Strongly ω-Gorenstein modules

Jianmin Xing   Wei Shao

Abstract—We introduce the notion of strongly ω -Gorenstein modules, where ω is a faithfully balanced self-orthogonal module. This gives a common generalization of both Gorenstein projective (injective) modules and ω-Gorenstein modules. We investigate some characterizations of strongly ω -Gorenstein modules. Consequently, some properties under change of rings are obtained.

Keywords—faithfully balanced self-orthogonal module, ω-Gorenstein module, strongly ω-Gorenstein module, finite generated.

I. INTRODUCTION

LET R be an associative ring with identity, and R-Mod(respectively,Mod-R) be the category of left (respectively,right)R-modules. For any R-module M, we denote by $pd_R(M)$, $id_R(M)$ and $fd_R(M)$ the projective dimension, injective dimension and flat dimension of M respectively.

When R is a two-sided noetherian ring, Auslander and Bridger introduced in [1] the notion of finitely generated R-modules having Gorenstein dimension zero. Enochs and Jenda in [2,3] extended these ideas and introduced Gorenstein projective and its dual notion which is called Gorenstein injective for any rings. As a generalization of the notion of flat modules, Enochs, Jenda and Martsinkovsky in [4,5] introduced the notion of Gorenstein flat modules. These Gorenstein homological modules have been extensively studied, we refer to [3,6,7] for details.

D.Bennis and N.Mahdou in [8] introduced a particular case of Gorenstein projective, injective and flat modules, which are called strongly Gorenstein projective, injective and flat modules respectively. They also proved that every Gorenstein projective (resp. Gorenstein injective, Gorenstein flat) module is a direct summand of a strongly Gorenstein projective (resp. strongly Gorenstein injective, strongly Gorenstein flat) module.

When R is a left noetherian ring, let ω be a faithfully balanced self-orthogonal module and $add_R\omega$ the subcategory of direct summands of finite sums of copies of ω. In [9], wei,j. introduce the notion of ω-Gorenstein modules which generalize finitely generated Gorenstein projective modules in noetherian rings. By the definition, a left R-module M is ω-Gorenstein if there is an exact sequence $\cdots \xrightarrow{f_{-2}} \omega_{-1} \xrightarrow{f_{-1}} \omega_0 \xrightarrow{f_0} \omega_1 \xrightarrow{f_1} \cdots$ with each $\omega_i \in add_R\omega$ such that $M = \mathrm{Im}f_i$ for some integer i and the induced sequences after application of the functors $\mathrm{Hom}_R(\omega,*)$ and $\mathrm{Hom}_R(*,\omega)$ are also exact. wei,j. also study properties and characterizations

Jianmin Xing is with the College of Mathematics and Physics, Qingdao University of Science and Technology , Qingdao, 266061, P.R.China e-mail: xingjianmin1978@163.com..

Wei Shao is with the College of Automation and Electronic Engineering Science, Qingdao University of Science and Technology , Qingdao, 266042, P.R.China. e-mail: greatshao@126.com.

of ω-Gorenstein modules, such that the subcategory of all ω-Gorenstein R-modules is closed under extensions and direct summands. But the subcategory is not always closed under direct sums.

In this paper, in order to extend the notion of the ω-Gorenstein , we redefine the notion of ω-Gorenstein modules . Let ω be a faithfully balanced self-orthogonal R-module,a left R-module M is ω-Gorenstein if there is an exact sequence $\cdots \xrightarrow{f_{-2}} \omega_{-1} \xrightarrow{f_{-1}} \omega_0 \xrightarrow{f_0} \omega_1 \xrightarrow{f_1} \cdots$ with each $\omega_i \in \mathrm{Add}_R\omega$. such that $M = \mathrm{Im}f_i$ for some integer i and the induced sequences after application of the functors $\mathrm{Hom}_R(\omega,*)$ and $\mathrm{Hom}_R(*,\omega)$ are also exact,and study properties and characterizations of ω-Gorenstein modules. Then we introduce a particular case of ω-Gorenstein modules which are called strongly ω -Gorenstein. we prove that a module is ω-Gorenstein if, and only if, it is a direct summand of a strongly ω-Gorenstein module. We also study these Gorenstein homological properties under change of rings.

This paper is organized as follows. In Section 2 we give the definitions of ω-Gorenstein modules, and collect some known results needed for our further research. In Section 3 we give the definitions of strongly ω-Gorenstein modules and study the properties of strongly ω-Gorenstein modules. In Section 4, we investigate these Gorenstein homological properties under change of rings.

II. DEFINITION AND GENERAL RESULTS

At first, we collect some known results needed for our further research. Let R (respectively, S)be a left (respectively,right) noetherian ring, modules are left modules. By a subcategory we mean a full subcategory closed under isomorphisms. Let $\mathcal{C}$ be a subcategory and T be an R-module. Denote by $\mathrm{Add}_R T$ the subcategory of direct summands of sums of copies of T . An infinite exact sequence $\cdots \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} T \longrightarrow 0 (0 \longrightarrow T \xrightarrow{f_0} C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} \cdots)$ with $C_i \in \mathcal{C}$ is called a $\mathcal{C}$-resolution ($\mathcal{C}$-coresolution, respectively) of T.we denote by $_\mathcal{C}\mathcal{X}, (\mathcal{X}_\mathcal{C}$, respectively) the subcategory of all R-modules T such that T has a $\mathcal{C}$-resolution ($\mathcal{C}$ - coresolution, respectively) with $syz_\mathcal{C}^i(T) \in \mathcal{C}^\perp$ $(cosyz_\mathcal{C}^i(T) \in^\perp \mathcal{C}$, respectively) for all $i \geq 0$.

A left R-module T is a faithfully balanced self-orthogonal (i.e., the generalized tilting module in sense of wakamatsu [10]) provided: (i) $R \cong \mathrm{End}_S(T)$ where $S = \mathrm{End}_R(T)$ (S acts on T on the right); (ii) $\mathrm{Ext}_R^1(T,T) = \mathrm{Ext}_S^1(T,T) = 0$.

A left R-module T is small if $\mathrm{Hom}_R(T,-)$ preserves sums. It is knows that a finitely generated module is small. Let T be a finitely generated faithfully balanced self-orthogonal R-module and $\mathcal{C} = \mathrm{Add}_R T$, then $\mathrm{Add}_R T$ is a self-orthogonal subcategory . Therefore $_\mathcal{C}\mathcal{X}$ is closed under extensions, kernels

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of epimorphisms, and direct summands, and $\mathcal{X}_C$ is closed under extensions, cokernels of monomorphisms, and direct summands [11, Proposition 5.1 and its dual].

Next We give the following definition of ω-Gorenstein.

Definition 2.1 Let ω be a faithfully balanced self-orthogonal R-module. An R-module M is said to be ω-Gorenstein provided there is an exact sequence with $\omega_i \in \text{Add}_R\omega$

$$\cdots \xrightarrow{f_{-2}} \omega_{-1} \xrightarrow{f_{-1}} \omega_0 \xrightarrow{f_0} \omega_1 \xrightarrow{f_1} \cdots (*)$$

such that:

(1) $M = \text{Im} f_i$ for some integer i; and

(ii)$\text{Hom}_R(\omega, *)$ and $\text{Hom}_R(*, \omega)$ are also exact sequences.

Clearly, finitely generated Gorenstein projective modules are just R-Gorenstein module.If there is a finitely generated injective cogenerator in R-mod,say Q, then finitely generated Gorenstein injective modules are just Q-Gorenstein modules.We also note that all modules in $\text{Add}_R\omega$ are obviously ω-Gorenstein R -modules .

We denote $\mathcal{G}_\omega$ by the subcategory of all ω-Gorenstein R-modules.

Using the definitions, we immediately get the following charaterization of ω-Gorenstein R-modules.

Proposition 2.2 Let ω be faithfully balanced self-orthogonal R-module, Then $\mathcal{G}_\omega =_\omega \mathcal{X} \bigcap \mathcal{X}_\omega$.

Proposition 2.3 Let ω be finitely generated faithfully balanced self-orthogonal R-module, then $\mathcal{G}_\omega$ is closed under extensions , direct sum and summands.

Proof We can proof $\mathcal{G}_\omega$ is closed under extensions and summands by Proposition 2.2. Since ω is finitely generated faithfully balanced self-orthogonal R-module, we can proof $\mathcal{G}_\omega$ is closed under direct sum by the definition of ω-Gorenstein R-modules.

III. Strongly ω-Gorenstein modules

In this section, we give the definitions of strongly ω-Gorenstein modules and study the properties of strongly ω-Gorenstein modules.

Definition 3.1 Let ω be a faithfully balanced self-orthogonal R-module. An R-module M is said to be strongly ω-Gorenstein provided there is an exact sequence with $\bar{\omega} \in \text{Add}_R\omega$

$$\cdots \xrightarrow{f} \bar{\omega} \xrightarrow{f} \bar{\omega} \xrightarrow{f} \bar{\omega} \xrightarrow{f} \cdots (**)$$

such that:

(1) $M = \text{Im} f$; and

(ii)$\text{Hom}_R(\omega, **)$ and $\text{Hom}_R(**, \omega)$ are also exact sequences.

Using the definition, we immediately get the following result

Proposition 3.2 Let ω be a finitely generated faithfully balanced self-orthogonal R-module. If $(M_i)_{i \in I}$ is a family of strongly ω- Gorenstein modules,then $\oplus M_i$ is strongly ω-Gorenstein .

It is straightforward that the strongly ω-Gorenstein modules are a particular case of the ω-Gorenstein modules. And it is well known that every module in $\text{Add}_R\omega$ is ω-Gorenstein.

The next result shows that the class of all strongly ω-Gorenstein is between $\text{Add}_R\omega$ and the class of all ω-Gorenstein modules.

Proposition 3.3 Let ω be a faithfully balanced self-orthogonal R-module, then every R-module ω_i in $\text{Add}_R\omega$ is strongly ω-Gorenstein.

Proof Consider the exact sequence

$$\cdots \xrightarrow{f} \omega_i \oplus \omega_i \xrightarrow{f} \omega_i \oplus \omega_i \xrightarrow{f} \omega_i \oplus \omega_i \xrightarrow{f} \cdots$$

where $f : \omega_i \oplus \omega_i \longrightarrow \omega_i \oplus \omega_i, (x, y) \longrightarrow (0, x)$,for $(x, y) \in \omega_i \oplus \omega_i$. we have $0 \oplus \omega_i = \ker f = \text{Im} f \cong \omega_i$.

Applying the functor $\text{Hom}_R(\omega, -)$ and $\text{Hom}_R(-, \omega)$ to the above sequence, respectively . We get the morphism :

$$Hom_R(\omega_i \oplus \omega_i, \omega) \xrightarrow{Hom_R(f, \omega)} Hom_R(\omega_i \oplus \omega_i, \omega)$$

$$Hom_R(\omega, \omega_i \oplus \omega_i) \xrightarrow{Hom_R(\omega, f)} Hom_R(\omega, \omega_i \oplus \omega_i)$$

and the isomorphisms

$$Hom_R(\omega_i \oplus \omega_i, \omega) \simeq Hom_R(\omega_i, \omega) \oplus Hom_R(\omega_i, \omega)$$

$$Hom_R(\omega, \omega_i \oplus \omega_i,) \simeq Hom_R(\omega, \omega_i) \oplus Hom_R(\omega, \omega_i)$$

Hence the proposition follows.

Now , we give the main result in this section .

Theorem 3.4 Let ω be a finitely generated faithfully balanced self-orthogonal R-module. A module is ω-Gorenstein if, and only if, it is a direct summand of a strongly ω-Gorenstein module.

Proof By Proposition 2.3, it remains to prove the direct implication.

Let M be a ω-Gorenstein R-module. Then, there exists a exact sequence with $\omega_i \in \text{Add}_R\omega$

$$(*)\mathcal{P} : \cdots \xrightarrow{f_{-2}} \omega_{-1} \xrightarrow{f_{-1}} \omega_0 \xrightarrow{f_0} \omega_1 \xrightarrow{f_1} \cdots$$

such that:

(1) $M = \text{Im} f_i$ for some integer i; and

(ii)$\text{Hom}_R(\omega, *)$ and $\text{Hom}_R(*, \omega)$ are also exact sequences.

For all $m \in \mathcal{Z}$, denote $\Sigma^m \mathcal{P}$ the exact sequence obtained from $\mathcal{P}$ by increasing all indexes by m:

$(\Sigma^m \mathcal{P})_i = \omega_{i+m}$ and $f_i^{\Sigma^m \mathcal{P}} = f_{i+m}^{\mathcal{P}}$ for all $i \in \mathcal{Z}$.

considering the exact sequence

$$\oplus(\Sigma^m \mathcal{P}) : \cdots \longrightarrow \oplus\omega_i \xrightarrow{\oplus f_i^{\mathcal{P}}} \oplus\omega_i \xrightarrow{\oplus f_i^{\mathcal{P}}} \cdots$$

with $\oplus\omega_i \in \text{Add}_R\omega$. Since $\text{Im}\oplus f_i \cong \oplus\text{Im} f_i$, M is a direct summand of $\text{Im}\oplus f_i$.

Moreover , from [1, Proposition 20.2]

$$\text{Hom}_R(\oplus(\Sigma^m \mathcal{P}), \omega) \cong \prod_{m \in \mathcal{Z}} \text{Hom}(\Sigma^m \mathcal{P}, \omega).$$

and since ω be a finitely generated faithfully balanced self-orthogonal R-module, then

$$\text{Hom}_R(\omega, \oplus\Sigma^m \mathcal{P}) \cong \oplus\text{Hom}(\omega, \Sigma^m \mathcal{P}).$$

which is an exact sequence . Therefore M is a direct summand of the strongly $\omega-$ Gorenstein module $\text{Im}\oplus f_i$, as desired.

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The next result gives a simple characterization of the strongly ω-Gorenstein modules.

Theorem 3.5 Let ω be a finitely generated faithfully balanced self-orthogonal R-module. For any module M, the following are equivalent:

(1) M is strongly ω-Gorenstein ;

(2) there exists a short exact sequence $0 \longrightarrow M \longrightarrow \bar{\omega} \longrightarrow M \longrightarrow 0$ where $\bar{\omega} \in \text{Add}_R \omega$ and $\text{Ext}_R^1(M,\omega) = \text{Ext}_R^1(\omega, M) = 0$;

(3) there exists a short exact sequence $0 \longrightarrow M \longrightarrow \bar{\omega} \longrightarrow M \longrightarrow 0$ where $\bar{\omega} \in \text{Add}_R \omega$ and $\text{Ext}_R^1(M,Q) = \text{Ext}_R^1(Q',M) = 0$ for any module Q with finite $\text{Add}_R\omega$-resolution and for any module Q' with finite $\text{Add}_R\omega$-coresolution.

(4) There exists a short exact sequence $0 \longrightarrow M \longrightarrow \bar{\omega} \longrightarrow M \longrightarrow 0$ where $\bar{\omega} \in \text{Add}_R\omega$ such that $0 \longrightarrow \text{Hom}_R(M,\omega) \longrightarrow \text{Hom}_R(\bar{\omega},\omega) \longrightarrow \text{Hom}_R(M,\omega) \longrightarrow 0$ and $0 \longrightarrow \text{Hom}_R(\omega,M) \longrightarrow \text{Hom}_R(\omega,\bar{\omega}) \longrightarrow \text{Hom}_R(\omega,M) \longrightarrow 0$ are exact sequences;

(5) There exists a short exact sequence $0 \longrightarrow M \longrightarrow \bar{\omega} \longrightarrow M \longrightarrow 0$ where $\bar{\omega} \in \text{Add}_R\omega$ such that $0 \longrightarrow \text{Hom}_R(M,Q) \longrightarrow \text{Hom}_R(\bar{\omega},Q) \longrightarrow \text{Hom}_R(M,Q) \longrightarrow 0$ and $0 \longrightarrow \text{Hom}_R(Q',M) \longrightarrow \text{Hom}_R(Q',\bar{\omega}) \longrightarrow \text{Hom}_R(Q',M) \longrightarrow 0$ are exact sequences for any module Q with finite $\text{Add}_R\omega$-resolution and for any module Q' with finite $\text{Add}_R\omega$-coresolution.

Proof Using standard arguments, this follows immediately from the definition of strongly ω-Gorenstein modules.

IV. CHANGE OF RINGS

In this section, we consider the properties of strongly ω-Gorenstein modules under the change of rings, when ω is a special faithfully balanced self-orthogonal R-module.

Lemma 4.1([12]) Let R be commutative Noetherian with subset S, and let A, B be R-modules with A finitely generated . Then there is a natural isomorphism, for all $n \geq 0$,

$$\text{Ext}_{S^{-1}R}^n(S^{-1}A, S^{-1}B) \cong S^{-1}\text{Ext}_R^n(A,B)$$

Theorem 4.2 Let R be commutative Noetherian with subset S, and ω be a finitely generated faithfully balanced self-orthogonal R-module, then $S^{-1}\omega$ is a finitely generated faithfully balanced self-orthogonal $S^{-1}R$-module.

Proof Since ω is a finitely generated faithfully balanced self-orthogonal R-module, $R \cong \text{End}_T(\omega)$ where $T = \text{End}_R(\omega)$ and $\text{Ext}_R^1(\omega,\omega) = \text{Ext}_T^1(\omega,\omega) = 0$. Then we have the following result by Lemma 4.1 :

$$\text{Hom}_{S^{-1}R}(S^{-1}\omega, S^{-1}\omega) \cong S^{-1}\text{Hom}_R(\omega,\omega) = S^{-1}T$$

$$\text{Hom}_{S^{-1}T}(S^{-1}\omega, S^{-1}\omega) \cong S^{-1}\text{Hom}_T(\omega,\omega) \cong S^{-1}R$$

and

$$\text{Ext}_{S^{-1}R}^1(S^{-1}\omega, S^{-1}\omega) \cong S^{-1}\text{Ext}_R^1(\omega,\omega) = 0$$

$$\text{Ext}_{S^{-1}T}^1(S^{-1}\omega, S^{-1}\omega) \cong S^{-1}\text{Ext}_T^1(\omega,\omega) = 0$$

Therefore $S^{-1}\omega$ is a finitely generated faithfully balanced self-orthogonal $S^{-1}R$-module.

Theorem 4.3 Let R be a commutative Noetherian ring and S a multiplicatively closed set of R. If ω is a finitely generated faithfully balanced self-orthogonal R-module and if A is an finitely generated strongly ω-Gorenstein R-module , then $S^{-1}A$ is a finitely generated strongly $S^{-1}\omega$-Gorenstein $S^{-1}R$-module.

Proof Since ω is a finitely generated faithfully balanced self-orthogonal R-module , $S^{-1}\omega$ is a faithfully balanced self-orthogonal $S^{-1}R$-module by Theorem 4.2.

There exists a short exact sequence $0 \longrightarrow A \longrightarrow \bar{\omega} \longrightarrow A \longrightarrow 0$ where $\bar{\omega} \in \text{Add}_R\omega$. Then $0 \longrightarrow S^{-1}A \longrightarrow S^{-1}\bar{\omega} \longrightarrow S^{-1}A \longrightarrow 0$ is exact in $S^{-1}R$-Mod and $S^{-1}\bar{\omega} \in \text{Add}_R S^{-1}\omega$. So $\text{Ext}_{S^{-1}R}^1(S^{-1}\omega, S^{-1}A) \cong S^{-1}\text{Ext}_R^1(\omega,A) = 0$ and $\text{Ext}_{S^{-1}R}^1(S^{-1}A, S^{-1}\omega) \cong S^{-1}\text{Ext}_R^1(A,\omega) = 0$ by Lemma 4.1. It is obvious that $S^{-1}A$ is a finitely generated $S^{-1}R$-module, since A is an finitely generated R-module. Therefore $S^{-1}A$ is a finitely generated strongly $S^{-1}\omega$-Gorenstein $S^{-1}R$-module.

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