Prime Cordial Labeling on Graphs

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Abstract—A prime cordial labeling of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to $\{1, 2, \ldots, |V|\}$ such that each edge $uv$ is assigned the label $1$ if $gcd(f(u), f(v)) = 1$ and $0$ if $gcd(f(u), f(v)) > 1$, then the number of edges labelled with $0$ and the number of edges labelled with $1$ differ by at most $1$. In this paper we exhibit some characterization results and new constructions on prime cordial graphs.

Keywords—Prime cordial, tree, Euler, bijective, function.

I. INTRODUCTION

Graph labeling is a strong relation between numbers and structure of graphs. A useful survey to know about the numerous graph labeling methods is given by J.A. Gallian [7]. By combining the relatively prime concept in number theory and cordial labeling [6] concept in graph labeling, Sundaram, Ponraj and Somasundaram [9] have introduced the concept called prime cordial labeling. A prime cordial labeling of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to $\{1, 2, \ldots, |V|\}$ such that each edge $uv \in E$ is

$$f(uv) = \begin{cases} 1 & \text{if } gcd(f(u), f(v)) = 1 \\ 0 & \text{if } gcd(f(u), f(v)) > 1 \end{cases}$$

then $|e(0) - e(1)| \leq 1$ where $e(0)$ is the number of edges labeled with $0$ and $e(1)$ is the number of edges labeled with $1$. In [4], [5], [9], the following graphs are proved to have prime cordial labeling: $C_n$ if and only if $n \equiv 6 \pmod{10}$; $P_n$ if and only if $n \neq 3$ or $5$; $K_{1,n}$ (n odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \equiv 3 \pmod{4}$; bistars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders, sun graph, kite graph $(n \geq 10)$ and coconut tree, $S_n^{(1)} : S_n^{(2)}$, full binary trees from second level, $K_2 \Theta C_n$ $(C_n)$ and $K_{1,1,2}$ for $n \geq 3$.

S.K. Vaidya in [11], [12] proved that the square graph of path $P_n$ is a prime cordial graph for $n = 6$ and $n \geq 8$ while the square graph of cycle $C_n$ is a prime cordial graph for $n \geq 10$. Also they show that the shadow graph of $K_{1,n}$ for $n \geq 4$ and the shadow graph of $B_{n,n}$ are prime cordial graphs, certain cycle related graphs, the graphs obtained by mutual duplication of a pair of edges as well as mutual duplication of a pair of vertices from each of two copies of cycle $C_n$ admit prime cordial labeling. In [8] Haque proved that prime cordial labeling of generalized Petersen graph. Also in [3], prime cordial labeling for some class of cactus graphs and discuss with duality of prime cordial labeling. Sundaram and Somasundaram [10] and Youssef observed that for $n > 3$, $K_n$ is not prime cordial provided that the inequality $\varphi(2) + \varphi(3) + \cdots + \varphi(n) > n(n+1)/4 + 1$ is valid for $n > 3$. The definitions and results we used in our paper is given as follows

**Theorem 1:** [13] In any binary edge labeling the following are equivalent. (i) $|e(0) - e(1)| \leq 1$

(ii) $e(1) = \begin{cases} \frac{q}{2} & \text{if } q \text{ is even} \\ \lfloor \frac{q}{2} \rfloor \text{or} \lfloor \frac{q}{2} \rfloor - 1 & \text{if } q \text{ is odd} \end{cases}$

where $[x]$ denotes the smallest integer greater than or equal to $x$. (iii) $\sum_{j=1}^{q} e_j + (q \mod 2)e_d = \lfloor \frac{q}{2} \rfloor$ where $e_d$ is the binary label of a dummy edge which is introduced only when $q$ is odd.

**Definition 1:** [2] If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two connected graphs, $G_1 \circ G_2$ is obtained by superimposing any selected vertex of $G_2$ on any selected vertex of $G_1$. The resultant graph $G = G_1 \circ G_2$ consists of $p_1 + p_2 - 1$ vertices and $q_1 + q_2$ edges.

II. CHARACTERIZATION RESULTS ON PRIME CORDIAL GRAPHS

In this section we give some characterization results for prime cordial graphs.

**Theorem 2:** A $(p,q)$-graph $G$ is prime cordial. Let $f : V(G) \to \{1, 2, \ldots, p\}$ be a prime cordial labeling then for each edge $uv \in E(G)$,

$$\sum_{uv \in E(G)} f(u) + f(uv) + f(v) = \sum_{uv \in E(G)} f(u) + \sum_{uv \in E(G)} f(uv) + \sum_{uv \in E(G)} f(v)$$

Proof: Let $f$ be a prime cordial labeling of $G(p,q)$. Consider

$$\sum_{uv \in E(G)} f(u) + f(uv) + f(v) = \sum_{uv \in E(G)} f(u) + \sum_{uv \in E(G)} f(uv) + \sum_{uv \in E(G)} f(v)$$

Since $f$ is a bijective function, each vertex label $f(u), u \in V(G)$ occurs exactly $deg(u)$ times once in the summation and each label $uv \in E(G)$ admits binary labeling $\{0, 1\}$ by the concept of prime cordial which implies $\sum_{uv \in E(G)} f(uv) = e(1)$. Since by (ii) of 1.1,

$$\sum_{i=1}^{p} f(v_i)\deg(v_i) + e(1)$$
\[ \sum_{uv \in E(G)} f(u) + f(uv) + f(v) = \begin{cases} \sum_{i=1}^{p} f(v_i) \deg(v_i) + \frac{q}{2} & \text{if } q \text{ is even} \\ \sum_{i=1}^{p} f(v_i) \deg(v_i) + \left(\left\lfloor \frac{q}{2} \right\rfloor \text{ or } \left\lceil \frac{q}{2} \right\rceil - 1 \right) & \text{if } q \text{ is odd} \end{cases} \]

**Corollary 1:** Let \( G \) be an \( r \)-regular prime cordial graph on \( p \) vertices and \( S = \sum_{uv \in E(G)} f(u) + f(uv) + f(v) \) then

\[ S = \begin{cases} r \left( \frac{p(p+1)}{2} \right) + \frac{q}{2} & \text{if } q \text{ is even} \\ r \left( \frac{p(p+1)}{2} \right) + \left( \left\lfloor \frac{q}{2} \right\rfloor \text{ or } \left\lceil \frac{q}{2} \right\rceil - 1 \right) & \text{if } q \text{ is odd} \end{cases} \]

**Corollary 2:** If a \( (p,q) \)-graph \( G \) is Prime cordial and

\[ S = \sum_{uv \in E(G)} f(u) + f(uv) + f(v) \]

\[ S = \frac{p(p+1)}{2} \]

\[ = \begin{cases} \sum_{i=1}^{p} f(v_i) \deg(v_i - 1) + \frac{q}{2} & \text{if } q \text{ is even} \\ \sum_{i=1}^{p} f(v_i) \deg(v_i - 1) + \left(\left\lfloor \frac{q}{2} \right\rfloor \text{ or } \left\lceil \frac{q}{2} \right\rceil - 1 \right) & \text{if } q \text{ is odd} \end{cases} \]

**Theorem 3:** If \( G \) is a prime cordial graph, then \( G - e \) is also prime cordial

(i) for all \( e \in E(G) \) when \( q \) is even

(ii) for some \( e \in E(G) \) when \( q \) is odd

**Proof:** Consider the prime cordial graph \( G \) with \( p \) vertices and \( q \) edges.

**Case (i):** when \( q \) is even

Let \( q \) be the even size of the prime cordial graph \( G \). Then it follows that \( e(0) = e(1) = q/2 \). Let \( e \) be any edge in \( G \) which is labeled either 0 or 1. Then in \( G - e \), we have either

\[ e(0) = e(1) + 1 \text{ or } e(1) = e(0) + 1 \text{ and hence } |e(0) - e(1)| \leq 1. \]

Thus \( G - e \) is prime cordial for all \( e \in E(G) \).

**Case (ii):** when \( q \) is odd

Let \( q \) be the odd size of the prime cordial graph \( G \). Then it follows that either \( e(0) = e(1) + 1 \) or \( e(1) = e(0) + 1 \). If \( e(0) = e(1) + 1 \) then remove the edge \( e \) which is labeled as 0 and if \( e(1) = e(0) + 1 \) then remove the edge \( e \) which is labeled 1 from \( G \). Then it follows that \( e(0) = e(1) \). Thus \( G - e \) is prime cordial for some \( e \in E(G) \).

**Remark 1:** \( G \) is a prime cordial graph. Similarly we can prove primeness for the graph which is obtained by adding one edge to \( G \).

It becomes an interesting problem to investigate the maximum number of possible edges that can be constructed in a graph with \( n \) vertices having prime cordial labeling property. We interpret this problem in number theory to find total number of relatively prime pair of integers in the set \( \{1,2,3,\ldots,n\} \) and find the maximum number of edges in prime cordial graph using Euler’s phi function \( \varphi(n) \). In [1], the total number of relatively prime pairs for the set \( S_n = \{1,2,3,\ldots,n\} \) is given by \( |S_n| = \sum_{k=1}^{n} \varphi(k) \). For \( n \) vertices, the number of possible edges labeled with 1 is \( e(1) = \sum_{k=1}^{n} \varphi(k) \) then the number of possible edges labeled with 0 is \( e(0) = nC_2 - \sum_{k=1}^{n} \varphi(k) \). By the definition of prime cordiality, if \( e(0) \) is \( nC_2 - \sum_{k=1}^{n} \varphi(k) \) then \( e(1) \) is at most \( nC_2 - \sum_{k=1}^{n} \varphi(k) + 1 \).

**Theorem 4:** A maximum number of edges in a simple prime cordial graph with \( n \) vertices is \( n^2 - n + 1 + 2 \sum_{k=1}^{n} \varphi(k) \).

**Proof:** Consider a graph \( G(V,E) \) with \( n \) vertices is a simple graph with no loops and parallel edges. The vertices \( \{v_1,v_2,\ldots,v_n\} \) are labeled with the integers \( \{1,2,\ldots,n\} \) such that each edge \( v_iv_j \in E \) is assigned the label 1 if \( \gcd(f(u),f(v)) = 1 \) and 0 if \( \gcd(f(u),f(v)) > 1 \), then the number of edges labeled 0 and the number of edges labeled with 1 differ by at most 1.

Number of relatively prime pairs for the set \( \{1,2,\ldots,n\} \) is \( \sum_{k=1}^{n} \varphi(k) \). For \( n \) vertices, the number of possible non relatively prime pairs is \( nC_2 - \sum_{k=1}^{n} \varphi(k) \). By the definition of prime cordial, if \( e(0) \) is \( nC_2 - \sum_{k=1}^{n} \varphi(k) \) then \( e(1) \) is at most \( nC_2 - \sum_{k=1}^{n} \varphi(k) + 1 \). For any prime cordial graph, the maximal number of edges is

\[ e(0) + e(1) = nC_2 - \sum_{k=1}^{n} \varphi(k) + nC_2 - \sum_{k=1}^{n} \varphi(k) + 1 \]

\[ = 2nC_2 - 2 \sum_{k=1}^{n} \varphi(k) - n(n-1) + 1 \]

\[ = n^2 - n + 1 \]

\[ = n^2 - n + 1 - 2 \sum_{k=2}^{n} \varphi(k). \]

Hence maximum number of edges in a simple prime cordial graph with \( n \) vertices is \( n^2 - n + 1 - 2 \sum_{k=2}^{n} \varphi(k) \).

**III. PRIME CORDIAL LABELING FOR \( G_1\text{G}_2 \)**

In the following theorems we consider the prime cordial graph \( G \) and glue a vertex of some class of graphs to one of the selected vertex of \( G \) and check whether the new graph retain the property of prime cordial. For that we need the theorem given by [13] wieslet.

**Theorem 5:** If \( G \) has prime cordial labeling then, \( G\overline{G}_1 \) admits prime cordial labeling for

(i) \( m \) is even and \( G \) is of any size \( q \).

(ii) \( m \) is odd and

(a) \( G \) is of even size \( q \).

(b) \( G \) is of odd size \( q \) with \( e(1) = \left\lfloor \frac{q}{2} \right\rfloor \) and order is odd.

(c) \( G \) is of odd size \( q \) with \( e(1) = \left\lceil \frac{q}{2} \right\rceil - 1 \) and order is even.

**Proof:** Let \( G(p,q) \) be a prime cordial graph. Let \( w \in V \) be the vertex whose label is \( f(w) = 2 \). Consider the star \( K_{1,m} \) with vertex set \( \{x_0,x_1 : 1 \leq i \leq m\} \) and edge set \( \{x_0x_i : 1 \leq i \leq m\} \). We superimpose the vertex \( x_0 \) of the star \( K_{1,m} \) graph on the vertex \( w \in V \) of \( G \). Now we define the new graph called \( G_1 = G\overline{G}_1K_{1,m} \) with vertex set \( V_1(G_1) = V(G) \cup \{x_0 : 1 \leq i \leq m\} \) and \( E_1(G_1) = E(G) \cup \{xx_1 : 1 \leq i \leq m\} \). Consider the bijection function \( g : V_1 \rightarrow \{1,2,\ldots,p,p+1,p+2,\ldots,p+m\} \) defined
by
 \[ g(v_i) = f(v_i); \quad 1 \leq i \leq p \]

By our construction, \( f(w) = g(x_0) = 2 \).

\[ g(x_i) = p + i, \quad \text{for} \quad 1 \leq i \leq m. \]

We have to show that the graph \( G_1 = G\hat{o}K_{1,m} \) has prime cordial labeling in various cases.

**Case (i):** \( m \) is even and \( G \) is of any size \( q \)

The edge set is defined as \( g(uv) = f(uv) \) for all \( uv \in E(G) \).

If \( G \) is of even order then \( g(wx_{2i-1}) = 1 \) for \( 1 \leq i \leq m/2 \);

\[ g(wx_{2i}) = 0 \quad \text{for} \quad 1 \leq i \leq m/2. \]

If \( G \) is of odd order then \( g(wx_{2i-1}) = 0 \) for \( 1 \leq i \leq m/2 \);

\[ g(wx_{2i}) = 1 \quad \text{for} \quad 1 \leq i \leq m/2. \]

The total number of edges labeled with 1's are given by \( e(1) = \frac{m}{2} \) and the total number of edges labeled with 0's are given by \( e(0) = \frac{m}{2} - 1 \). Since \( G \) is prime cordial labeling of odd size \( q \), the total number of edges labeled with 0's are given by \( e(0) = \frac{q}{2} - 1 \) and the total number of edges labeled with 0's are given by \( e(0) = \frac{q}{2} - 1 + \frac{p}{2} \).

Therefore the total difference between 1's and 0's is given by \( |e(1)| - |e(0)| = \frac{q}{2} - 1 + \frac{p}{2} - \frac{q}{2} + \frac{p}{2} = \).

**Subcase (ii):** \( G \) is of odd size \( q \)

The bijective function \( g : V_1 \rightarrow \{1, 2, \ldots, p, p + 1, p + 2, \ldots, p + m\} \) defined as given above.

The edge set is defined as \( g(uv) = f(uv) \) for all \( uv \in E(G) \).

If \( G \) is of even order then \( g(wx_{2i-1}) = 1 \) for \( 1 \leq i \leq \frac{m}{2} \);

\[ g(wx_{2i}) = 0 \quad \text{for} \quad 1 \leq i \leq \frac{m}{2}. \]

If \( G \) is of odd order then \( g(wx_{2i-1}) = 0 \) for \( 1 \leq i \leq \frac{m}{2} \);

\[ g(wx_{2i}) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{m}{2}. \]

The total number of edges labeled with 1's are given by \( e(1) = \frac{m}{2} \) and the total number of edges labeled with 0's are given by \( e(0) = \frac{m}{2} - 1 \). Since \( G \) is prime cordial labeling of even size \( q \), the total number of edges labeled with 0's are given by \( e(0) = \frac{q}{2} - 1 \) and the total number of edges labeled with 0's are given by \( e(0) = \frac{q}{2} - 1 + \frac{p}{2} \).

Therefore the total difference between 1's and 0's is given by \( |e(1)| - |e(0)| = \frac{q}{2} - 1 + \frac{p}{2} - \frac{q}{2} + \frac{p}{2} = \).

**Theorem 6:** If \( G \) has prime cordial labeling then, \( G\hat{o}Fm \) admits prime cordial labeling for

(i) \( m \) is even and \( G \) is of size \( q \).

(ii) \( m \) is odd and

(a) \( G \) is of even size \( q \).

(b) \( G \) is of odd size \( q \) with \( e(1) = \frac{q}{2} \) and order is odd.

(c) \( G \) is of odd size \( q \) with \( e(1) = \frac{q}{2} - 1 \) and order is even.

**Proof:** Let \( G(p, q) \) be a prime cordial graph. Let \( w \in V \) be the vertex whose label is \( f(w) = 2 \). Consider the path \( P_m \) with vertex set \( \{x_1 : 1 \leq i \leq m\} \) and edge set \( \{(x_i, x_{i+1} : 1 \leq i \leq m - 1\} \). We superimpose one of the pendent vertex of the path \( P_m \) say \( x_0 \) on the vertex \( w \in V \) of \( G \). Now we define the new graph called \( G_1 = G\hat{o}Fm \) with vertex set \( V_1(G_1) = V(G) \cup \{x_0 \} \) and edge set \( E_1(G_1) = E(G) \cup \{wx_2, \ldots, x_{m-1}\} \) defined by

\[ g(v_i) = f(v_i); \quad 1 \leq i \leq p \]

By our construction, \( g(w) = g(x_0) = 2 \).

\[ g(x_i) = p + i - 3, \quad \text{for} \quad 2 \leq i \leq \frac{m+2}{2}; \quad g(x_p+i) = p + m - 2(i - 1), \quad \text{for} \quad 2 \leq i \leq \frac{m}{2}. \]

When \( p \) is odd and \( m \) is even, \( g(x_0) = p + 2i - 3 \), for \( 2 \leq i \leq \frac{m+2}{2} \).

\[ g(x_{p+i}) = p + m - 2(i - 1), \quad \text{for} \quad 2 \leq i \leq \frac{m}{2}. \]

When \( p \) is odd and \( m \) is odd, \( g(x_0) = p + 2i - 3 \), for \( 2 \leq i \leq \frac{m+2}{2} \).

\[ g(x_{p+i}) = p + m - 2i + 1, \quad \text{for} \quad 2 \leq i \leq \frac{m}{2}. \]

When \( p \) is even and \( m \) is even, \( g(x_0) = p + 2(i - 1), \quad \text{for} \quad 2 \leq i \leq \frac{m}{2}; \quad g(x_{p+i}) = p + m - 2(i - 1), \quad \text{for} \quad 2 \leq i \leq \frac{m}{2}. \]
Using the above labeling and similar method of Theorem ??, one can easily verify that the graph $G_1 = G\tilde{O} F_m$ has prime cordial labeling, all the cases given in hypothesis.

**Theorem 7:** If $G$ has prime cordial labeling then, $G\tilde{O} F_{1,m}$ admits prime cordial labeling for

(i) $m$ is even and
   
   (a) $G$ is of even size $q$.
   
   (b) $G$ is of odd size with $e(1) = \left\lceil \frac{q}{2} \right\rceil - 1$.

(ii) $m$ is odd and

(c) $G$ is of even size $q$ and odd order.

(d) $G$ is of odd size $q$ with $e(1) = \left\lceil \frac{q}{2} \right\rceil$ and order is odd.

**Proof:** Let $G(p, q)$ be a prime cordial graph. Let $w \in V$

\[ g(uv) = f(uv) \] for all $uv \in E(G)$.

For any $p$, $g(v_i) = 0$ for $1 \leq i \leq m/2$; $g(v_i) = 1$ for

\[ \frac{m}{2} + 1 \leq i \leq m \] and

\[ g(x_{i+1}) = 0 \] for $1 \leq i \leq \frac{m}{2} - 1$; $g(x_{i+1}) = 1$ for

\[ \frac{m}{2} \leq i \leq m - 1. \]

Hence the total number of edges labeled with 1's are given by $e(1) = m$ and the total number of edges labeled with 0's are given by $e(0) = m - 1$. Since $G$ is prime cordial labeling of even size, $e(1) = \frac{q}{2}$ and $e(0) = \frac{q}{2}$. Hence for the graph $G_1 = G\tilde{O} F_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \frac{q}{2} + m$ and the total number of edges and 0's is given by $e(0) = \frac{q}{2} + m - 1$.

Therefore the total difference between 1's and 0's is given by $e(1) - e(0) = \left\lceil \frac{q}{2} + m - \left( \frac{q}{2} + m - 1 \right) \right\rceil = 1$.

**Case (i):** $m$ is odd.

**Subcase (i):** $G$ is of even size $q$ and order is odd.

The bijective function $g: V_1 \rightarrow \{1, 2, \ldots, p, p + 1, p + 2, \ldots, p + m\}$ defined as given in above.

The edge set is defined as $g(uv) = f(uv)$ for all $uv \in E(G)$.

\[ g(v_i) = 0 \] for $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$; $g(v_i) = 1$ for

\[ \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq m \] and

\[ g(x_{i+1}) = 0 \] for $1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$; $g(x_{i+1}) = 1$ for

\[ \left\lfloor \frac{m}{2} \right\rfloor \leq i \leq m - 1. \]

The total number of edges labeled with 1’s are given by $e(1) = m - 1$ and the total number of edges labeled with 0’s are given by $e(0) = m$. Since $G$ is prime cordial labeling of even size, $e(1) = \frac{q}{2}$ and $e(0) = \frac{q}{2}$. Hence for the graph $G_1 = G\tilde{O} F_{1,m}$, the total number of edges labeled with 1’s are given by $e(1) = \frac{q}{2} + m - 1$ and the total number of edges labeled with 0’s are given by $e(0) = \frac{q}{2} + m - 1$.

Therefore the total difference between 1’s and 0’s is given by $e(1) - e(0) = \left\lfloor \frac{q}{2} - 1 + m - \left( \frac{q}{2} + m - 1 \right) \right\rfloor = 0$.

**Case (ii):** $m$ is odd.

**Subcase (ii):** $G$ is of odd size $q$ with $e(1) = \left\lceil \frac{q}{2} \right\rceil$ and order is odd.

Since $G$ is prime cordial labeling of odd size $q$ with $e(1) = \left\lceil \frac{q}{2} \right\rceil$, then $e(0) = \left\lfloor \frac{q}{2} \right\rfloor - 1$. Hence for the graph $G_1 = G\tilde{O} F_{1,m}$, the total number of edges labeled with 1’s are given by $e(1) = \left\lceil \frac{q}{2} \right\rceil + m - 1$ and the total number of edges labeled with 0’s are given by $e(0) = \left\lfloor \frac{q}{2} \right\rfloor - 1 + m$.

Therefore the total difference between 1’s and 0’s is given by $e(1) - e(0) = \left\lfloor \frac{q}{2} + m - 1 - \left( \frac{q}{2} - 1 + m \right) \right\rfloor = 0$.

Hence the graph $G\tilde{O} F_{1,m}$ has prime cordial labeling with above conditions.
IV. PRIME CORDIAL LABELING FOR TREE RELATED GRAPHS

Theorem 8: The tree $K_{1,m} \cap P_n$ ($n, m > 2$) obtained by replacing each edge of $K_{1,m}$ by equal length of path has prime cordial labeling.

Proof: Consider the tree $K_{1,m} \cap P_n$ with vertex set $V = \{v_i : i = 1, 2, 3, \ldots, nm + 1\}$ and the edge set $E = \{E_1 \cup E_2 \cup E_3 \cup E_4 \cup \cdots \cup E_{m-1} \cup E_{m+1}\}$ where $E_1 = \{v_{i+1} : 2 \leq i \leq m\}$, $E_2 = \{v_{i+2} : n + 2 \leq i \leq 2n\}$, $E_3 = \{v_i v_{i+1} : n(m - 1) + 2 \leq i \leq nm\}$ and $E_{m+1} = \{v_{(i-1)n+2} : 1 \leq i \leq m\}$. We prove this theorem in three cases.

Case (i): For even $m$ and any value of $n$.
The bijective function $f : V \rightarrow \{1, 2, \ldots, nm + 1\}$ defined as $f(v_i) = 2i$ for $1 \leq i \leq \lfloor \frac{nm}{2} \rfloor$.
$f(v_i) = nm + 1$ and $f(v_{i+1}) = 1$; $f(v_{i+2}) = 3$; $f(v_{i+3}) = 9$; $f(v_{i+4}) = 5$; $f(v_{i+5}) = 7$; $f(v_{i+6}) = 2i - 3$ for $7 \leq i \leq \lfloor \frac{nm}{2} \rfloor - 1$.

By the above labeling, the $\frac{nm}{2}$ edges have label 0 and the $\frac{nm}{2}$ edges have label 1. Hence the difference between the total number of edges labeled with 1’s and 0’s is given by $\frac{nm}{2} - \frac{nm}{2} = 0$.

Case (ii): For odd $m$ and even $n$.
The bijective function $f : V \rightarrow \{1, 2, \ldots, nm + 1\}$ defined as $f(v_i) = 2i$ for $1 \leq i \leq \lfloor \frac{nm}{2} \rfloor$.
$f(v_{i+1}) = 2i - 1$ for $6 \leq i \leq \lfloor \frac{nm}{2} \rfloor - 1$.

By the above labeling, the $\frac{nm}{2}$ edges have label 0 and the $\frac{nm}{2}$ edges have label 1. Hence the difference between the total number of edges labeled with 1’s and 0’s is given by $\frac{nm}{2} - \frac{nm}{2} = 0$.

Case (iii): For odd $m$ and odd $n$.
The bijective function $f : V \rightarrow \{1, 2, \ldots, nm + 1\}$ defined as $f(v_i) = 2i$ for $1 \leq i \leq \lfloor \frac{nm}{2} \rfloor$.
$f(v_{i+1}) = 2i - 1$ for $6 \leq i \leq \lfloor \frac{nm}{2} \rfloor - 1$.

By the above labeling, the $\frac{nm}{2}$ edges have label 0 and the $\frac{nm}{2}$ edges have label 1. Hence the difference between the total number of edges labeled with 1’s and 0’s is given by $\frac{nm}{2} - \frac{nm}{2} = 0$.

Theorem 9: The tree $P_n \cap P_m$ $(n, m \geq 2)$ obtained by replacing each vertex of $P_n$ by equal length of path has prime cordial labeling.

Proof: Consider the tree $P_n \cap P_m$ with the vertex set $V = \{v_1, v_2, \ldots, v_{mn}, v_{mn+1}, \ldots, v_{2mn}, v_{2mn+1}, \ldots, v_{(m-1)n} m, v_{(m-1)n+1} + \ldots, v_{nm}\}$ and the edge set $E = \{E_1 \cup E_2 \cup \cdots \cup E_{m-1} \cup E_{m+1}\}$ where $E_1 = \{v_i v_{i+1} : 1 \leq i \leq m - 1\}$; $E_2 = \{v_{i+2} : n + 2 \leq i \leq 2n\}$; $E_{m-1} = \{v_{i+2} : (n-2)m + 1 \leq i \leq (n-1)m - 1\}$; $E_n = \{v_i v_{i+1} : (n-1)m + 1 \leq i \leq nm - 1\}$ and $E_{m+1} = \{v_{(i-1)n+2} : 1 \leq i \leq n - 2\}$. We prove this theorem in three cases.

Case (i): For any value of $n$ and $m = 2$.
The bijective function $f : V \rightarrow \{1, 2, \ldots, nm\}$ defined as $f(v_{i-1}) = 2i$ for $1 \leq i \leq n$; $f(v_i) = 2i - 1$ for $1 \leq i \leq n$.
In view of the labeling pattern defined above, one can easily verify that $|e(0) - e(1)| = 1$.

Case (ii): For even $n$ and any value of $m > 2$.
The bijective function $f : V \rightarrow \{1, 2, \ldots, nm\}$ defined as $f(v_{i-1}) = 2i$ for $1 \leq i \leq \frac{nm}{2}$; $f(v_{i+1}) = 2i - 1$ for $1 \leq i \leq \frac{nm}{2}$.
By the definition of prime cordial and the pattern of labeling given above, we have the edges in the sets $E_1 \cup E_2 \cup \cdots \cup E_{m+1}$ and the edges in the set $\{v_{im+1} v_{(i+1)m+1} : 0 \leq i \leq \frac{nm}{2}\}$ have edge label 0. The edge $v_{\left(\frac{m}{2}\right)} - 1)v_{\left(\frac{m}{2}\right)}$ have induced label 1.
Let $E_{n+1} = \{v_{im+1} v_{(i+1)m+1} : 0 \leq i \leq n - 2\}$. The vertices from $v_1$ to $v_{\left(\frac{m}{2}\right)} - 1)$ in $E_{n+1}$ have even numbers and the vertices from $v_{\left(\frac{m}{2}\right)}$ to $v_{(\frac{m}{2})}$ have odd numbers.

Claim: Prove that the labeling $f$ for any edge $w \in \{v_{im+1} v_{(i+1)m+1} : \frac{m}{2} + 1 \leq i \leq n - 2\}$ is 1 (i.e., $f(u)$ and $f(v)$ is relatively prime).

For any integer $p$,
$f(u) \equiv 0 \mod p$ and $f(v) \equiv 0 \mod p$.
$f(u) - f(v) \equiv 0 \mod p$.

Since both $f(u)$ and $f(v)$ are odd, $p$ is also odd and according to our labeling,
$|f(u) - f(v)| \equiv 0 \mod 2$.

By (1) and (2), $|f(u) - f(v)| > 0$.

Case (iii): For odd $n$ and any value of $m > 2$.

Subcase (i): For $m$ is even.
The bijective function $f : V \rightarrow \{1, 2, \ldots, nm\}$ defined as $f(v_{i-1}) = 2i$ for $1 \leq i \leq \frac{nm}{2}$; $f(v_{i+1}) = nm + 2i - 2$ for $1 \leq i \leq \frac{nm}{2}$; $f(v_{i+1}) = 2i - 1$ for $1 \leq i \leq \frac{nm}{2}$.
By the above labeling, the total number of edges having edge label 0 is $\frac{nm}{2} - 1$. The remaining $\frac{nm}{2}$ edges in the edge set $E$ have edge label 1. Hence the difference between the total number of edges labeled with 1’s and 0’s is given by $\frac{nm}{2} - 1$. Hence the difference between the total number of edges labeled with 1’s and 0’s is given by $\frac{nm}{2} - 1$.

Case (ii): For $m$ is odd.
The bijective function $f : V \rightarrow \{1, 2, \ldots, |V|\}$ defined as $f(v_{i-1}) = 2i$ for $1 \leq i \leq \frac{nm}{2}$.
By the above labeling, the total number of edges having edge label 0 is \( e(0) = \left\lfloor \frac{nm}{2} \right\rfloor \) and the total number of edges having edge label 1 is \( e(1) = \left\lceil \frac{nm}{2} \right\rceil - \left\lfloor \frac{nm}{2} \right\rfloor \). Hence the difference between the total number of edges labeled with 1’s and 0’s is given by \( |e(1) - e(0)| = \left| \left\lceil \frac{nm}{2} \right\rceil - \left\lfloor \frac{nm}{2} \right\rfloor \right| = 0 \).

The edge \( v_i^{(\frac{1}{2})m+1} v_i^{(\frac{1}{2})m+1} \) have induced label 1. Let \( E_{n+1} = \{ v_i^{(\frac{1}{2})m+1} v_i^{(\frac{1}{2})m+1} : 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \} \). The edges from \( v_1 \) to \( v_i^{(\frac{1}{2})m+1} \) in \( E_{n+1} \) have even numbers and the edges from \( v_i^{(\frac{1}{2})m+1} \) to \( v_i^{(\frac{1}{2})m+1} {\frac{1}{2}} - 1 \) have odd numbers. We can prove that the labeling \( f \) for any edge \( uv \in \{ v_i^{(\frac{1}{2})m+1} v_i^{(\frac{1}{2})m+1} : \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - 2 \} \) is relatively prime similar to the claim given in case (ii). Hence \( P_n \circ P_n \) has prime cordial labeling.

**Corollary 3:** The comb graph \( P_n \circ K_2 \) has prime cordial labeling.

**V. CONCLUSION**

According to literature survey, more work has been done in prime cordial labeling for cycle related graphs. In our work we determine the prime cordial labeling for certain classes of trees and also exhibit some characterization results and new constructions of prime cordial graph.

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