Hybrid function method for solving nonlinear Fredholm integral equations of the second kind

Jianhua Hou, Changqing Yang, and Beibo Qin

Abstract—A numerical method for solving nonlinear Fredholm integral equations of second kind is proposed. The Fredholm type equations which have many applications in mathematical physics are then considered. The method is based on hybrid function approximations. The properties of hybrid of block-pulse functions and Chebyshev polynomials are presented and are utilized to reduce the computation of nonlinear Fredholm integral equations to a system of nonlinear. Some numerical examples are selected to illustrate the effectiveness and simplicity of the method.

Keywords—Hybrid functions, Fredholm integral equation, Block-pulse, Chebyshev polynomials, Product operational matrix.

I. INTRODUCTION

INTTEGRAL equations are often involved in the mathematical formulation of physical phenomena. Integral equations can be encountered in various fields such as physics [1], biology [2] and engineering. But we can also use it in numerous applications, such as biomechanics, control, economics, elasticity, electrical engineering, fluid dynamics, heat and mass transfer, oscillation theory, queuing theory, etc. Fredholm and Volterra integral equations of the second kind show up in studies that include airfoil theory, elastic contact problems, fracture mechanics, combined infrared radiation and molecular conduction [3] and so on.

The problem of finding numerical solutions for Fredholm integral equations of the second kind is one of the oldest problems in the applied mathematics literature and many computational methods are introduced in this field. One may find in the references [4], [5], [6], [7], a collection of the best numerical methods for solving Fredholm integral equations appeared after 1960. Also, a functional analysis framework for these methods can be found in [8]. The classical methods for finding approximate solutions, dependent on the definition of the approximate solution, are mostly classified into two types, collocation methods and Galerkin methods.

In this study, we are concerned with the application of hybrid block-pulse function and Chebyshev polynomials to the numerical solution of Fredholm integral equation of the form

$$y(x) = f(x) + \int_0^1 k(x, t)[y(t)]^m dt, \quad 0 \leq x \leq 1. \qquad (1)$$

The function $f(x)$ and $k(x, t)$ are known, $y(x)$ is unknown function to be determined and $m \leq 1$ is a positive integer. For $m = 1$, (1) is a linear and $m \geq 2$ is a nonlinear Fredholm integral equation.

The article is organized as follows: In Section 2, we describe the basic formulation of hybrid block-pulse function and Chebyshev polynomials required for our subsequent. Section 3 is devoted to the solution of equation (1) by using hybrid functions. In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

II. PROPERTIES OF HYBRID FUNCTIONS

A. Hybrid functions of block-pulse and Chebyshev polynomials

Hybrid function $b_{nm}(x), n = 1, 2, \ldots, N, \; m = 0, 1, \ldots, M - 1$, are defined on the interval $[0, 1]$ as

$$b_{nm}(x) = \begin{cases} T_m(2Nx - 2n + 1), & x \in \left[\frac{n-1}{N}, \frac{n}{N}\right], \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

where $n$ and $m$ are the orders of block-pulse functions and Chebyshev polynomials, respectively. Here $T_m(x)$ are the well-known Chebyshev polynomials which are orthogonal in the interval $[0, 1]$ with respect to the weight function $\omega(x) = 1/\sqrt{1-x^2}$ and satisfy the following recursive formula:

$$T_0 = 1, \quad T_1 = x, \quad T_{m+1} = 2xT_m(x) - T_{m-1}(x), \; m = 1, 2, \ldots.$$ 

Since $b_{nm}$ consists of block-pulse functions and Chebyshev polynomials, which are both complete and orthogonal, the set of hybrid functions is complete orthogonal set.

B. Function approximation

A function $y(x)$ defined over the interval $[0, 1]$ may be expanded as

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} b_{nm}, \quad (3)$$

where

$$c_{nm} = \langle f(x), b_{nm}(x) \rangle,$$

in which $\langle , \rangle$ denotes the inner product.

If $y(x)$ in (3) is truncated, then (3) can be written as

$$y(x) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm} = C^TB(x) = B^T(x)C, \quad (4)$$

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where
\[ C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{NM-1}]^T \]
and
\[ B(x) = [b_{10}(x), b_{11}(x), \ldots, b_{1M-1}(x), c_{20}(x), \ldots, b_{2M-1}(x), \ldots, b_{NM-1}(x)]^T \]
In (5) and (6), \( c_{nm}, n = 1, 2, \ldots, N, m = 0, 1, 2, \ldots, M - 1, \) are the coefficients expansions of the function \( y(x) \) in the subinterval \( [(n - 1)/N, n/N] \) and \( b_{nm} \) are defined in (2).

C. The product operational matrix of the hybrid of block-pulse and Chebyshev polynomials

The following property of the product of two hybrid function vectors will also be used. Let
\[ B(x)B^T(x)C = \tilde{C}B(x) \]
where \( \tilde{C} = \text{diag}(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_N) \) is an \( MN \times MN \) product operational matrix. And \( \tilde{C}_i, i = 1, 2, \ldots, N \) are \( M \times M \) matrices given in [9]. We also define the matrix \( D \) as follows:
\[ D = \int_0^1 B(x)B^T(x)dx \]
For the hybrid functions of block-pulse and Chebyshev polynomials, \( D \) has the following form:
\[ D = \begin{pmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{pmatrix} \]
where \( L \) is \( M \times M \) nonsingular symmetric matrix given in [10].

III. NONLINEAR FREDHOLM INTEGRAL EQUATIONS

Consider the following integral equation
\[ y(x) = f(x) + \int_0^1 k(x, t)[y(t)]^m dt, \quad 0 \leq x \leq 1. \]
We approximate \( f(x), y(x), k(x, t), \) and \( [y(t)]^m \) by the way mentioned in Section 2 as
\[ f(x) = B^T(x)F, \quad y(x) = B^T(x)C, \quad k(x, t) = B^T(x)K(B(t), \]
and
\[ [y(t)]^m = [B^T(x)C]^m = C^T B(x) \cdot B^T(x)C[B^T(x)C]^{m-2}. \]
Applying (7), equation (10) becomes
\[ [y(t)]^m = C^T \tilde{C}^{m-1} B(t) = C^* B(t). \]
With substituting in (9) we have
\[ B^T(x)C = B^T(x)F + B^T(x)K \left( \int_0^1 B(t)B^T(t)dt \right) C^T. \]
Applying (8), then we get
\[ C = F + KDC^* \]
which is a nonlinear system of equations. By solving this equation we can find the vector \( C \).
with the exact solution \( y(x) = e^x \). The comparison among the hybrid solution, Haar wavelets solution and the analytic solution for \( x \in [0, 1] \) is shown in Table 2 for \( N = 2, M = 10 \), which confirms that the hybrid function method in section 4 gives almost the same solution as the analytic method.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Analytic solution</th>
<th>Hybrid function solution</th>
<th>Haar wavelets solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.01510790187565</td>
<td>1.01321329898095</td>
<td>1.107217811</td>
</tr>
<tr>
<td>0.2</td>
<td>1.22140723150107</td>
<td>1.2214975177075</td>
<td>1.218102916</td>
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<tr>
<td>0.3</td>
<td>1.34089580075760</td>
<td>1.3500010299025</td>
<td>1.341165462</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4918269794127</td>
<td>1.492105438253</td>
<td>1.476198603</td>
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<tr>
<td>0.5</td>
<td>1.64872127070013</td>
<td>1.648586294478</td>
<td>1.66702633</td>
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<tr>
<td>0.6</td>
<td>1.82211880039051</td>
<td>1.82240326946041</td>
<td>1.833861053</td>
</tr>
<tr>
<td>0.7</td>
<td>2.01375270474988</td>
<td>2.0140845179746</td>
<td>2.014679830</td>
</tr>
<tr>
<td>0.8</td>
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<td>2.22592019109086</td>
<td>2.21795630</td>
</tr>
<tr>
<td>0.9</td>
<td>2.49560311115695</td>
<td>2.4602971282342</td>
<td>2.437978177</td>
</tr>
</tbody>
</table>

**Example3**

The exact solution is \( y(x) = f(x) \). Table 3 illustrates the numerical results of Example3 with \( N = 2, M = 10 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>Present method</th>
<th>Exact solution</th>
<th>Method in [13], [26, 32]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1652988399936</td>
<td>0.16529888822159</td>
<td>0.1625177701</td>
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<tr>
<td>0.2</td>
<td>0.20189649145656</td>
<td>0.201896517946</td>
<td>0.1921647474</td>
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<tr>
<td>0.3</td>
<td>0.24659692037010</td>
<td>0.246996939161</td>
<td>0.2290236855</td>
</tr>
<tr>
<td>0.4</td>
<td>0.30191547507771</td>
<td>0.3019142191220</td>
<td>0.2744778506</td>
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<tr>
<td>0.5</td>
<td>0.36787933070220</td>
<td>0.3678794417144</td>
<td>0.3204321944</td>
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<tr>
<td>0.6</td>
<td>0.2578794563142</td>
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<td>0.9</td>
<td>-0.19212058662314</td>
<td>-0.19212058882856</td>
<td>-0.2238779332</td>
</tr>
</tbody>
</table>

**VI. Conclusion**

We have solved the nonlinear Fredholm integral equations of second kind by using hybrid of block-pulse functions and Chebyshev polynomials. The properties of hybrid of block-pulse functions and Chebyshev polynomials are used to reduce the equation to the solution of nonlinear algebraic equations. Illustrative examples are given to demonstrate the validity and applicability of the proposed method. The advantages of hybrid functions are that the values of \( N \) and \( M \) are adjustable as well as being able to yield more accurate numerical solutions than Haar wavelets functions [13], for the solutions of integral equations. Also hybrid functions have good advantage in dealing with piecewise continuous functions, as are shown. The method can be extended and applied to the system of nonlinear integral equations, linear and nonlinear integro-differential equations, but some modifications are required.

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**References**


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