_j(z_j(t)) &= z^T(t) A g(z(t))\\ &\le \|z(t)\|_2\|A\|_2\|g(z(t))\|_2\\ &\le \ell_M\|A\|_2\|z(t)\|_2^2\\ &\le \ell_M\|Q\|_2\|z(t)\|_2^2\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\sum_{i=1}^n\sum_{j=1}^n b_{ij} z_i(t) g_j(z_j(t-\tau_{ij}))\\ &\le \sum_{i=1}^n\sum_{j=1}^n \mid b_{ij} z_i(t) g_j(z_j(t-\tau_{ij}))\mid\\ &\le \sum_{i=1}^n\sum_{j=1}^n \ell_M \mid b_{ij}\mid\mid z_i(t)\mid\mid z_j(t-\tau_{ij})\mid\\ &\le \frac{1}{2}\ell_M\sum_{i=1}^n\sum_{j=1}^n \mid b_{ij}\mid (\beta z_j^2(t-\tau_{ij})+\frac{1}{\beta}z_i^2(t))\end{aligned}$$

Thus,we can get

$$\begin{aligned}&\sum_{i=1}^n\sum_{j=1}^n b_{ij} z_i(t) g_j(z_j(t-\tau_{ij}))\\ &+\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \ell_M\mid b_{ij}\mid \beta(z_j^2(t)-z_j^2(t-\tau_{ij}))\\ &=\frac{1}{2}\ell_M\sum_{i=1}^n\sum_{j=1}^n(\mid b_{ij}\mid \beta z_j^2(t)+\mid b_{ij}\mid \frac{1}{\beta}z_i^2(t))\\ &=\frac{1}{2}\ell_M\sum_{i=1}^n\sum_{j=1}^n(\frac{1}{\beta}\mid b_{ij}\mid +\beta\mid b_{ji}\mid)z_i^2(t)\end{aligned}$$

Let $\beta=\sqrt{\frac{\|B\|_\infty}{\|B\|_1}}$, we have

$$\begin{aligned}&\sum_{i=1}^n\sum_{j=1}^n b_{ij} z_i(t) g_j(z_j(t-\tau_{ij}))\\ &+\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \ell_M\mid b_{ij}\mid \beta(z_j^2(t)-z_j^2(t-\tau_{ij}))\\ &=\frac{1}{2}\ell_M(\sqrt{\frac{\|B\|_\infty}{\|B\|_1}}\|B\|_1+\sqrt{\frac{\|B\|_1}{\|B\|_\infty}}\|B\|_\infty)\|z(t)\|_2^2\\ &=\ell_M\sqrt{\|B\|_1\|B\|_\infty}\|z(t)\|_2^2\\ &\le \ell_M\sqrt{\|\hat{B}\|_1\|\hat{B}\|_\infty}\|z(t)\|_2^2\end{aligned} \tag{14}$$

then

$$\begin{aligned}&\sum_{i=1}^n\sum_{j=1}^n d_{ij} z_i(t)\int_{t-\sigma}^t g_j(z_j(s))ds\\ &\le \sum_{i=1}^n\sum_{j=1}^n\int_{t-\sigma}^t \mid z(t)\mid\mid d_{ij}\mid \ell_j\mid z_j(s)\mid ds\\ &=\int_{t-\sigma}^t\begin{pmatrix}\mid z_1(t)\mid\\ \mid z_2(t)\mid\\ \vdots\\ \mid z_n(t)\mid\end{pmatrix}^T \frac{D^*}{\sigma}\begin{pmatrix}\mid z_1(s)\mid\\ \mid z_2(s)\mid\\ \vdots\\ \mid z_n(s)\mid\end{pmatrix}ds\\ &\le \int_{t-\sigma}^t\frac{1}{2\sigma}\Bigg(\begin{pmatrix}\mid z_1(t)\mid\\ \mid z_2(t)\mid\\ \vdots\\ \mid z_n(t)\mid\end{pmatrix}^T D^*\begin{pmatrix}\mid z_1(t)\mid\\ \mid z_2(t)\mid\\ \vdots\\ \mid z_n(t)\mid\end{pmatrix}\end{aligned} \tag{15}$$

$$\begin{aligned}&+\begin{pmatrix}\mid z_1(s)\mid\\ \mid z_2(s)\mid\\ \vdots\\ \mid z_n(s)\mid\end{pmatrix}^T D^*\begin{pmatrix}\mid z_1(s)\mid\\ \mid z_2(s)\mid\\ \vdots\\ \mid z_n(s)\mid\end{pmatrix}\Bigg)ds\\ &=\frac{1}{2}\mid z(t)\mid^T D^*\mid z(t)\mid\\ &+\frac{1}{2\sigma}\int_{t-\sigma}^t\mid z(s)\mid^T D^*\mid z(s)\mid ds\end{aligned} \tag{16}$$

$$\begin{aligned}\dot{V}_2(z(t)) = &\frac{1}{2}\mid z(t)\mid^T D^*\mid z(t)\mid\\ &-\frac{1}{2\sigma}\int_{t-\sigma}^t\mid z(s)\mid^T D^*\mid z(s)\mid ds\end{aligned} \tag{17}$$

Using (11)-(16), we have

$$\begin{aligned}\dot{V}(z(t)) &= \dot{V}_1(z(t))+\dot{V}_2(z(t))\\ &= -c_m\parallel z(t)\parallel_2^2+\ell_M\parallel Q\parallel_2\parallel z(t)\parallel_2^2\\ &+\ell_M\sqrt{\parallel\hat{B}\parallel_1\parallel\hat{B}\parallel_\infty}\parallel z(t)\parallel_2^2+\mid z(t)\mid^T D^*\mid z(t)\mid\\ &\le -c_m\parallel z(t)\parallel_2^2+\ell_M\parallel Q\parallel_2\parallel z(t)\parallel_2^2\\ &+\ell_M\sqrt{\parallel\hat{B}\parallel_1\parallel\hat{B}\parallel_\infty}\parallel z(t)\parallel_2^2+\lambda_{max}(D^*)\parallel z(t)\parallel_2^2\\ &= -(c_m-\ell_M\parallel Q\parallel_2-\ell_M\sqrt{\parallel\hat{B}\parallel_1\parallel\hat{B}\parallel_\infty}\\ &-\lambda_{max}(D^*))\parallel z(t)\parallel_2^2\\ &= -\varepsilon\parallel z(t)\parallel_2^2\end{aligned}$$

From which we can observe that $\dot{V}(z(t)) < 0$, for all $z(t)\neq 0$.Assume that $z(t)=0$, we can get $\dot{V}(z(t))<0$ easily. If $z(t)=0$ and $z_j(t-\tau_{ij})=0$ for all i, j implying that $\dot{V}(z(t))=0$, and $\dot{V}(z(t))<0$ otherwise. It is easy to prove that $V(z(t))$ is radially unbound since $V(z(t))\to\infty$ as $\|z(t)\|\to\infty$. So,from Lyapunav theorems, we can conclude that the system (3) or equivalently the equilibrium point of system (1) is globally asymptotically stable. ■

IV. Examples

Example 1. According to the model (1), giving the the following matrices:

$$\underline{A}=\begin{bmatrix}-1&-1&-1&-1\\-1&-1&-1&-1\\-1&-1&-1&-1\\-1&-1&-1&-1\end{bmatrix},\overline{A}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix},$$

$$\overline{B}=\begin{bmatrix}1&2&2&4\\1&2&2&4\\1&2&2&4\\1&2&2&4\end{bmatrix}\underline{B}=\begin{bmatrix}-1&-2&-2&-4\\-1&-2&-2&-4\\-1&-2&-2&-4\\-1&-2&-2&-4\end{bmatrix},$$

$$D=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

$$\ell_1=\ell_2=\ell_3=\ell_4=1,\underline{c}_1=\underline{c}_2=\underline{c}_3=\underline{c}_4=1,\sigma=1.$$

We can calculate the follow matrices:

$$A^*=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},A_*=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix},$$

$$\hat{A}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\hat{B}=\begin{bmatrix}1&2&2&4\\1&2&2&4\\1&2&2&4\\1&2&2&4\end{bmatrix},$$

$$D^*=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

from which we can directly calculate the norms: $\| A^* \|_2 + \| A_2 \|_2 = 4$, $\sqrt{\| A^* \|_2^2 + \| A_* \|_2^2 + 2 \| A_*^T \mid A^* \| \|_2} = 4$, $\| \hat{A} \|_2 = 4. So \| q \|_2 = 4$. We can also get that $\| \hat{B} \|_1 = 16$, $\| \hat{B} \|_\infty = 9$, $\lambda_{max}(D^*) = 4$.

Firstly, we apply the result of Theorem 1 to the neural network employing the network parameters of this example, we get

$$\varepsilon = c_m - \ell_M \| Q \|_2 - \ell_M \sqrt{\| \hat{B} \|_1 \| \hat{B} \|_\infty} - \lambda_{max}(D^*)$$
$$= c_m - 20.$$

Thus, the constraint condition imposed on c_m by Theorem 1 for robust stability of system (1) is determined to be $c_m > 20$.

Example 2. According to the model (1), giving the the following matrices:

$$\underline{A}=\begin{bmatrix}0&0&0&0\\-1&-1&0&0\\0&-1&0&-1\\-1&-0&-0&-1\end{bmatrix},\overline{A}=\begin{bmatrix}1&1&1&1\\0&0&1&1\\1&0&0&1\\0&1&1&0\end{bmatrix},$$

$$\overline{B}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\underline{B}=\begin{bmatrix}-1&-1&-1&-1\\-1&-1&-1&-1\\-1&-1&-1&-1\\-1&-1&-1&-1\end{bmatrix},$$

$$D=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

$$\ell_1 = \ell_2 = \ell_3 = \ell_4 = 1, \underline{c}_1 = \underline{c}_2 = \underline{c}_3 = \underline{c}_4 = 1, \sigma = 1.$$

We can calculate the follow matrices:

$$A^*=\frac{1}{2}\begin{bmatrix}1&1&1&1\\-1&-1&1&1\\1&-1&1&-1\\-1&1&1&-1\end{bmatrix},A_*=\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix},$$

$$\hat{A}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}\hat{B}=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix},$$

$$D^*=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

from which we can directly calculate the norms: $\| A^* \|_2 + \| A_2 \|_2 = 3$, $\sqrt{\| A^* \|_2^2 + \| A_* \|_2^2 + 2 \| A_*^T \mid A^* \| \|_2} = 3.6$, $\| \hat{A} \|_2 = 4. So \| q \|_2 = 3$. We can also get that $\| \hat{B} \|_1 = 16$, $\| \hat{B} \|_\infty = 4$, $\lambda_{max}(D^*) = 4$.

Firstly, we apply the result of Theorem 1 to the neural network employing the network parameters of this example, we get

$$\varepsilon = c_m - \ell_M \| Q \|_2 - \ell_M \sqrt{\| \hat{B} \|_1 \| \hat{B} \|_\infty} - \lambda_{max}(D^*)$$
$$= c_m - 11 > 0$$

Hence, $c_m > 11$ proved to be a sufficient condition for robust stability of the neural network parameters of this example.

V. Conclusion

In this letter,a improved delay-dependent global robust exponential stability criterion for uncertain stochastic discrete-time neural networks with time-varying delay is proposed. A suitable Lyapunov functional has been proposed to derive some less conservative delay-dependent stability criteria by using the free-weighting matrices method and the convex combination theorem. Finally, two numerical examples have been given to demonstrate the effectiveness of the proposed method.

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