Two iterative algorithms to compute the bisymmetric solution of the matrix equation

\[ A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \]

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Abstract—In this paper, two matrix iterative methods are presented to solve the matrix equation \( A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \) the minimum residual problem \( \| \sum_{i=1}^{l} A_iX_iB_i - C \|_F = \min_{X_i \in BSR^{n \times n_i}} \| \sum_{i=1}^{l} A_iX_iB_i - C \|_F \) and the matrix nearness problem \( [X_1, X_2, \ldots, X_l] = \min_{X_i \in S_R} \| [X_1, X_2, \ldots, X_l] - [X_1, X_2, \ldots, X_l] \|_F \), where \( BSR^{n \times n_i} \) is the set of bisymmetric matrices, and \( S_R \) is the solution set of above matrix equation or minimum residual problem. These matrix iterative methods have faster convergence rate and higher accuracy than former methods. Paige's algorithms are used as the frame method for deriving these matrix iterative methods. The numerical example is used to illustrate the efficiency of these new methods.

Keywords—Bisymmetric matrices, Paige's algorithms, Least square.

I. INTRODUCTION

In this work, we will use the following notations. Let \( R^{m \times n} \) and \( BSR^{m \times n} \) denote the set of \( m \times n \) real matrices and \( m \times n \) real bisymmetric matrices, respectively. \( S_R(S_n = (e_n, e_{n-1}, \ldots, e_1)) \) denotes the \( n \times n \) reverse identity matrix (\( e_i \) denotes the \( i \)-th column of \( n \times n \) identity matrix). The superscript \( T \) represents the transpose of a matrix. In space \( R^{m \times n} \), we define inner product as: \( \langle A, B \rangle = trace(B^T \cdot A) \) for all \( A, B \in R^{m \times n} \) which represents the Frobenius norm \( \| A \|_F = \sqrt{\langle A, A \rangle} \). Notation \( A \otimes B \) is Kronecker product. The symbol \( vec(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T \) is a vector formed by the columns of given matrix \( A = (a_1, a_2, \ldots, a_n) \). The bisymmetric matrices play an important role in information theory, linear system theory, linear estimate theory and numerical analysis [3, 13], which can be defined as follows:

Definition 1.1: Let \( S_n \in R^{n \times n} \) be a reverse identity matrix. A matrix \( X \in R^{n \times n} \) is said to be bisymmetric matrix if \( X = X^T = S_n \cdot X \cdot S_n \).

In this paper, we consider the following three problems.

Problem I: Given \( A_i \in R^{m \times n_i}, i = 1, 2, \ldots, l \) and \( C \in R^{p \times q} \), find matrix group \( [X_1, X_2, \ldots, X_l] \) with \( X_i \in BSR^{m \times n_i}, i = 1, 2, \ldots, l \) such that

\[ A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C. \]  (1)

Problem II: Given \( A_i \in R^{m \times n_i}, i = 1, 2, \ldots, l \) and \( C \in R^{p \times q} \), find matrix group \( [X_1, X_2, \ldots, X_l] \) with \( X_i \in BSR^{m \times n_i} \), \( i = 1, 2, \ldots, l \) such that

\[ BSR^{n_i \times n_i}, i = 1, 2, \ldots, l \] such that

\[ \sum_{i=1}^{l} A_i \hat{X}_iB_i - C \|_F = \min_{X_i \in BSR^{n \times n_i}} \| \sum_{i=1}^{l} A_iX_iB_i - C \|_F \]

Problem III: When problem I or II is consistent. Let \( S_E \) denote its solution set of the minimum residual problem for given matrix group \( [X_1, X_2, \ldots, X_i] \) with \( X_i \in R^{n_i \times n_i}, i = 1, 2, \ldots, l \), find \( [X_1, X_2, \ldots, X_i] \) in \( S_E \) with \( \hat{X}_i \in BSR^{n \times n_i}, i = 1, 2, \ldots, l \) such that

\[ [X_1, X_2, \ldots, X_i] = \min_{X_i \in S_R} \| [X_1, X_2, \ldots, X_i] - [X_1, X_2, \ldots, X_i] \|_F \]  (2)

In many areas of computational mathematics, control and system theory, matrix equations can be encountered. In recent years, there has been an increased interest in solving matrix equations; for example, Dai [2], Huang [4], have studied the linear matrix equation \( AXB = C \) with a symmetric and skew-symmetric condition on the solution, Peng [7], [6], Shim [12], Chu [1] have studied the linear matrix equation \( AXB + CYD = E \) with unknown matrices \( X \) and \( Y \) being real or complex. The methods used in these papers included generalized inverse, generalized singular value decomposition (GSVD) and canonical decomposition (CCD) of matrices. Peng [10], [11] has studied the equation \( A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \) with bisymmetric conditions on the solutions. Peng [11] has studied the conjugate gradient method, and show that the solvability of the matrix equation can be judged automatically. By using Paige’s algorithms [5], Peng [9], [8] proposed two matrix iterative methods to get the constrained solutions of \( AXB = C \) and the constrained least squares solutions of \( AXB + CYD = E \), and to solve general coupled matrix equations, respectively. Motivated by the work of Peng [9], [8], we propose two iterative methods to solve the matrix equation \( A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C \) with bisymmetric condition on the solution, and matrix nearness problem II. These matrix iterative methods have faster convergence rate and higher accuracy than the iterative methods proposed in above references in some cases. We will use Paige’s algorithms [5], which are based on the bidiagonalization procedure of Golub and Kahan [3] as the framework for deriving these matrix-form iterative methods. The basic idea is that we first transform the problem I into the unconstrained linear problem in vector form which can be solved by Paige’s algorithms by the Kronecker product of matrices, and finally,
we transform the vector-form iterative methods into matrix-
form iterative methods.
This paper is organized as follows. In section 2, we shortly
recall Paige’s algorithms for solving linear systems and least-
squares problem, and so based on Paige’s algorithms, we
propose two iterative algorithms to solve problems I, II, III.
Finally, in section 3, one numerical example is presented to
support the theoretical results of this paper.

II. TWO MATRIX ITERATIVE METHODS

In this section, by extending the idea of Paige’s algorithms,
we construct two algorithms for solving problem I, II. We
first shortly recall Paige’s algorithms for solving the minimum
norm solution of the following unconstrained linear system:

\[ Ax = b, \]

where \( A \in R^{m \times n} \) and \( b \in R^m \), Paige’s algorithms are based
on the Bidiagonalization procedure of Golub and Kahan [3],
which are summarized as follows.

Paige’s Algorithm 1
1. \( \tau_0 = 1; \xi_0 = -1; \omega_0 = 0; z_0 = 0; w_0 = 0; \)
\( \beta_1 u_1 = b; \alpha_1 v_1 = A^T u_1; \)
2. For \( i = 1, 2, \ldots \)
   a. \( \xi_i = -\xi_{i-1} \beta_i / \alpha_i; \)
   b. \( z_i = z_{i-1} + \xi_i v_i; \)
   c. \( \omega_i = (\tau_{i-1} - \beta_i \omega_{i-1}) / \alpha_i; \)
   d. \( w_i = w_{i-1} + \omega_i v_i; \)
   e. \( \beta_{i+1} u_{i+1} = A v_i - \beta_i u_i; \)
   f. \( \tau_i = -\tau_{i-1} + \beta_i; \)
   g. \( \alpha_{i+1} v_{i+1} = A^T u_{i+1} - \beta_{i+1} v_{i+1}; \)
   h. \( \gamma_i = \beta_{i+1} \xi_i / (\beta_i \omega_i - \tau_i); \)
   i. \( x_i = x_{i-1} - \gamma_i \omega_i; \)
   j. Exit if a stopping criterion has been met.

Paige’s Algorithm 2
1. \( \theta_1 v_1 = A^T b; \rho_1 u_1 = A v_1; v_1 = v_1 / \rho_1; \xi_1 = \theta_1 / \rho_1; x_1 = \xi_1 w_1; \)
2. For \( i = 1, 2, \ldots \)
   a. \( \theta_{i+1} v_{i+1} = A^T u_i - \rho_i v_i; \)
   b. \( \rho_{i+1} u_{i+1} = A v_{i+1} - \theta_{i+1} u_i; \)
   c. \( \omega_{i+1} = (v_{i+1} - \theta_{i+1} \omega_i) / \rho_{i+1}; \)
   d. \( \xi_{i+1} = -\xi_i \theta_{i+1} / \rho_i; \)
   e. \( x_{i+1} = x_i + \xi_{i+1} w_{i+1}; \)
   f. Exit if a stopping criterion has been met.

The real scalars \( \alpha_i, \beta_i, \rho_i, \) and \( \theta_i \) are chosen to be nonnegative and such that \( \| u_i \|_2 = \| v_i \|_2 = 1 \) in Paige’s algorithms,
respectively. The stopping criterion may be chosen as
\( \| x_i - x_{i-1} \|_2 \leq \epsilon \) or \( \| x_i - x_{i-1} \|_2 \leq \epsilon, \) where \( \epsilon > 0 \)
is a small tolerance.

Based on Paige’s algorithms 1 and 2, we propose two matrix
iterative algorithms to solve problem I and II.

We can show that problem I is equivalent to the linear matrix
equation

\[ Ax = b \]  \( (4) \)

where,
\[
A = \begin{pmatrix}
    n_1^2 \bigotimes A_1 & n_2^2 \bigotimes A_2 & \cdots & n_l^2 \bigotimes A_l \\
    (sn_1^2 A_1, 1) & (sn_2^2 A_2, 1) & \cdots & (sn_l^2 A_l, 1) \\
    * & * & \cdots & * \\
    * & * & \cdots & * \\
    * & * & \cdots & * \\
    * & * & \cdots & * \\
    * & * & \cdots & * \\
    * & * & \cdots & * \\
\end{pmatrix}
\]

Therefore, the vector form of \( \beta_1 u_1 = b; \alpha_1 v_1 = A^T u_1; \)
\( \beta_{i+1} u_{i+1} = A v_i - \alpha_{i+1} u_i, \) and \( \alpha_{i+1} v_{i+1} = A^T u_{i+1} - \beta_{i+1} v_i, \)
i = 1, 2, \ldots \) in Paige’s algorithm 1 can be written in the matrix
form

\[
\begin{align*}
\beta_i &= 2 \| C \|_F, U_{i,1} = C / \beta_1, U_{i,2} = C / \beta_1, U_{i,3} = C^T / \beta_1, U_{i,4} = C^T / \beta_1, \\
\alpha_i &= \{ \sum_{k=1}^{l} \| A^T U_{k+1} B_i + S_n A^T U_{k+1} B_i^T S_n + B_i U_{k+1} A_i + S_n B_i U_{k+1} A_i S_n \|_F \}^{1/2}, \\
\alpha_i V_{i,1} &= A^T U_{i,1} B_i^T + S_n A^T U_{i,1} B_i^T S_n + B_i U_{i,1} A_i + S_n B_i U_{i,1} A_i S_n, \\
\alpha_i V_{i,2} &= \sum_{k=i}^{l} A_i V_{k-1} B_i^T, A_i = \alpha_k U_{k-1} B_i, \\
\beta_{i+1} &= \sum_{k=i}^{l} \| A_i S_n V_{k-1} B_i S_n = \beta_{k+1} U_{k+1} A_i, \beta_{k+1} U_{k+1} A_i - \alpha_k U_{k+1} B_i, \\
\beta_{i+1} V_{i,2} &= \sum_{k=i}^{l} A_i S_n V_{k-1} B_i S_n - \alpha_k U_{k+1} B_i, \\
\beta_{i+1} V_{i,3} &= \sum_{k=i}^{l} B_i U_{k+1} A_i^T, \beta_{k+1} U_{k+1} A_i - \alpha_k U_{k+1} A_i, \\
\beta_{i+1} V_{i,4} &= \sum_{k=i}^{l} B_i U_{k+1} A_i^T, \beta_{k+1} U_{k+1} A_i - \alpha_k U_{k+1} A_i, \\
\alpha_k &= \{ \sum_{k=1}^{l} \| A_i S_n V_{k-1} B_i S_n = \beta_{k+1} U_{k+1} A_i, \beta_{k+1} U_{k+1} A_i - \alpha_k U_{k+1} B_i, \\
\alpha_k V_{i,1} &= A^T U_{i,1} B_i^T + S_n A^T U_{i,1} B_i^T S_n + B_i U_{i,1} A_i + S_n B_i U_{i,1} A_i S_n, \\
\theta_i &= \{ \sum_{k=1}^{l} \| A_i S_n V_{k-1} B_i S_n = \beta_{k+1} U_{k+1} A_i, \beta_{k+1} U_{k+1} A_i - \alpha_k U_{k+1} B_i, \\
\theta_i V_{i,1} &= A^T U_{i,1} B_i^T + S_n A^T U_{i,1} B_i^T S_n + B_i U_{i,1} A_i + S_n B_i U_{i,1} A_i S_n, i = 1, 2, \ldots . \}
\end{align*}
\]
\[ \rho_1 U_{1,1} = \sum_{i=1}^l A_i X_{1,i} B_i, \quad \rho_1 U_{1,2} = \sum_{i=1}^l A_i S_n X_{1,i} S_n B_i, \]
\[ \rho_1 U_{1,3} = \sum_{i=1}^l B_i^T X_{1,i} A_i^T, \quad \rho_1 U_{1,4} = \sum_{i=1}^l B_i^T S_n X_{1,i} S_n A_i^T, \]
\[ \theta_{k+1} = \sum_{i=1}^l (A_i^T U_{k+1} B_i^T + S_n A_i^T U_{k+1} B_i^T + B_i U_{k+1} A_i + S_n B_i U_{k+1} A_i - \rho_k V_{k+1,1}^2)^{1/2}, \]
\[ \theta_{k+1} U_{k+1,1} = A_i^T U_{k+1} B_i^T + S_n A_i^T U_{k+1} B_i^T + B_i U_{k+1} A_i + S_n B_i U_{k+1} A_i - \rho_k V_{k+1,1}^2, \]
\[ \beta_{k+1} = \sum_{i=1}^l (A_i^T U_{k+1} B_i^T + S_n A_i^T U_{k+1} B_i^T + B_i U_{k+1} A_i + S_n B_i U_{k+1} A_i - \rho_k V_{k+1,1}^2)^{1/2}, \]
\[ \theta_{k+1}V_{k+1,i} = A^T U_{k,i} B^T_{i} + S_n A^T U_{k,i} B^T_{i} \]

\[ + S_n B_i U_{k,i} A_i - \rho_k V_{k,i} \quad i=1,2,\ldots,l; \]

(b) \[ \rho_{k+1} = \{ \sum_{i=1}^l A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1} \}^2 + \]

\[ \sum_{i=1}^l A_i S_n V_{k+1,i} B_i - \theta_{k+1} U_{k,2} \}^2 + \]

\[ \sum_{i=1}^l B^T_{i} V_{k+1,i} A^T_{i} - \theta_{k+1} U_{k,3} \}^2 \]

\[ \rho_{k+1} U_{k+1,1} = \sum_{i=1}^l A_i V_{k+1,i} B_i - \theta_{k+1} U_{k,1}; \]

\[ \rho_{k+1} U_{k+1,2} = \sum_{i=1}^l A_i S_n V_{k+1,i} B_i - \theta_{k+1} U_{k,2}; \]

\[ \rho_{k+1} U_{k+1,3} = \sum_{i=1}^l B^T_{i} V_{k+1,i} A^T_{i} - \theta_{k+1} U_{k,3}; \]

(c) \[ W_{k+1,i} = (V_{k+1,i} - \theta_{k+1} W_{k,i})/\rho_{k+1}, \quad i=1,2,\ldots,l; \]

(d) \[ \xi_{k+1} = -\xi_k/\rho_{k+1}; \]

(e) \[ X_{k+1,i} = X_{k,i} + \xi_{k+1} W_{k+1,i}, \quad i=1,2,\ldots,l; \]

(f) Exit if a stopping criterion has been met.

Now, we consider the matrix nearness problem III. Suppose \(X_i\), l=1,2,\ldots,l are the initial iterative matrix groups in the Peng’s method as zero matrix group in suitable size. All the following examples are used to illustrate the performance of three methods to compute the minimum Frobenius norm bisymmetric solution group \([X_1, X_2, \ldots, X_l]\) of the matrix equation \(A^T X_i B_i + A X_i S_n = C\) and \(C\) are given

\[ A1 = \begin{pmatrix} 1 & 3 & 1 & 3 & 1 \\ 3 & -7 & 3 & -7 & 3 \\ 3 & -2 & 3 & -2 & 3 \\ 11 & 6 & 11 & 6 & 11 \\ -5 & 5 & -5 & 5 & -5 \\ 9 & 4 & 9 & 4 & 9 \end{pmatrix}, \]

\[ B1 = \begin{pmatrix} -1 & 4 & 1 & 4 & 1 \\ 5 & -1 & 5 & -1 & 5 \\ 3 & 9 & 3 & 9 & 3 \\ 7 & -8 & 7 & -8 & 7 \end{pmatrix}, \]

\[ A2 = \begin{pmatrix} 3 & 4 & 3 & 4 & 3 \\ 3 & 4 & 3 & 4 & 3 \\ 3 & 5 & 3 & 5 & 3 \\ 3 & 5 & 3 & 5 & 3 \\ 3 & 5 & 3 & 5 & 3 \end{pmatrix} \]

\[ B2 = \begin{pmatrix} -5 & 4 & -1 & -5 & 4 \\ -2 & 3 & 5 & 2 & -3 \\ 1 & 11 & 7 & 1 & 11 \\ 4 & 1 & 4 & 5 & 4 \end{pmatrix} \]


Example 3.1: Suppose that the matrices \(A_1, B_1, A_2, B_2, \) and \(C\) are given

\[ \text{and} \quad C = \begin{pmatrix} -136 & 878 & 419 & -510 & 1216 \\ 898 & 481 & 701 & 1321 & 82 \\ 499 & 1779 & 943 & 406 & 1840 \\ 1088 & 1278 & 1643 & -110 & 2440 \\ -974 & -1855 & -1171 & -1015 & -1790 \\ 973 & 1431 & 1417 & 58 & 2314 \end{pmatrix} \]

The above given matrices \(A_1, B_1, A_2, B_2, \) and \(C\) are such that the matrix equation \(A_1 X_1 B_1 + A_2 X_2 B_2 = C\) have bisymmetric solution pairs \([X_1, X_2]\). Figure 1 describes the convergence rate of the function \(R(k) = ||C - A_1 X_1 B_1 - A_2 X_2 B_2||_F\) of the above two methods and conjugate gradient method.

III. NUMERICAL EXAMPLES

In this section, we compare Paige 1 B.S and Paige 2 B.S numerically with the method proposed in [11], denoted by Peng-M. All the tests were performed by Matlab 7.1. We choose the initial iterative matrix groups in the Peng’s method as zero matrix group in suitable size. All the following examples are used to illustrate the performance of three methods to compute the minimum Frobenius norm bisymmetric solution group \([X_1, X_2, \ldots, X_l]\) of the matrix equation 1 an the minimum residual 2.
Fig. 1. The results obtained for Example 3.1

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