The Statistical Properties of Filtered Signals

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Abstract—In this paper, the statistical properties of filtered or convolved signals are considered by deriving the resulting density functions as well as the exact mean and variance expressions given a *prior* knowledge about the statistics of the individual signals in the filtering or convolution process. It is shown that the density function after linear convolution is a mixture density, where the number of density components is equal to the number of observations of the shortest signal. For circular convolution, the observed samples are characterized by a single density function, which is a sum of products.

Keywords—Circular Convolution, linear Convolution, mixture density function.

NOTATION

A signal is a group of observations, and these are represented in vector form. For example, $\mathbf{x}_i(n) = [x_i(1), x_i(2), \cdots, x_i(K_i)]$ is a vector for the ith signal, for $i\epsilon[1, N]$, whose observations are $x_i(n)$, for $n\epsilon[1, K_i]$, where K_i is the length of the ith signal. Given a vector $\mathbf{x}_i(n)$, the variable X_i (capital letter with a corresponding subscript) is defined where its possible values are the vector elements or observations, and for X_i we define the density function $p(x_i)$ summarizing the statistical properties of its observations.

I. INTRODUCTION

IT is well known that the probability density function of the sum of signals $\sum_{i=1}^{N} \mathbf{x}_i(n)$ is the convolution of their density functions $p(x_1) * p(x_2) * \cdots * p(x_N)$, where * is the convolution operator and $p(x_i)$ is the density function of the signal $\mathbf{x}_i(n)$ [1]. However the reverse problem, meaning the study of the statistical properties of the convolution or filtered signal $\mathbf{z}(n) = \mathbf{x}_1(n) * \mathbf{x}_2(n) * \cdots * \mathbf{x}_N(n)$ has received little to no attention. As filtering is a typical process in signal processing applications, it is the focus of this paper to derive the density functions, as well the exact mean and variance expressions, for the resulting linear and circular convolution signals given a prior knowledge about the statistics of the individual signals in the convolution process. The aim is to provide an understanding of the statistical properties and transformations to a signal once it has gone through some filtering or convolution process.

In engineering applications, convolution is usually between an input signal and some vector of coefficients, such as a filter, and it is sometimes necessary to estimate the first and second moments of the underlying signals for subsequent processing. For example, in signal denoising some methods rely on

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estimating the noise variance [2]-[5], and in blind single channel source separation knowledge of the variances of the underlying source signals can be crucial [6]-[9]. Most single channel source separation algorithms assume instantaneous mixing, such as the one in [6], yet in practical situations like with the cocktail party problem the source signals are filtered by the room impulse responses [9]. A strong prior knowledge is usually required in these ill-determined conditions such as the variance of the signals [7], or certain time-domain basis functions [8]. For filtering processes, then an understanding of the statistical properties of the filtered signals is essential as the separation algorithms are very sensitive to modeling errors.

Prediction applications rely on the use of filters, such as wavelet [10] or Kalman [11] based filtering. These are useful in finance, economics, biomedical engineering for prediction of certain processes such as the short-time prediction of glucose concentrations [12] and intelligent transportation systems [13], [14] for traffic prediction which helps to optimize routing algorithms for controlling traffic build up. Since any sequence prediction process is basically point estimation, there is always some uncertainty associated with the future expected points where the lower bound in the uncertainty is governed by the Fisher-Cramer-Rao (FCR). If the FCR bounds of the predicted sequences are known, then the model which offers the least uncertainty is usually preferred. This is perhaps one of the most driving reasons to study the statistical properties of filtered signals as by the law of large numbers the exact variance expression is essentially the FCR bound to a known scale. Given the mean and variance of the input signal as well as for the prediction model's coefficients are known, the lower bound in uncertainty associated with the prediction can be calculated as will be shown in this paper, offering invaluable information for the choice of the least risk/variance prediction process.

This paper is structured as follows: In Section II we derive the density function for the linear convolution of two signals, followed by derivations of the exact mean and variance expressions. Section III considers a more general case of N signals. In Section IV circular convolution of N signals is considered. Discussions and summary remarks are in Section V.

II. LINEAR CONVOLUTION OF TWO SIGNALS

Let $\mathbf{z}(n) = \mathbf{x}_1(n) * \mathbf{x}_2(n)$ be the convolution of the two signals $\mathbf{x}_1(n)$ and $\mathbf{x}_2(n)$. The observations of $\mathbf{z}(n)$ are given by:

$$
\begin{aligned}
z(n) = \sum_r x_1(r)x_2(n-r) &= x_1(1)x_2(n-1) \\
&+ x_1(2)x_2(n-2) + \cdots + x_1(n-1)x_2(1) \quad (1)
\end{aligned}
$$

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for $n \in [1, K_1 + K_2 - 1]$, where K_i is the total number of observations of the signal $\mathbf{x}_i(n)$, for $i\epsilon[1, N]$, and $x_i(r)$ is the rth observation of $\mathbf{x}_i(n)$, for $r\epsilon[1, K_i]$. Assuming that $K_1 \geq K_2$, an observation of $\mathbf{z}(n)$ is the sum of the product of the observations of the signals $\mathbf{x}_1(n)$ and $\mathbf{x}_2(n)$ as illustrated in (1), where K_2 is the limiting length of the number of sum terms. Let $p_j(z)$ be the generating density function of an observation derived from the sum of j products. For example, $z(1)$ and $z(K_1 + K_2 - 1)$ would be drawn from $p_1(z)$, and $z(K_2 - 1)$ and $z(K_1 + 1)$ from $p_{K_2-1}(z)$. It follows that the variable Z, realised by the observations of the signal $\mathbf{z}(n)$, is a mixture of K_2 variables where $p_1(z), p_2(z), \dots, p_{K_2-1}(p)$ constitute two observations each and $p_{K_2}(z)$ constitutes $K_1 - K_2 + 1$ observations. Therefore the density function of Z is the mixture

$$p(z) = \sum_{j=1}^{K_2} w_j p_j(z)$$

where w_j is the relative weight of $p_j(z)$ in the mixture and $\sum_{j=1}^{K_1} w_j = 1$. As the number of observations from each density function is known, it is clear that $w_1 = w_2 = \dots = w_{K-1} = \frac{2}{K_1+K_2-1}$ and $w_{K_2} = \frac{K_1-K_2+1}{K_1+K_2-1}$, thus the density function of th signal $\mathbf{z}(n)$ resulting from the linear convolution of two signals is

$$p(z) = \frac{2}{K_1 + K_2 - 1} \sum_{j=1}^{K_2} p_j(z) + \frac{K_1 - K_2 + 1}{K_1 + K_2 - 1} p_{K_2}(z) \quad (2)$$

A. The Mean of a Sum of Identical Products

If $\mu_{X_i} = E[X_i]$ is the mean of X_i, the variable realized by the observations of the signals $\mathbf{x}_i(n)$, for $i\epsilon[1, N]$, where $E[\cdot]$ is mathematical expectation, the mean of the product variable $X_1 X_2$ is $\mu_{X_1 X_2} = E[X_1 X_2]$. Thus, the mean of the density function $p_j(z)$ is

$$\mu_j = j \cdot \mu_{X_1,X_2}, \qquad for \quad j\epsilon[1, K_2] \quad (3)$$

since the expectation of a sum is the sum of expectations.

B. The Variance of a Sum of Identical Products

Let $\sigma_{X_i}^2 = E[X_i^2] - \mu_{X_i}^2$ be the variance of X_i, for $i\epsilon[1, N]$. In [15, 16], the exact variance of a product is derived for independent variables X_1 and X_2 as:

$$\sigma_{X_1 X_2}^2 = \mu_{X_1}^2 \sigma_{X_2}^2 + \mu_{X_2}^2 \sigma_{X_1}^2 + \sigma_{X_1}^2 \sigma_{X_2}^2,$$

and if they are not necessarily independent

$$\sigma_{X_1 X_2}^2 = \mu_{X_1}^2 \sigma_{X_2}^2 + \mu_{X_2}^2 \sigma_{X_1}^2 + 2\mu_{X_1}\mu_{X_2} E_{11} \\ + 2\mu_{X_1} E_{12} + 2\mu_{X_2} E_{21} + E_{22} - E_{11}^2$$

where $E_{ij} = E[(X_1 - \mu_{X_1})^i (X_2 - \mu_{X_2})^j]$. Taking note that the variances of the products in the sum given by (1) are identical (because all observations of X_1 and X_2 are drawn from $p(x_1)$ and $p(x_2)$ respectively), if the observations of X_1

are drawn independently from $p(x_1)$ as well as those of X_2 from $p(x_2)$, the variance of $p_j(z)$ is given by:

$$\sigma_j^2 = j \cdot \sigma_{X_1,X_2}^2, \qquad for \quad j\epsilon[1, K_2] \quad (4)$$

because the variance of a sum is the sum of variances for independent and identical products [17]. If the observations are not drawn independently from their respective distributions, then it is shown in [17] that the net variance is the sum of the permutations of the covariance matrices of all the product variables in the sum. Since in this case the sum terms are identically distributed,

$$\sigma_j^2 = j^2 \cdot \sigma_{X_1,X_2}^2, \qquad for \quad j\epsilon[1, K_2] \quad (5)$$

C. The Mean of a Linear Convolution of Two Signals

The mean μ_Z of the variable Z, representing the signal $\mathbf{z}(n) = \mathbf{x}_1(n) * \mathbf{x}_2(n)$, is given by

$$\mu_Z = E[Z] = \int_{-\infty}^{\infty} z \cdot p(z) dz$$

Using (2),

$$\mu_Z = \frac{2}{K_1 + K_2 - 1} \sum_{j=1}^{K_2-1} \int_{-\infty}^{\infty} z \cdot p_j(z) dz$$
$$+ \frac{K_1 - K_2 + 1}{K_1 + K_2 - 1} \int_{-\infty}^{\infty} z \cdot p_{K_2}(z) dz$$
$$= \frac{2}{K_1 + K_2 - 1} \sum_{j=1}^{K_2-1} \mu_j + \frac{K_1 - K_2 + 1}{K_1 + K_2 - 1} \mu_{K_2}$$

By (3),

$$\mu_Z = \frac{2\mu_{X_1 X_2}}{K_1 + K_2 - 1} \sum_{j=1}^{K_2-1} j + \frac{K_2(K_1 - K_2 + 1)}{K_1 + K_2 - 1} \mu_{X_1 X_2}$$

The sum $\sum_{j=1}^{K_2-1} j = \frac{K_2(K_2-1)}{2}$, therefore the mean of the signal $\mathbf{z}(n)$ can be expressed as

$$\mu_Z = \frac{K_1 K_2}{K_1 + K_2 - 1} \mu_{X_1 X_2} \quad (6)$$

D. The Variance of a Linear Convolution of Two Signals

The variance σ_Z^2 of the variable Z, realized by the observations of the signal $\mathbf{z}(n) = \mathbf{x}_1(n) * \mathbf{x}_2(n)$, is given by

$$\sigma_Z^2 = E[Z^2] - \mu_Z^2 = \int_{-\infty}^{\infty} z^2 . p(z) dz - \mu_Z^2$$

The integral $\int_{-\infty}^{\infty} z^2 . p(z) dz = \sum_{j=1}^{K_2} w_j \int_{-\infty}^{\infty} z^2 p_j(z) dz$, and since

$$\sigma_j^2 = \int_{-\infty}^{\infty} (z - \mu_j)^2 p_j(z) dz = \int_{-\infty}^{\infty} z^2 p_j(z) dz - \mu_j^2,$$

it implies that $\int_{-\infty}^{\infty} z^2 p_j(z) dz = \sigma_j^2 + \mu_j^2$, thus

$$\sigma_Z^2 = \sum_{j=1}^{K_2} w_j (\sigma_j^2 + \mu_j^2) - \mu_Z^2.$$

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For independent observations of X_i from $p(x_i)$ (for the signal $\mathbf{x}_i(n)$), using (2), (3) and (4)

$$\sigma_Z^2 = \frac{2}{K_1 + K_2 - 1} \left(\sigma_{X_1 X_2}^2 \sum_{j=1}^{K_2-1} j + \mu_{X_1 X_2}^2 \sum_{j=1}^{K_2-1} j^2 \right)$$
$$+ \frac{K_1 - K_2 + 1}{K_1 + K_2 - 1} \left(K_2 \sigma_{X_1 X_2}^2 + K_2^2 \mu_{X_1 X_2}^2 \right) - \mu_Z^2$$

Noting that $\sum_{j=1}^{K_2} j = \frac{K_2(K_2-1)}{2}$ and the sum of squares $\sum_{j=1}^{K_2-1} j^2 = \frac{K_2(K_2-1)(2K_2-1)}{6}$, the variance of the signal $\mathbf{z}(n)$ can be expressed as

$$\sigma_Z^2 = \frac{K_2}{K_1 + K_2 - 1} \varsigma - \mu_Z^2 \qquad (7)$$

where $\quad \varsigma = \left(K_1 \sigma_{X_1 X_2}^2 + \frac{3K_1 K_2 - K_2^2 + 1}{3} \mu_{X_1 X_2}^2 \right)$

If the observations of are not drawn independently, then by (2), (3) and (5)

$$\sigma_Z^2 = (\sigma_{xy}^2 + \mu_{xy}^2)\chi - \mu_Z^2$$

where $\quad \chi = \frac{2}{K_1 + K_2 - 1} \sum_{j=1}^{K_2-1} j^2 + \frac{K_2^2(K_1 - K_2 + 1)}{K_1 + K_2 - 1}$

After simplification,

$$\sigma_Z^2 = \frac{K_2(3K_1 K_2 - K_2^2 + 1)}{3(K_1 + K_2 - 1)}(\sigma_{xy}^2 + \mu_{xy}^2) - \mu_Z^2 \qquad (8)$$

III. LINEAR CONVOLUTION OF TWO OR MORE SIGNALS

Let $\mathbf{z}^{(N)}(n) = \mathbf{x}_1(n) * \mathbf{x}_2(n) * \cdots * \mathbf{x}_N(n)$ where the superscript (N) in $\mathbf{z}^{(N)}(n)$ is used to indicate the convolution of N signals. Based on this notation, it follows that $\mathbf{z}^{(N)}(n) = \mathbf{z}^{(N-1)}(n) * \mathbf{x}_N(n)$ which means that the observations of $\mathbf{z}^{(N)}(n)$ are given by

$$z^{(N)}(n) = \sum_r z^{(N-1)}(r) x_N(n-r), \text{ for } n \in \left[1, K^{(N)}\right] \quad (9)$$

with $K^{(v)}$ as the length of the resulting convolution signal $\mathbf{z}^{(v)}(n)$, for $v \in [2, N]$. It can be shown that

$$K^{(v)} = \sum_{i=1}^{v} K_i - v + 1, \qquad (10)$$

where K_i is the length of the signal $\mathbf{x}_i(n)$, for $i \in [i, N]$, and it is assumed that $K_1 \geq K_2 \geq ... \geq K_N$. In order to use equations (3), (4) and (5) it is necessary to know the exact number of product terms in the sum (9) for each and every observation of $\mathbf{z}^{(N)}(n)$. If we let $num\left[z^{(v)}(n)\right]$ be the total number of product terms for the observation $z^{(v)}(n)$, then in general

$$q_n^{(N)} = \begin{cases} \sum_{i=1}^{n} num\left[z^{(N-1)}(i)\right], \\ \qquad\qquad \text{for } n \in [1, K_N - 1] \\ \sum_{i=1}^{K_N} num\left[z^{(N-1)}(i)\right], \\ \qquad\qquad \text{for } n \in \left[K_N, K^{(N-1)}\right] \\ \sum_{i=1}^{n} num\left[z^{(N-1)}(K^{(N-1)} + K_N - i)\right], \\ \qquad\qquad \text{for } n \in \left[K^{(N-1)} + 1, K^{(N)}\right] \end{cases}$$

where $q_n^{(N)} = num[z_n^{(N)}]$ given $N \geq 3$, because it has already been shown that for $N = 2$,

$$q_n^{(2)} = \begin{cases} n, & \text{for } n \in [1, K_2 - 1] \\ K_2, & \text{for } n \in [K_2, K_1] \\ K_1 + K_2 - n, & \text{for } n \in [K_2 + 1, K_1] \end{cases} \qquad (11)$$

It follows that,

$$q_n^N = \begin{cases} \sum_{i=1}^{n} q_i^{(N-1)}, \\ \qquad\qquad \text{for } n \in [1, K_N - 1], \\ \sum_{i=1}^{K^N} q_i^{(N-1)}, \\ \qquad\qquad \text{for } n \in \left[K_N, K^{(N-1)}\right], \\ \sum_{i=1}^{n} q_{K^{(N-1)} + K_N - i}^{(N-1)}, \\ \qquad\qquad \text{for } n \in \left[K^{(N-1)} + 1, K^{(N)}\right] \end{cases} \qquad (12)$$

for $N \geq 3$. From (12), it is observed that the total number of product terms in each observation of $\mathbf{z}^{(N)}(n)$ on expanding in equation (9) from $\mathbf{z}^{(N-1)}(n) \rightarrow \mathbf{z}^{(N-2)}(n) * \mathbf{z}_{N-1}(n) \rightarrow \ldots \rightarrow \mathbf{x}_1(n) * \mathbf{x}_2(n) * \ldots * \mathbf{x}_{N-1}(n)$ is evaluated in a recursive manner with the help of (11). Based on this result, then mean of $p_j(z^{(N)})$ is

$$\mu_{Z_j^{(N)}} = q_j^{(N)} \mu_{X_1 X_2 \ldots X_N}, \qquad (13)$$

and for observations of X_i drawn independently from $p(x_i), \forall i \in [1, N]$, the variance is given by

$$\sigma_{Z_j^{(N)}}^2 = q_j^{(N)} \sigma_{X_1 X_2 \ldots X_N}^2, \qquad (14)$$

otherwise

$$\sigma_{Z_j^{(N)}}^2 = \left[q_j^{(N)}\right]^2 \sigma_{X_1 X_2 \ldots X_N}^2 \qquad (15)$$

for $j \in [1, K_N]$. The evaluation of the exact variance of the product $\prod_{i=1}^{N} X_i$, given $N \geq 2$, for independent and not necessarily independent variables is fully covered in [6]. The density function of the signal $\mathbf{z}^{(N)}(n)$ is the mixture model

$$p(z^{(N)}) = \sum_{j=1}^{K_N} w_j p_j(z^{(N)}) \qquad (16)$$

Using (12), it follows that $w_1 = w_2 = \ldots = w_{K_N-1} = \frac{2}{K^{(N)}}$ since the first expression is a reversal of the last expression. That is, the density functions $p_1(z^{(N)}), p_2(z^{(N)}), \ldots, p_{K_N-1}(z^{(N)})$ constitute two observations each towards the final mixture density given by (16). The last relative weight is given by $w_{K_N} = \frac{K^{(N-1)} - K_N + 1}{K^{(N)}}$. The mean of $\mathbf{z}^{(N)(n)}$ is

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$$\mu_{Z^{(N)}} = \int_{-\infty}^{\infty} z^{(N)} p(z^{(N)}) dz^{(N)} = \sum_{j=1}^{K_N} w_j \mu_{Z_j^{(N)}}$$

Substituting for the relative weights and using (13),

$$\mu_{Z^{(N)}} = \frac{\mu_{X_1 X_2 \ldots X_N}}{K^{(N)}} \zeta = \frac{\mu_{X_1 X_2 \ldots X_N}}{K^{(N)}} \varrho \qquad (17)$$

where $\quad \zeta = 2 \sum_{j=1}^{K_N - 1} q_j^{(N)} + (K^{(N-1)} - K_N + 1) q_{K_N}^{(N)}$

and $\quad \varrho = 2 q_{(K_N - 1)}^{(N+1)} + (K^{(N-1)} - K_N + 1) q_{K_N}^{(N)}$

since $\sum_{j=1}^{K_N - 1} q_j^{(N)} = q_{K_N - 1}^{(N+1)}$ based on the recursive definitions of (12). Using

$$\sigma_{Z^{(N)}} = \int_{-\infty}^{\infty} \left(z^{(N)} \right)^2 p(z^{(N)}) dz^{(N)} - \mu_{Z^{(N)}}^2$$
$$= \sum_{j=1}^{K_N} w_j \left(\sigma_{Z_j^{(N)}}^2 + \mu_{Z_j^{(N)}}^2 \right) - \mu_{Z^{(N)}}^2$$

the variance of $z^{(N)}(n)$ given the observations of $x_i(n)$ are drawn independently from $p(x_i)$, $\forall i \in N$, is given by

$$\sigma_{Z^{(N)}}^2 = \frac{1}{K^{(N)}} \{ \sigma_{X_1 X_2 \ldots X_N}^2 \varphi + \mu_{X_1 X_2 \ldots X_N}^2 \xi \} - \mu_{Z_j^{(N)}}^2 \quad (18)$$

where $\quad \varphi = 2 q_{K_N - 1}^{N+1} + (K^{N-1} - K_N + 1) q_{K_N}^{(N)}$

and $\quad \xi = 2 \sum_{j=1}^{K_N - 1} (q_j^{(N)})^2 + (K^{(N-1)} - K_N + 1)(q_{K_N}^{(N)})^2$

otherwise,

$$\sigma_{Z^{(N)}}^2 = \left(\frac{\sigma_{X_1 X_2 \ldots X_N}^2 + \mu_{X_1 X_2 \ldots X_N}^2}{K^{(N)}} \right) \psi - \mu_{Z_j^{(N)}}^2 \qquad (19)$$

where $\quad \psi = 2 \sum_{j=1}^{K_N - 1} (q_j^{(N)})^2 + (K^{(N-1)} - K_N + 1)(q_{K_N}^{(N)})^2$

and $\mu_{Z^{(N)}}$ is given by (17).

IV. Circular Convolution

With circular convolution, the analysis is somewhat simpler because the number of product terms is constant for all observations of the signal $\mathbf{z}^{(N)}(n)$. Due to zero padding, it follows that $z^{(N)}(n)$, for $n \in \left[1, K^{(N)} \right]$, has K_N product terms in the sum (where K_N is the length of the shortest variable). That is, all observations are drawn from the same density function $p(z^{(N)}) = p_{K_n}(z^{(N)})$. Therefore, the mean of the signal $\mathbf{z}^{(N)}(n)$ is given by

$$\mu_{Z^{(N)}} = K_N \cdot \mu_{X_1 X_2 \ldots X_N} \qquad (20)$$

If the observations are drawn independently,

$$\sigma_{Z^{(N)}}^2 = K_N \cdot \sigma_{X_1 X_2 \ldots X_N}^2 \qquad (21)$$

otherwise,

$$\sigma_{Z^{(N)}}^2 = K_N^2 \cdot \sigma_{X_1 X_2 \ldots X_N}^2 \qquad (22)$$

V. Conclusion

The generalized density functions, and mathematical expressions for the mean and variance of the convolution (linear and circular) of signals have been derived based on the assumption that the statistical properties of the individual signals being processed are a known prior. The variance expressions rely on the statistical dependence between observations of a given signal, whereas the mean expression is not. It has been shown that the linear convolution signal has a mixture density, with the number of density components equal to the length of the shortest signal. The circular convolution signal has a relatively simpler description as observations are characterized by a single density function.

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