World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:7, No:5, 2013

Two iterative algorithms to compute the bisymmetric solution of the matrix equation $A_1X_1B_1 + A_2X_2B_2 + ... + A_lX_lB_l = C$

A.Tajaddini

Abstract—In this paper, two matrix iterative methods are presented to solve the matrix equation $\boldsymbol{A_1X_1B_1 + A_2X_2B_2 + ... + A_lX_lB_l = C}$ the minimum residual problem $\boldsymbol{\| \sum_{i=1}^{l} A_i\widehat{X_i}B_i - C\|_F = min_{X_i \in BR^{n_i \times n_i}} \| \sum_{i=1}^{l} A_iX_iB_i - C\|_F}$ and the matrix nearness problem $\boldsymbol{[\widehat{X_1}, \widehat{X_2}, ..., \widehat{X_l}] = \min_{[X_1, X_2, ..., X_l] \in S_E} \|[X_1, X_2, ..., X_l] - [\widetilde{X_1}, \widetilde{X_2}, ..., \widetilde{X_l}]\|_F}$, where $\boldsymbol{BR^{n_i \times n_i}}$ is the set of bisymmetric matrices, and $\boldsymbol{S_E}$ is the solution set of above matrix equation or minimum residual problem. These matrix iterative methods have faster convergence rate and higher accuracy than former methods. Paige's algorithms are used as the frame method for deriving these matrix iterative methods. The numerical example is used to illustrate the efficiency of these new methods.

Keywords—Bisymmetric matrices, Paige's algorithms , Least square.

I. Introduction

In this work, we will use the following notations. Let $R^{m\times n}$ and $BSR^{n\times n}$ denote the set of $m \times n$ real matrices and $n \times n$ real bisymmetric matrices, respectively. $S_n(S_n = (e_n, e_{n-1}, ..., e_1))$denotes the $n \times n$ reverse identity matrix (e_i denotes ith column of $n \times n$ identity matrix). The superscript T represents the transpose of a matrix. In space $R^{m\times n}$, we define inner product as: $< A, B >= trace(B^T A)$ for all $A, B \in R^{m\times n}$ which generates the Frobenius norm $\|A\|_F = \sqrt{< A, A >}$. Notation $A \bigotimes B$ is Kronecker product. The symbol $vec(A) = (a_1^T, a_2^T, ..., a_n^T)^T$ is a vector formed by the columns of given matrix $A = (a_1, a_2, ..., a_n)$. The bisymmetric matrices play an important role in information theory, linear system theory, linear estimate theory and numerical analysis [3], [13], which can be defined as follows:

Definition 1.1: Let $S_n \in R^{n\times n}$ be a reverse identity matrix. A matrix $X \in R^{n\times n}$ is said to be bisymmetric matrix if $X = X^T = S_nXS_n$.

In this paper, we consider the following three problems.

Problem I. Given $A_i \in R^{p\times n_i}$, $B_i \in R^{n_i\times q}$, $i = 1, 2, ...l$ and $C \in R^{p\times q}$, find matrix group $[X_1, X_2, ..., X_l]$ with $X_i \in BSR^{n_i\times n_i}$, $i = 1, 2, ..., l$ such that

$$A_1X_1B_1 + A_2X_2B_2 + ... + A_lX_lB_l = C. \qquad (1)$$

Problem II. Given $A_i \in R^{p\times n_i}$, $B_i \in R^{n_i\times q}$, $i = 1, 2, ...l$ and $C \in R^{p\times q}$, find matrix group $[\widehat{X_1}, \widehat{X_2}, ..., \widehat{X_l}]$ with $\widehat{X_i} \in BSR^{n_i\times n_i}$, $i = 1, 2, ..., l$ such that

$$\| \sum_{i=1}^{l} A_i\widehat{X_i}B_i - C\|_F = min_{X_i \in BR^{n_i\times n_i}} \| \sum_{i=1}^{l} A_iX_iB_i - C\|_F \qquad (2)$$

Problem III. When problem I or II is consistent. Let S_E denote its solution group set, of the minimum residual problem for given matrix group $[\widetilde{X_1}, \widetilde{X_2}, ..., \widetilde{X_l}]$ with $\widetilde{X_i} \in R^{n_i\times n_i}$, $i = 1, 2, ..., l$, find $[\widehat{X_1}, \widehat{X_2}, ..., \widehat{X_l}] \in S_E$ with $\widehat{X_i} \in BSR^{n_i\times n_i}$, such that

$$[\widehat{X_1}, \widehat{X_2}, ..., \widehat{X_l}] = \min_{[X_1, X_2, ..., X_l] \in S_E} \|[X_1, X_2, ..., X_l] - [\widetilde{X_1}, \widetilde{X_2}, ..., \widetilde{X_l}]\|_F \qquad (3)$$

In many areas of computational mathematics, control and system theory, matrix equations can be encountered. In recent years, there has been an increased interest in solving matrix equations; for example, Dai [2], Huang [4], have studied the linear matrix equation $AXB = C$ with a symmetric and skew-symmetric condition on the solution, Peng [7], [6], Shim [12], Chu [1] have studied the linear matrix equation $AXB + CYD = E$ with unknown matrices X and Y being real or complex. The methods used in these papers included generalized inverse, generalized singular value decomposition (GSVD) and canonical decomposition (CCD) of matrices. Peng [10], [11] has studied the equation $A_1X_1B_1 + A_2X_2B_2 + ... + A_lX_lB_l = C$ with the bisymmetric conditions on the solutions. Peng [11] has studied the conjugate gradient method, and show that the solvability of the matrix equation can be judged automatically. By using Paige's algorithms [5], Peng [9], [8] proposed two matrix iterative methods to get the constrained solutions of $AXB = C$ and the constrained least squares solutions of $AXB + CYD = E$, and to solve general coupled matrix equations, respectively. Motivated by the work of Peng [9], [8], we propose two iterative methods to solve the matrix equation$A_1X_1B_1 + A_2X_2B_2 + ... + A_lX_lB_l = C$ with bisymmetric condition on the solution, and matrix nearness problem II. These matrix iterative methods have faster convergence rate and higher accuracy than the iterative methods proposed in above references in some cases. We will use Paige's algorithms [5], which are based on the bidiagonalization procedure of Golub and Kahan [3] as the framework for deriving these matrix-form iterative methods. The basic idea is that we first transform the problem I into the unconstrained linear problem in vector form which can be solved by Paige's algorithms by the Kronecker product of matrices, and finally,

A. Tajaddini is with the Department of Mathematics, Shahid Bahonar University, Kerman, Iran e-mail: (atajadini@uk.ac.ir)

we transform the vector-form iterative methods into matrix-form iterative methods.
This paper is organized as follows. In section 2, we shortly recall Paige's algorithms for solving linear systems and least-squres problem, and so based on Paige's algorithms, we propose two iterative algorithms to solve problems I, II, III. Finally, in section 3, one numerical example are presented to support the theorical results of this paper.

II. Two matrix iterative methods

In this section, by extending the idea of Paige's algorithms, we construct two algorithms for solving problem I, II. We first shortly recall Paige's algorithms for solving the minimum norm solution of the following unconstrained linear system:

$$Ax = b,$$

where $A \in R^{m\times n}$ and $b \in R^m$. Paige's algorithms are based on the Bidiagonalization procedure of Golub and Kahan [3], which are summarized as follows.

Paige's Algorithm 1

1. $\tau_0 = 1; \xi_0 = -1; \omega_0 = 0; z_0 = 0; w_0 = 0;$
$\beta_1 u_1 = b; \alpha_1 v_1 = A^T u_1;$
2. For i=1,2,...
(a) $\xi_i = -\xi_{i-1}\beta_i/\alpha_i;$
(b) $z_i = z_{i-1} + \xi_i v_i;$
(c) $\omega_i = (\tau_{i-1} - \beta_i\omega_{i-1})/\alpha_i;$
(d) $w_i = w_{i-1} + \omega_i v_i;$
(e) $\beta_{i+1}u_{i+1} = Av_i - \alpha_i u_i;$
(f) $\tau_i = -\tau_{i-1}\alpha_i/\beta_{i+1};$
(g) $\alpha_{i+1}v_{i+1} = A^T u_{i+1} - \beta_{i+1}v_i;$
(h) $\gamma_i = \beta_{i+1}\xi_i/(\beta_{i+1}\omega_i - \tau_i);$
(i) $x_i = z_i - \gamma_i\omega_i;$
(j) Exit if a stopping criterion has been met.

Paige's Algorithm 2

1. $\theta_1 v_1 = A^T b; \rho_1 u_1 = Av_1; w_1 = v_1/\rho_1; \xi_1 = \theta_1/\rho_1; x_1 = \xi_1 w_1;$
2. For i=1,2,...
(a) $\theta_{i+1}v_{i+1} = A^T u_i - \rho_i v_i;$
(b) $\rho_{i+1}u_{i+1} = Av_{i+1} - \theta_{i+1}u_i;$
(c) $\omega_{i+1} = (v_{i+1} - \theta_{i+1}\omega_i)/\rho_{i+1};$
(d) $\xi_{i+1} = -\xi_i\theta_{i+1}/\rho_{i+1};$
(e) $x_{i+1} = x_i + \xi_{i+1}w_{i+1};$
(f) Exit if a stopping criterion has been met.

The real scalars α_i, β_i, ρ_i, and θ_i are chosen to be nonnegative and such that $\|u_i\|_2 = \|v_i\|_2 = 1$ in Paige's algorithms, respectively. The stopping criterion may be chosen as $\|r_i\|_2 = \|b - Ax_i\|_2 \le \epsilon$ or $\|x_i - x_{i-1}\|_2 \le \epsilon$, where $\epsilon > 0$ is a small tolerance.
Based on Paige's algorithms 1 and 2, we propose two matrix iterative algorithms to solve problem I and II.
We can show that problem I is equivalent to the linear matrix equation

$$Ax = b \tag{4}$$

where,

$$A = \begin{pmatrix} B_1^T \bigotimes A_1 & B_2^T \bigotimes A_2 & \cdots & B_l^T \bigotimes A_l \\ (S_{n_1}B_1)^T \bigotimes A_1 S_{n_1} & (S_{n_2}B_2)^T \bigotimes A_2 S_{n_2} & \cdots & (S_{n_l}B_l)^T \bigotimes A_l S_{n_l} \\ A_1 \bigotimes B_1^T & A_2 \bigotimes B_2^T & \cdots & A_l \bigotimes B_l^T \\ (S_{n_1}A_1^T)^T \bigotimes B_1^T S_{n_1} & (S_{n_2}A_2^T)^T \bigotimes B_2^T S_{n_2} & \cdots & (S_{n_l}A_l^T)^T \bigotimes B_l^T S_{n_l} \end{pmatrix}$$

$$x = \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ . \\ . \\ . \\ vec(X_l) \end{pmatrix}, \quad b = \begin{pmatrix} vec(C) \\ vec(C) \\ vec(C^T) \\ vec(C^T) \end{pmatrix}.$$

Therefore, the vector form of $\beta_1 u_1 = b$, $\alpha_1 v_1 = A^T u_1$, $\beta_{i+1}u_{i+1} = Av_i - \alpha_i u_i$, and $\alpha_{i+1}v_{i+1} = A^T u_{i+1} - \beta_{i+1}v_i$, $i = 1, 2, ...$ in Paige's algorithm 1 can be written in the matrix form

$\beta_1 = 2\|C\|_F$, $U_{1,1} = C/\beta_1$, $U_{1,2} = C/\beta_1$, $U_{1,3} = C^T/\beta_1$, $U_{1,4} = C^T/\beta_1$,

$\alpha_1 = \{\sum_{i=1}^l \|A_i^T U_{1,1} B_i^T + S_{n_i} A_i^T U_{1,2} B_i^T S_{n_i} + B_i U_{1,3} A_i + S_{n_i} B_i U_{1,4} A_i S_{n_i}\|_F^2\}^{1/2}$,

$\alpha_1 V_{1,i} = A_i^T U_{1,1} B_i^T + S_{n_i} A_i^T U_{1,2} B_i^T S_{n_i} + B_i U_{1,3} A_i + S_{n_i} B_i U_{1,4} A_i S_{n_i}$, $i = 1, 2, ..., l$,

$\beta_{k+1} = \{\|\sum_{i=1}^l A_i V_{k,i} B_i - \alpha_k U_{k,1}\|_F^2 + \sum_{i=1}^l \|A_i S_{n_i} V_{k,i} S_{n_i} B_i - \alpha_k U_{k,2}\|_F^2 + \|\sum_{i=1}^l B_i^T V_{k,i} A_i^T - \alpha_k U_{k,3}\|_F^2 + \sum_{i=1}^l \|B_i^T S_{n_i} V_{k,i} S_{n_i} A_i^T - \alpha_k U_{k,4}\|_F^2\}^{\frac{1}{2}}$

$\beta_{k+1} U_{k+1,1} = \sum_{i=1}^l A_i V_{k,i} B_i - \alpha_k U_{k,1}$,

$\beta_{k+1} U_{k+1,2} = \sum_{i=1}^l A_i S_{n_i} V_{k,i} S_{n_i} B_i - \alpha_k U_{k,2}$,

$\beta_{k+1} U_{k+1,3} = \sum_{i=1}^l B_i^T V_{k,i} A_i^T - \alpha_k U_{k,3}$,

$\beta_{k+1} U_{k+1,4} = \sum_{i=1}^l B_i^T S_{n_i} V_{k,i} S_{n_i} A_i^T - \alpha_k U_{k,4}$,

$\alpha_{k+1} = \{\sum_{i=1}^l \|A_i^T U_{k+1,1} B_i^T + S_{n_i} A_i^T U_{k+1,2} B_i^T S_{n_i} + B_i U_{k+1,3} A_i + S_{n_i} B_i U_{k+1,4} A_i S_{n_i} - \beta_{k+1} V_{k,i}\|_F^2\}^{1/2}$,

$\alpha_{k+1} V_{k+1,i} = A_i^T U_{k+1,1} B_i^T + S_{n_i} A_i^T U_{k+1,2} B_i^T S_{n_i} + B_i U_{k+1,3} A_i + S_{n_i} B_i U_{k+1,4} A_i S_{n_i} - \beta_{k+1} V_{k,i}$, i=1,2,...,l.

Also, the vector form of $\theta_1 v_1 = A^T b$, $\rho_1 u_1 = Av_1$, $\theta_{i+1}v_{i+1} = A^T u_i - \rho_i v_i$, $\rho_{i+1}u_{i+1} = Av_{i+1} - \theta_{i+1}u_i$, $i = 1, 2, ...$ in paige's algorithm 2 can be written as:

$\theta_1 = \{\sum_{i=1}^l \|A_i^T C B_i^T + S_{n_i} A_i^T C B_i^T S_{n_i} + B_i C^T A_i + S_{n_i} B_i C^T A_i S_{n_i}\|_F^2\}^{1/2}$,

$\theta_1 V_{1,i} = A_i^T C B_i^T + S_{n_i} A_i^T C B_i^T S_{n_i} + B_i C^T A_i + S_{n_i} B_i C^T A_i S_{n_i}$, i=1,2,...,l,

$\rho_1 = \{\|\sum_{i=1}^l A_i X_{1,i} B_i\|_F^2 + \|\sum_{i=1}^l A_i S_{n_i} X_{1,i} S_{n_i} B_i\|_F^2 + \|\sum_{i=1}^l B_i^T X_{1,i} A_i^T\|_F^2 + \|\sum_{i=1}^l B_i^T S_{n_i} X_{1,i} S_{n_i} A_i^T\|_F^2\}^{1/2}$,

$\rho_1 U_{1,1} = \sum_{i=1}^{l} A_i X_{1,i} B_i$, $\rho_1 U_{1,2} = \sum_{i=1}^{l} A_i S_{n_i} X_{1,i} S_{n_i} B_i$, $\rho_1 U_{1,3} = \sum_{i=1}^{l} B_i^T X_{1,i} A_i^T$, $\rho_1 U_{1,4} = \sum_{i=1}^{l} B_i^T S_{n_i} X_{1,i} S_{n_i} A_i^T$,

$\theta_{k+1} = \sum_{i=1}^{l} \{\|A_i^T U_{k,i} B_i^T + S_{n_i} A_i^T U_{k,2} B_i^T S_{n_i} + B_i U_{k,3} A_i + S_{n_i} B_i U_{k,4} A_i S_{n_i} - \rho_k V_{k,i}\|_F^2\}^{1/2}$,

$\theta_{k+1} V_{k+1,i} = A_i^T U_{k,i} B_i^T + S_{n_i} A_i^T U_{k,2} B_i^T S_{n_i} + B_i U_{k,3} A_i + S_{n_i} B_i U_{k,4} A_i S_{n_i} - \rho_k V_{k,i}$, i=1,2,...,l,

$\rho_{k+1} = \{\| \sum_{i=1}^{l} A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1}\|_F^2 + \| \sum_{i=1}^{l} A_i S_{n_i} V_{k+1,i} S_{n_i} B_i - \theta_{k+1} U_{k,2}\|_F^2 + \| \sum_{i=1}^{l} B_i^T V_{k+1,i} A_i^T - \theta_{k+1} U_{k,3}\|_F^2 + \| \sum_{i=1}^{l} B_i^T S_{n_i} V_{k+1,i} S_{n_i} A_i^T - \theta_{k+1} U_{k,4}\|_F^2 + \}^{1/2}$

$\rho_{k+1} U_{k+1,1} = \sum_{i=1}^{l} A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1}$,

$\rho_{k+1} U_{k+1,2} = \sum_{i=1}^{l} A_i S_{n_i} V_{k+1,i} S_{n_i} B_i - \theta_{k+1} U_{k,2}$,

$\rho_{k+1} U_{k+1,3} = \sum_{i=1}^{l} B_i^T V_{k+1,i} A_i^T - \theta_{k+1} U_{k,3}$,

$\rho_{k+1} U_{k+1,4} = \sum_{i=1}^{l} B_i^T S_{n_i} V_{k+1,i} S_{n_i} A_i^T - \theta_{k+1} U_{k,4}$,

Analogous results can be obtained about the minimum residual problem 1. According to above discussion, we introduce two iterative algorithms to compute the unique minimum Frobenius norm solution $[X_1, X_2, ..., X_l]$ of the problem I as:

Paige 1 B.S.

1. $\tau_0 = 1; \xi_0 = -1; \omega_0 = 0; Z_{0,1} = ... = Z_{0,l} = 0; W_{0,1} = ... = W_{0,l} = 0$;

$\beta_1 = 2\|C\|_F$, $U_{1,1} = C/\beta_1$,$U_{1,2} = C/\beta_1$, $U_{1,3} = C^T/\beta_1$, $U_{1,4} = C^T/\beta_1$;

$\alpha_1 = \{\sum_{i=1}^{l} \|A_i^T U_{1,1} B_i^T + S_{n_i} A_i^T U_{1,2} B_i^T S_{n_i} + B_i U_{1,3} A_i + S_{n_i} B_i U_{1,4} A_i S_{n_i}\|_F^2\}^{1/2}$,

$V_{1,i} = A_i^T U_{1,1} B_i^T + S_{n_i} A_i^T U_{1,2} B_i^T S_{n_i} + B_i U_{1,3} A_i + S_{n_i} B_i U_{1,4} A_i S_{n_i}$, $i = 1, 2, ..., l$,

2. For k=1,2,...

(a) $\xi_k = -\xi_{k-1} \beta_k / \alpha_k$;

(b) $Z_{k,i} = Z_{k-1,i} + \xi_k V_{k,i} = Z_{k-1,i} + \xi_k/\alpha_k (A_i^T U_{k,1} B_i^T + S_{n_i} A_i^T U_{k,2} B_i^T S_{n_i} + B_i U_{k,3} A_i + S_{n_i} B_i U_{k,4} A_i S_{n_i} - \beta_k V_{k-1,i})$, i=1,2 ...,l;

(c) $\omega_k = (\tau_{k-1} - \beta_k \omega_{k-1})/\alpha_k$;

(d) $W_{k,i} = W_{k-1,i} + \omega_k V_{k,i} = W_{k-1,i} + \omega_k/\alpha_k (A_i^T U_{k,1} B_i^T + S_{n_i} A_i^T U_{k,2} B_i^T S_{n_i} + B_i U_{k,3} A_i + S_{n_i} B_i U_{k,4} A_i S_{n_i} - \beta_k V_{k-1,i})$, i=1,2,...,l;

(e) $\beta_{k+1} = \{\| \sum_{i=1}^{l} A_i V_{k,i} B_i - \alpha_k U_{k,1}\|_F^2 + \| \sum_{i=1}^{l} A_i S_{n_i} V_{k,i} S_{n_i} B_i - \alpha_k U_{k,2}\|_F^2 + \| \sum_{i=1}^{l} B_i^T V_{k,i} A_i^T - \alpha_k U_{k,3}\|_F^2$

$+ \| \sum_{i=1}^{l} B_i^T S_{n_i} V_{k,i} S_{n_i} A_i^T - \alpha_k U_{k,4}\|_F^2\}^{\frac{1}{2}}$;

(f) $\beta_{k+1} U_{k+1,1} = \sum_{i=1}^{l} A_i V_{k,i} B_i - \alpha_k U_{k,1}$;

$\beta_{k+1} U_{k+1,2} = \sum_{i=1}^{l} A_i S_{n_i} V_{k,i} S_{n_i} B_i - \alpha_k U_{k,2}$;

$\beta_{k+1} U_{k+1,3} = \sum_{i=1}^{l} B_i^T V_{k,i} A_i^T - \alpha_k U_{k,3}$;

$\beta_{k+1} U_{k+1,4} = \sum_{i=1}^{l} B_i^T S_{n_i} V_{k,i} S_{n_i} A_i^T - \alpha_k U_{k,4}$;

(g) $\tau_k = -\tau_{k-1} \alpha_k / \beta_{k+1}$;

(h) $\alpha_{k+1} = \{\sum_{i=1}^{l} \|A_i^T U_{k+1,1} B_i^T + S_{n_i} A_i^T U_{k+1,2} B_i^T S_{n_i} + B_i U_{k+1,3} A_i + S_{n_i} B_i U_{k+1,4} A_i S_{n_i} - \beta_{k+1} V_{k,i}\|_F^2\}^{1/2}$,

(i) $\alpha_{k+1} V_{k+1,i} = A_i^T U_{k+1,1} B_i^T + S_{n_i} A_i^T U_{k+1,2} B_i^T S_{n_i} + B_i U_{k+1,3} A_i + S_{n_i} B_i U_{k+1,4} A_i S_{n_i} - \beta_{k+1} V_{k,i}$, i=1,2,...,l;

(j) $\gamma_k = \beta_{k+1} \xi_k / (\beta_{k+1} \omega_k - \tau_k)$;

(k) $X_{k,i} = Z_{k,i} - \gamma_k W_{k,i}$,i=1,2,...,l;

(l) Exit if a stopping criterion has been met.

Paige 2 B.S.

1. $\theta_1 = \{\sum_{i=1}^{l} \|A_i^T C B_i^T + S_{n_i} A_i^T C B_i^T S_{n_i} + B_i C^T A_i + S_{n_i} B_i C^T A_i S_{n_i}\|_F^2\}^{1/2}$;

$\theta_1 V_{1,i} = A_i^T C B_i^T + S_{n_i} A_i^T C B_i^T S_{n_i} + B_i C^T A_i + S_{n_i} B_i C^T A_i S_{n_i}$, i=1,2,...,l;

$\rho_1 = \{\| \sum_{i=1}^{l} A_i V_{1,i} B_i\|_F^2 + \| \sum_{i=1}^{l} A_i S_{n_i} V_{1,i} S_{n_i} B_i\|_F^2 + \| \sum_{i=1}^{l} B_i^T V_{1,i} A_i^T\|_F^2 + \| \sum_{i=1}^{l} B_i^T S_{n_i} V_{1,i} S_{n_i} A_i^T\|_F^2\}^{1/2}$;

$\rho_1 U_{1,1} = \sum_{i=1}^{l} A_i V_{1,i} B_i$;

$\rho_1 U_{1,2} = \sum_{i=1}^{l} A_i S_{n_i} V_{1,i} S_{n_i} B_i$;

$\rho_1 U_{1,3} = \sum_{i=1}^{l} B_i^T V_{1,i} A_i^T$;

$\rho_1 U_{1,4} = \sum_{i=1}^{l} B_i^T S_{n_i} V_{1,i} S_{n_i} A_i^T$;

$W_{1,i} = 1/\rho_1 V_{1,i}$, i=1,2,...,l;

$\xi_1 = \theta_1 / \rho_1$;

$X_{1,i} = \xi_1 W_{1,i}$, i=1,2,...,l;

2. For k=1,2,...

(a) $\theta_{k+1} = \sum_{i=1}^{l} \{\|A_i^T U_{k,i} B_i^T + S_{n_i} A_i^T U_{k,2} B_i^T S_{n_i} + B_i U_{k,3} A_i + S_{n_i} B_i U_{k,4} A_i S_{n_i} - \rho_k V_{k,i}\|_F^2\}^{1/2}$;

$\theta_{k+1}V_{k+1,i} = A_i^T U_{k,i} B_i^T + S_{n_i} A_i^T U_{k,2} B_i^T S_{n_i} + B_i U_{k,3} A_i + S_{n_i} B_i U_{k,4} A_i S_{n_i} - \rho_k V_{k,i}$, i=1,2,...,l;

(b) $\rho_{k+1} = \{\| \sum_{i=1}^{l} A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1}\|_F^2 + \| \sum_{i=1}^{l} A_i S_{n_i} V_{k+1,i} S_{n_i} B_i - \theta_{k+1} U_{k,2}\|_F^2 + \| \sum_{i=1}^{l} B_i^T V_{k+1,i} A_i^T - \theta_{k+1} U_{k,3}\|_F^2 + \| \sum_{i=1}^{l} B_i^T S_{n_i} V_{k+1,i} S_{n_i} A_i^T - \theta_{k+1} U_{k,4}\|_F^2\}^{1/2}$;

$\rho_{k+1} U_{k+1,1} = \sum_{i=1}^{l} A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1}$;

$\rho_{k+1} U_{k+1,2} = \sum_{i=1}^{l} A_i S_{n_i} V_{k+1,i} S_{n_i} B_i - \theta_{k+1} U_{k,2}$;

$\rho_{k+1} U_{k+1,3} = \sum_{i=1}^{l} B_i^T V_{k+1,i} A_i^T - \theta_{k+1} U_{k,3}$;

$\rho_{k+1} U_{k+1,4} = \sum_{i=1}^{l} B_i^T S_{n_i} V_{k+1,i} S_{n_i} A_i^T - \theta_{k+1} U_{k,4}$;

(c) $W_{k+1,i} = (V_{k+1,i} - \theta_{k+1} W_{k,i})/\rho_{k+1}$, i= 1,2,...,l;

(d) $\xi_{k+1} = -\xi_k \theta_{k+1}/\rho_{k+1}$;

(e) $X_{k+1,i} = X_{k,i} + \xi_{k+1} W_{k+1,i}$, i=1,2,...,l;

(f) Exit if a stopping criterion has been met.

Now, we consider the matrix nearness problem III. Suppose X_i, i=1,2,...,l are bisymmetric matrices, and $\widetilde{X}_i \in R^{n_i \times n_i}$, it follows

$$min_{X_i \in R^{n_i \times n_i}} \|[X_1, X_2, ..., X_l] - [\widetilde{X_1}, \widetilde{X_2}, ..., \widetilde{X_l}\|_F^2 = min_{X_i \in R^{n_i \times n_i}} \|[X_1 - \frac{\widetilde{X_1} + \widetilde{X_1}^T + S_{n_1}\widetilde{X_1}S_{n_1} + S_{n_1}\widetilde{X_1}^T S_{n_1}}{4}, ..., X_l - \frac{\widetilde{X}_l + \widetilde{X}_l^T + S_{n_l}\widetilde{X}_l S_{n_l} + S_{n_l}\widetilde{X}_l^T S_{n_l}}{4}]\|_F^2 + \|\frac{\widetilde{X_1} - \widetilde{X_1}^T + S_{n_1}\widetilde{X_1}S_{n_1} - S_{n_1}\widetilde{X_1}^T S_{n_1}}{4}, ..., \frac{\widetilde{X}_l - \widetilde{X}_l^T + S_{n_l}\widetilde{X}_l S_{n_l} - S_{n_l}\widetilde{X}_l^T S_{n_l}}{4}\|_F^2.$$

Hence, finding the unique solution of the matrix nearness problem III is equivalent to first finding the minimum Frobenius norm bisymmetric solution of the matrix equation I or the least-squares problem II with $C - \sum_{i=1}^{l} A_i (\frac{\widetilde{X}_i + \widetilde{X}_i^T + S_{n_i}\widetilde{X}_i S_{n_i} + S_{n_i}\widetilde{X}_i^T S_{n_i}}{4}) B_i$ instead of C. Once the minimum Frobenius norm bisymmetric solution group $[X_1^*, X_2^*, ..., X_l^*]$ is obtained by Paige 1 B.S and Paige 2 B.S, the unique bisymmetric solution group $[\widehat{X_1}, \widehat{X_2}, ..., \widehat{X_l}]$ of the matrix nearness problem III can be obtained. In this case, the solution group $[\widehat{X_1}, \widehat{X_2}, ..., \widehat{X_l}]$can be expressed as $\widehat{X_i} = X_i^* + \frac{\widetilde{X}_i + \widetilde{X}_i^T + S_{n_i}\widetilde{X}_i S_{n_i} + S_{n_i}\widetilde{X}_i^T S_{n_i}}{4}$, i=1,2,...,l.

III. Numerical examples

In this section, we compare Paige 1 B.S and Paige 2 B.S numerically with the method proposed in [11], denoted by Peng-M. All the tests were performed by Matlab 7.1. We choose the initial iterative matrix groups in the Peng's method as zero matrix group in suitable size. All the following examples are used to illustrate the performance of three methods to compute the minimum Frobenius norm bisymmetric solution group $[X_1, X_2, ..., X_l]$ of the matrix equation 1 an the minimum residual 2.

Example 3.1: Suppose that the matrices $A1$, $B1$, $A2$, $B2$, and C are given

$$A1 = \begin{pmatrix} 1 & 3 & 1 & 3 & 1 \\ 3 & -7 & 3 & -7 & 3 \\ 3 & -2 & 3 & -2 & 3 \\ 11 & 6 & 11 & 6 & 11 \\ -5 & 5 & -5 & 5 & -5 \\ 9 & 4 & 9 & 4 & 9 \end{pmatrix},$$

$$B1 = \begin{pmatrix} -1 & 4 & -1 & 4 & -1 \\ 5 & -1 & 5 & -1 & 5 \\ -1 & -2 & -1 & -2 & -1 \\ 3 & 9 & 3 & 9 & 3 \\ 7 & -8 & 7 & -8 & 7 \end{pmatrix},$$

$$A2 = \begin{pmatrix} 3 & -4 & 3 & -4 & 1 & 6 \\ -1 & 3 & -1 & 3 & -3 & -1 \\ 3 & -5 & 3 & -5 & 2 & 5 \\ 3 & -4 & 3 & -4 & 1 & 6 \\ -1 & 3 & -1 & 3 & -3 & -1 \\ 3 & -5 & 3 & -5 & 2 & 5 \end{pmatrix},$$

$$B2 = \begin{pmatrix} -5 & 4 & -1 & -5 & 4 \\ -2 & 3 & 5 & -2 & 3 \\ 3 & 5 & -1 & 3 & 5 \\ 2 & -6 & 3 & 2 & -6 \\ 1 & 11 & 7 & 1 & 11 \\ 4 & -1 & 4 & -5 & 4 \end{pmatrix},$$

and

$$C = \begin{pmatrix} -136 & 878 & 419 & -510 & 1216 \\ 898 & 481 & 701 & 1321 & 82 \\ 499 & 1779 & 943 & 406 & 1840 \\ 1088 & 1278 & 1643 & -110 & 2440 \\ -974 & -1855 & -1171 & -1015 & -1790 \\ 973 & 1431 & 1417 & 58 & 2314 \end{pmatrix}.$$

The above given matrices $A1$, $B1$, $A2$, $B2$, and C are such that the matrix equation $A_1X_1B_1 + A_2X_2B_2 = C$ have bisymmetric solution pairs $[X_1, X_2]$. Figure 1 describes the convergence rate of the function $R(k) = \|C - A_1X_1B_1 - A_2X_2B_2\|_F$ of the above two methods and conjugate gradient method.

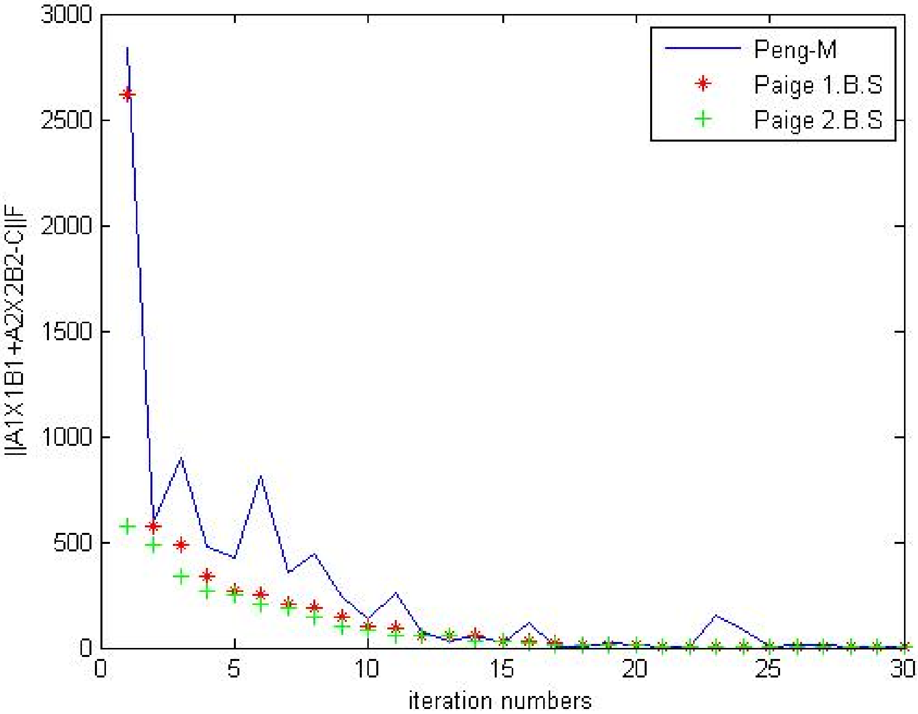

Fig. 1. The results obtained for Example 3.1

REFERENCES

[1] K. E. Chu, Singular value and generalized value decomposition and the solution of linear matrix equations, Linear Algebra Appl. , 87 (1987) 83-98.
[2] H. Dai, On the symmetric solutions of linear matrix equations, Linear Algebra Appl. , 131 (1990) 1-7.
[3] GH. Golub, W. Kahan, Calculating the singular values and pseudoinverse of a matrix, SIAM Journal on Numerical Analysis, (1965) 197-209.
[4] G. X. Huang, F. Yin, K. Guo., An iterative method for the skew-symmetric solution and the optimal approximate solution of the matrix equation $AXB = C$, J. Comput. Appl. Math. , 212 (2008) 231-244.
[5] C.C.paige, Bidiagonalization of matrices and solution of linear equation, SIAM. J. Numer. Anal. , 11 (1974) 197-209.
[6] Z.-Y. Peng, The inverse problem of bisymmetric matrices, Numer. Linear Algebra Appl. , 11 (2004) 59-73.
[7] Z.-Y. Peng, The solutions of matrix $AXC + BYD = E$ and its optimal approximation, Math. Theory Appl. , 22 (2) (2002) 99-103.
[8] Z.-Y. Peng, Solutions of symmetry-constrained least-squares problems, Numer. Linear Algebra Appl. , 15 (2008) 373-389.
[9] Z.-Y. Peng, New matrix iterative methods for constrained solutions of the matrix equation $AXB = C$, J. Comput. Appl. Math. 235 (2010) 726-735.
[10] Z.-Y. Peng, The nearest bisymmetric solutions of linear matrix equations, J. Comput. Math., 22 (6) (2004) 873-880.
[11] Z.-H. Peng, X.-Y. Hu, L. Zhang, The bisymmetric solutions of the matrix equation $A_1X_1B_1 + A_2X_2B_2 + ... + A_lX_lB_l = C$ and its optimal approximation, Linear Algebra Appl. , 426 (2007) 583-595.
[12] S.- Y. Shim, Y. chen, Least squares solution of matrix equation $AXB^* + CYD^* = E$, SIAM J. Matrix Anal.Appl., 3 (2003) 802-808.
[13] Q.W.Wang, H. Sun, S. Z.Li, Consistency for bi(skew) symmetric solutions to systems of generalized Sylvester equations over a finite central algebra, Linear Algebra Appl. 353 (2002) 169-182.