World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:3, No:4, 2009

Non-Polynomial Spline Solution of Fourth-Order Obstacle Boundary-Value Problems

Jalil Rashidinia, Reza Jalilian

Abstract— In this paper we use quintic non-polynomial spline functions to develop numerical methods for approximation to the solution of a system of fourth-order boundary-value problems associated with obstacle, unilateral and contact problems. The convergence analysis of the methods has been discussed and shown that the given approximations are better than collocation and finite difference methods. Numerical examples are presented to illustrate the applications of these methods, and to compare the computed results with other known methods.

Keywords—Quintic non-polynomial spline, Boundary formula, Convergence, Obstacle problems.

I. INTRODUCTION

In this paper, we apply non-polynomial spline functions to develop numerical methods for obtaining smooth approximations to the solution of a system of fourth-order boundary-value problem of the form:

$$u^{(4)} = \begin{cases} f(x), & a \leq x \leq c, \\ g(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \quad (1)$$

subjected to the boundary and continuity conditions

$$\begin{array}{ll} u(a) = u(b) = \alpha_1, & u''(a) = u''(b) = \alpha_2, \\ u(c) = u(d) = \beta_1, & u''(c) = u''(d) = \beta_2, \end{array} \quad (2)$$

where $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters r, α_i and $\beta_i, (i = 1, 2)$ are real constants. Such type of system arise in the study of obstacle, unilateral, moving and free boundary-value problems and has important applications in other branches of pure and applied science [1-5,10,12-14]. In general it is not possible to obtain the analytical solution of (1) for arbitrary choices of $f(x), g(x)$. A special form of problem (1) have been considered by the numbers of authors [1-4,13,14] they used finite difference, collocation and spline methods. In the present paper, we apply non-polynomial spline functions [16,17,20] that have a polynomial and trigonometric part to develop numerical methods for obtaining smooth approximation to the solutions of such system. These methods are based on a non-polynomial spline space. The spline functions we propose in this paper have the form

$$a \sin(kx) + b \cos(kx) + cx^3 + dx^2 + ex + f.$$

We develop the class of various methods. Our method perform better than the other collocation, finite difference and spline methods of same order. This approach has the advantage over finite difference methods that it provides continuous approximations to not only for $u(x)$ but also for $u^{(i)}(x), i = 1, 2, 3$, at every point of the range of integration. Also, the c^∞-differentiability of the trigonometric part of non-polynomial spline compensates for the loss smoothness inherited by polynomial spline. The spline function we propose in this paper has the form

$$\text{Span}\{1, x, x^2, x^3, \sin(|k|x), \cos(|k|x)\},$$

where k is the frequency of trigonometric part of the spline function,when $k \to 0$ our spline reduce to the form:

$$\text{Span}\{1, x, x,^2, x^3, x^4, x^5\}, \qquad \text{(when k=0)}.$$

The above fact is evident when correlation between polynomial and non-polynomial splines basis is investigated in the following manner,

$$T_5 = \text{span}\{1, x, x^2, x^3, \sin(kx), \cos(kx)\}$$
$$= \text{span}\{1, x, x^2, x^3, \frac{24}{k^4}(\cos(kx) - 1 + \frac{(kx)^2}{2}),$$
$$\frac{120}{k^5}(\sin(kx) - (kx) + \frac{(kx)^3}{6})\}.$$

J. Rashidinia is with the School of Mathematics, Iran University of Science & Technology Narmak, Tehran 16844, Iran e-mail: rashidinia@iust.ac.ir.

R. Jalilian is with the Department of Mathematics, Ilam University, PO Box 69315516, Ilam, Iran, e-mail: rezajalilian@iust.ac.ir

From the above equation it follows that $\lim_{k\to 0} T_5 = \{1, x, x^2, x^3, x^4, x^5\}$, so that the Usmani's method [18], based on quintic splines is a special case ($k = 0$) of our approach.

II. Class of Methods

For simplicity we first develop the quintic non-polynomial spline for solving the fourth-order boundary value problem

$$\frac{d^4u}{dx^4} = g(x)u + f(x), \qquad \text{for} \qquad x \in [c, d],$$
$$u(c) = u(d) = \beta_1, \qquad u''(c) = u''(d) = \beta_2. \qquad (3)$$

For this purpose, we divide the interval [c,d] into n equal subintervals using the grid points. Let $u(x)$ be the exact solution of the boundary-value problem (3) and u_j be an approximation to $u(x_j)$, in order to develop the numerical method for approximating solution of differential equations (3), we introduce the set $\{x_j\}$ so that $x_j = c + jh, h = \frac{d-c}{n}, j = 0, 1 \ldots, n$, the non-polynomial quintic spline $p_j(x)$ in subinterval $x_j \le x \le x_{j+1}$, has the form

$$\begin{aligned} p_j(x) &= a_j \sin k(x - x_j) + b_j \cos k(x - x_j) \\ &\quad + c_j(x - x_j)^3 + d_j(x - x_j)^2 \\ &\quad + e_j(x - x_j) + l_j, \; j = 0, 1, \ldots, n, \end{aligned} \qquad (4)$$

where a_j, b_j, c_j, d_j, e_j and l_j are constants and k is free parameter. If $k \to 0$ then $p_j(x)$ reduces to quintic spline in [c,d]. By using continuity conditions at the common nodes (x_j, u_j), and to derive expression for the coefficients of (4) in terms of $u_j, u_{j+1}, m_j, m_{j+1}, S_j$ and S_{j+1} we have:

$$\begin{aligned} p_j(x_j) = u_j, \; p_j(x_{j+1}) = u_{j+1}, \\ p_j^{(2)}(x_j) = m_j, p_j^{(2)}(x_{j+1}) = m_{j+1}, \\ p_j^{(4)}(x_j) = S_j, \; p_j^{(4)}(x_{j+1}) = S_{j+1}. \end{aligned} \qquad (5)$$

Using the (5) we get the following expressions:

$$\begin{aligned} b_j &= \frac{h^4 S_j}{\theta^4}, l_j = u_j - \frac{h^4 S_j}{\theta^4}, \\ a_j &= \frac{S_{j+1} - S_j \cos\theta}{k^4 \sin\theta}, d_j = \frac{k^2 m_j + S_j}{2k^2}, \\ c_j &= \frac{h(S_{j+1} - S_j) + \theta k(m_{j+1} - m_j)}{6\theta^2}, \end{aligned}$$

$$\begin{aligned} e_j &= \frac{u_{j+1} - u_j}{h} \\ &\quad + \frac{h^3[(6 - 2\theta^2)S_j - (6 + \theta^2)S_{j+1}]}{6\theta^4} \\ &\quad - \frac{h(m_{j+1} + 2m_j)}{6}, \end{aligned} \qquad (6)$$

and $\theta = kh$. Using the continuity of the first and third derivatives at (x_j, u_j), we get the following relation for $j = 1, 2, \ldots, n-1$:

$$\begin{aligned} m_{j+1} + 4m_j + m_{j-1} &= \frac{6(u_{j+1} - 2u_j + u_{j-1})}{h^2} \\ &+ \frac{6(S_{j+1} - 2S_j \cos\theta + S_j)}{hk^3 \sin\theta} - \frac{3(S_j + S_{j-1})}{k^2} \\ &- \frac{(6 + \theta^2)S_{j+1} - (12 - \theta^2)S_j}{k^2\theta^2} \\ &+ \frac{(6 - 2\theta^2)S_{j-1}}{k^2\theta^2}, \end{aligned} \qquad (7)$$

and

$$\begin{aligned} m_{j+1} - 2m_j + m_{j-1} &= \frac{2S_j - S_{j-1} - S_{j+1}}{k^2} \\ &+ \frac{h(S_{j+1} - 2S_j \cos\theta + S_{j-1})}{k \sin\theta}. \end{aligned} \qquad (8)$$

Using equations (7) and (8), we get the following scheme:

$$\begin{aligned} & u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2} \\ & = h^4[\alpha(S_{j+2} + S_{j-2}) \\ & + \beta(S_{j+1} + S_{j-1}) + \gamma S_j], \end{aligned} \qquad (9)$$

where $j = 2, 3, \ldots, n-2$ and

$$\alpha = \frac{\theta^3 - 6(\theta - \sin\theta)}{6\theta^4 \sin\theta},$$

$$\beta = \frac{12\theta(1 + \cos\theta) - 2\theta^3(\cos\theta - 2) - 24\sin\theta}{6\theta^4 \sin\theta},$$

$$\gamma = \frac{36\sin\theta - 12\theta(1 + 2\cos\theta) - 2\theta^3(4\cos\theta - 1)}{6\theta^4 \sin\theta}.$$

If $\theta \to 0$ then $(\alpha, \beta, \gamma) \to (\frac{1}{120}, \frac{26}{120}, \frac{66}{120})$.

Using $u_j^{(4)} = g_j u_j + f_j + r, f_j \equiv f(x_j), u_j \equiv u(x_j), g_j \equiv g(x_j)$, at nodal points x_j and by Taylor expansion, the local truncation errors $t_j, j =$

$2, 3, ...n-2$, associated with our scheme is:

$$\begin{aligned} t_j &= (1-2(\alpha+\beta)-\gamma)h^4u_j^{(4)} \\ &+(\frac{1}{6}-(4\alpha+\beta))h^6u_j^{(6)} \\ &+(\frac{1}{80}-(\frac{16}{12}\alpha+\frac{1}{12}\beta))h^8u_j^{(8)} \\ &+(\frac{17}{30240}-\frac{1}{360}(64\alpha+\beta))h^{10}u_j^{(10)} \\ &+(\frac{31}{1814400}-(\frac{4\alpha}{315}+\frac{2\beta}{20160}))h^{12}u_j^{(12)}(\zeta_j) \\ &+O(h^{13}). \end{aligned}$$

For different choices of parameters α, β and γ we get the class of methods such as:

(i) Second-Order Method

For $\alpha = \frac{1}{120}, \beta = \frac{26}{120}$ and $\gamma = 1-2\alpha-2\beta$ gives:

$$\delta^4 u_j = h^4 u_j^{(4)} - \frac{1}{12}h^6 M_6 + O(h^7), j = 2, 3, ..., n-2. \quad (10)$$

(ii)Second-Order Method

For $\alpha = \frac{-6}{4319}, \beta = \frac{72}{400}$ and $\gamma = 1-2\alpha-2\beta$ gives:

$$\begin{aligned} \delta^4 u_j &= h^4 u_j^{(4)} - \frac{2519}{323925}h^6 M_6 + O(h^7), \quad (11) \\ j &= 2, 3, ..., n-2. \end{aligned}$$

(iii) Fourth-Order Method

For $\alpha = 0, \beta = \frac{1}{6} - 4\alpha$ and $\gamma = 1-2\alpha-2\beta$ gives:

$$\begin{aligned} &\delta^4 u_j = h^4[u_{j+1}^{(4)} + 4u_j^{(4)} + u_{j-1}^{(4)}] - \frac{1}{720}h^8 M_8 \\ &+O(h^9), \ j = 2, 3, ..., n-2. \quad (12) \end{aligned}$$

(iv) Sixth-Order Method

For $\alpha = \frac{-1}{720}, \beta = \frac{3}{20} - 16\alpha$ and $\gamma = 1-2\alpha-2\beta$ gives:

$$\begin{aligned} &\delta^4 u_j = \frac{h^4}{720}[-(u_{j+2}^{(4)} + u_{j-2}^{(4)}) + 124(u_{j+1}^{(4)} + u_{j-1}^{(4)}) \\ &+474(u_j^{(4)})] + \frac{1}{3024}h^{10}M_{10} \\ &+O(h^{11}), \ j = 2, 3, ..., n-2, \quad (13) \end{aligned}$$

where

$M_6 = \max_{c \le x \le d} \mid u^{(6)}(x) \mid$,

$M_8 = \max_{c \le x \le d} \mid u^{(8)}(x) \mid$,

$M_{10} = \max_{c \le x \le d} \mid u^{(10)}(x) \mid$.

Each of the above recurrence relations gives $n-2$ linear equations in n unknowns, we need two more equations at each end of the rang of integration.

III. Development of the Boundary Formulas

For discretization of boundary conditions we define:

$$\begin{aligned} &(i) \ \sum_{k=0}^{3} b'_k u_k + c' h^2 u''_0 + h^4 \sum_{k=0}^{3} d'_k u_k^{(4)} + t_1 = 0, \\ &(ii) \ \sum_{k=0}^{3} b'_k u_{n-k} + c' h^2 u''_n + h^4 \sum_{k=0}^{3} d'_k u_{n-k}^{(4)} \\ &+t_n = 0, \quad (14) \end{aligned}$$

where b'_k, c' and d'_k are arbitrary parameters to be determined. In order to obtain the second-order method we find that:

$(b'_0, b'_1, b'_2, b'_3, c') = (-2, 5, -4, 1, 1)$,

$(d'_0, d'_1, d'_2, d'_3) = (\frac{1}{12}, -1, 0, 0)$.

We obtain the second order boundary formulas as follows:

$$\begin{aligned} &(5-h^4 g_1)u_1 - 4u_2 + u_3 = (2-\frac{1}{12}h^4 g_0)\beta_1 \\ &-h^2\beta_2 - h^4(\frac{1}{12}(f_0+r) - (f_1+r)) \\ &+\frac{59}{360}h^6 u^{(6)}(x_1) + O(h^7), \\ &u_{n-3} - 4u_{n-2} + (5-h^4 g_{n-1})u_{n-1} \\ &= (2-\frac{1}{12}h^4 g_n)\beta_1 - h^2\beta_2 - h^4(-(f_{n-1}+r) \\ &+\frac{1}{12}(f_n+r)) + \frac{59}{360}h^6 u^{(6)}(x_n) + O(h^7). \quad (15) \end{aligned}$$

For four-order method we find that:

$(b'_0, b'_1, b'_2, b'_3, c') = (-2, 5, -4, 1, 1)$,

$(d'_0, d'_1, d'_2, d'_3) = \frac{-1}{360}(28, 245, 56, 1)$,

and

$$\begin{aligned} &(5-\frac{245}{360}h^4 g_1)u_1 + (-4-\frac{56}{360}h^4 g_2)u_2 \\ &+(1-\frac{h^4}{360}g_3)u_3 = (2+\frac{28}{360}h^4 g_0)\beta_1 - h^2\beta_2 \\ &+\frac{h^4}{360}(28(f_0+r) + 245(f_1+r) + 56(f_2+r) \\ &+(f_3+r) - \frac{241}{60480}h^8 u^{(8)}(\zeta_1) + O(h^9), \\ &(1-\frac{1}{360}h^4 g_{n-3})u_{n-3} + (-4-\frac{56}{360}h^4 g_{n-2})u_{n-2} \\ &+(5-\frac{245}{360}h^4 g_{n-1})u_{n-1} = (2+\frac{28}{360}h^4 g_n)\beta_1 \\ &-h^2\beta_2 + \frac{h^4}{360}(28(f_n+r) + 245(f_{n-1}+r) \\ &+56(g_{n-1}+r) + (g_{n-3}+r)) \end{aligned}$$

$$-\frac{241}{60480}h^8u^{(8)}(\zeta_n)+O(h^9). \tag{16}$$

For discretization of boundary conditions for sixth-order method we define:

$$(i)\ \sum_{k=0}^{3}b'_ku_k=c'h^2u''_0+h^4\sum_{k=0}^{5}d'_ku_k^{(4)}+t_1,$$
$$(ii)\ \sum_{k=0}^{3}b'_ku_{n-k}=c'^*h^2u''_n$$
$$+h^4\sum_{k=0}^{5}d'_{5-k}u_{n-5+k}^{(4)}+t_n. \tag{17}$$

In order to obtain the truncation errors of t_1 and t_2 we find that:

$$(b'_0,b'_1,b'_2,b'_3,c',c'^*)=(-2,5,-4,1,1,-1),$$
$$(d'_0,d'_1,d'_2,d'_3,d'_4,d'_5)$$
$$=\frac{-1}{60480}(4233,43274,5662,3432,-1391,230),$$

and

$$(5-d'_1h^4g_1)u_1+(-4-d'_2h^4g_2)u_2$$
$$+(1-d'_3h^4g_3)u_3-(d'_4h^4g_4)u_4-(d'_5h^4g_5)u_5$$
$$=(2-d'_0h^4g_0)\beta_1-h^2\beta_2+h^4(d'_0(f_0+r)$$
$$+d'_1(f_1+r)+d'_2(f_2+r)+d'_3(f_3+r)$$
$$+d'_4(f_4+r)+d'_5(f_5+r))$$
$$-\frac{167}{50400}h^{10}u^{(10)}(\zeta_1)+O(h^{11}),$$
$$(5-d'_1h^4g_{n-1})u_{n-1}+(-4-d'_2h^4g_{n-2})u_{n-2}$$
$$+(1-d'_3h^4g_{n-3})u_{n-3}-(d'_4h^4g_{n-4})u_{n-4}$$
$$-(d'_5h^4g_{n-5})u_{n-5}=(2-d'_0h^4g_n)\beta_1+h^2\beta_2$$
$$+h^4(d'_0(f_n+r)+d'_1(f_{n-1}+r)+d'_2(f_{n-2}+r)$$
$$+d'_3(f_{n-3}+r)+d'_4(f_{n-4}+r)+d'_5(f_{n-5}+r))$$
$$-\frac{167}{50400}h^{10}u^{(10)}(\zeta_n)+O(h^{11}). \tag{18}$$

IV. Convergence Analysis

Here we prove the convergence of the methods. Let us write the error equation of the methods as follows:

$$AE=T, \tag{19}$$

where $E=(e_j)$, is the (n-1)-dimensional column vector with e_j, the error of discretization defined by $e_j=u(x_j)-u_j$. In other words e_j is the amount by which computed solution u_j deviates from the actual solution $u(x_j)$ at $x=x_j$ and A is nine-band matrix which can be described as

$$A=M+BG,\ G=h^4\text{diag}(g_j),\ j=1,2,\dots,n-1, \tag{20}$$

here $M=P^2$, where $P=(p_{ij})$, is a tridigonal and monotone matrix defined by:

$$p_{ij}=\begin{cases}2, & i=j=1,2,...,N-1,\\ -1, & |i-j|=1,\\ 0, & \text{otherwise},\end{cases} \tag{21}$$

and

$$B=\begin{pmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & & & & \\ \beta & \gamma & \beta & \alpha & & & & & \\ \alpha & \beta & \gamma & \beta & \alpha & & & & \\ & \alpha & \beta & \gamma & \beta & \alpha & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & \alpha & \beta & \gamma & \beta & \alpha \\ & & & & & \alpha & \beta & \gamma & \beta \\ & & & & d_5 & d_4 & d_3 & d_2 & d_1 \end{pmatrix}, \tag{22}$$

$$E=A^{-1}T=[M+BG]^{-1}T,$$
$$|E\|\le\|[I+M^{-1}BG]^{-1}\|\|M^{-1}\|\|T\|. \tag{23}$$

By using $\|(I+A)^{-1}\|\le(1-\|A\|)^{-1}$ and Usmani et. al. [19] we obtain

$$\|M^{-1}\|\le\frac{5(d-c)^4+4(d-c)^2h^2}{384h^4}, \tag{24}$$

$$\|E\|\le\frac{\|M^{-1}\|\|T\|}{1-\|M^{-1}\|\|B\|\|G\|}. \tag{25}$$

Provided that $\|M^{-1}BG\|<1$.
Also we can obtain

$$\|B\|\le\frac{724}{720},\|G\|\le h^4M_g,M_g=\max_{c\le x\le d}|g(x)|. \tag{26}$$

For second order we obtain

$$\|T\|\le\frac{59h^6}{360}M_6,M_6=\max_{c\le\zeta\le d}|u^{(6)}(\zeta)|,$$

using (24)-(26) we obtain

$$\|E\|\le\frac{118\omega h^6M_6}{276480h^4-724\omega\|G\|}\equiv O(h^2), \tag{27}$$

for fourth order we get

$$\|T\|\le\frac{241h^8}{60480}M_8,M_8=\max_{c\le\zeta\le d}|u^{(8)}(\zeta)|,$$

using (24)-(26 we obtain

$$\|E\| \leq \frac{17352\omega h^8 M_8}{1672151040h^4 - 4378752\omega\|G\|} \equiv O(h^4), \quad (28)$$

and for sixth order we get

$$\|T\| \leq \frac{167h^{10}}{50400} M_{10}, M_{10} = \max_{c\leq\zeta\leq d}|u^{(10)}(\zeta)|.$$

By using (24)-(26,we get

$$\|E\| \leq \frac{12024\omega h^{10} M_{10}}{1393459200h^4 - 3648960\omega\|G\|} \equiv O(h^6), \quad (29)$$

where $\omega = 5(d-c)^4 + 4(d-c)^2h^2, \|G\| = \max|g(x)|, c \leq x \leq d$ provided

$$\|G\| < \frac{69120h^4}{181\omega}.$$

It follows $\|E\| \to 0$ as $h \to 0$. Therefore the convergence of the methods have been established.

V. Numerical Results

We consider the system of differential equations [1-4,13-15]

$$u^{(4)} = \begin{cases} 1, & -1 \leq x \leq \frac{-1}{2},\ \frac{1}{2} \leq x \leq 1, \\ 2 - 4u, & \frac{-1}{2} \leq x \leq \frac{1}{2}, \end{cases} \quad (30)$$

with the boundary conditions:

$$u(-1) = u(\frac{-1}{2}) = u(\frac{1}{2}) = u(1) = 0,$$
$$u''(-1) = u''(\frac{-1}{2}) = u''(\frac{1}{2})$$
$$= u''(1) = 0, \quad (31)$$

and the conditions of continuity of u and u'' at$x = \frac{-1}{2}$ and $\frac{1}{2}$.

The analytical solution for this boundary value problem is

$$u(x) = \begin{cases} \Gamma_1(x), & -1 \leq x \leq \frac{-1}{2}, \\ \Gamma_2(x), & \frac{-1}{2} \leq x \leq \frac{1}{2}, \\ \Gamma_3(x), & \frac{1}{2} \leq x \leq 1, \end{cases} \quad (32)$$

where
$\Gamma_1(x) = \frac{1}{24}x^4 + \frac{1}{8}x^3 + \frac{1}{8}x^2 + \frac{3}{64}x + \frac{1}{192}$,
$\Gamma_2(x) = 0.5 - \frac{1}{\varphi_1}[\varphi_2 \sin x \sinh x + \varphi_3 \cos x \cosh x]$,
$\Gamma_3(x) = \frac{1}{24}x^4 - \frac{1}{8}x^3 + \frac{1}{8}x^2 - \frac{3}{64}x + \frac{1}{192}$,
$\varphi_1 = \cos(1) + \cosh(1)$, $\varphi_2 = \sin(\frac{1}{2})\sinh(\frac{1}{2})$,
$\varphi_3 = \cos(\frac{1}{2})\cosh(\frac{1}{2})$.
We solved this example over the whole interval [-1,1] by using the Quintic non-polynomial spline methods with step lengths $h = 2^{-m}, m = 3, 4, 5$. The maximum absolute errors in solution for our various methods are listed in tables 1 and also the maximum absolute errors in the solution at middle points of interval are tabulated in table 2. To compare our computed results obtained by second and fourth order methods with the results obtained by other known methods in [1-4,13,14], the maximum absolute errors in the solution of example 1 are listed in tables 3,4 and 5.

Spline approach has the advantage over finite difference method that it provides continuous approximations to $u^{(i)}(x), i = 1, 2, 3$, at every point of the range of integration beside approximation to $u(x)$. Following [20] to obtain the necessary formula for computing values of first, second and third derivatives of solution of example 1, by using equation (7), (8) and solving the resulting identity for $m_j,\ j = 1, ..., n$ we have

$$m_j = \frac{(u_{j+1} - 2u_j + u_{j-1})}{h^2} + \frac{h^2(S_{j+1} - 2S_j\cos\theta + S_{j-1})}{3\sin\theta}$$
$$- \frac{h^2(S_j - S_{j-1})}{2\theta^2} - \frac{h^2[(6+\theta^2)S_{j+1} - (12-\theta^2)S_j + (6-2\theta^2)S_{j-1}]}{6\theta^4}$$
$$- \frac{(2S_j - S_{j-1} - S_{j+1})}{6\theta^2} - \frac{h^2(S_{j+1} - 2S_j\cos\theta + S_{j-1})}{6\theta^2\sin\theta},$$

with $m_0 = m_{n+1} = \beta_2$ being known from the boundary conditions. Having computed $u_j, m_j, S_j, j = 0, ..., n+1$, it is possible to evaluate the coefficient of the spline function (4) as given by (6). Since $y'_j = p'_j(x_j), j = 0, ..., n$ and $y'_{n+1} = p'_n(x_{n+1})$, it follows that

$$y'_j \approx \begin{cases} a_j k + e_j, & j = 0, ..., n, \\ \Psi_n + 3c_n h^2 + 2d_n h + e_n, & j = n+1, \end{cases} \quad (33)$$

where $\Psi_n = a_n k\cos\theta - b_n k\sin\theta$. Similarly, from $y'''_j = p'''_j(x_j), j = 0, ..., n$ and $y'''_{n+1} = p'''_n(x_{n+1})$, we can obtain

$$y'''_j \approx \begin{cases} -a_j k^3 + 6c_j, & j = 0, ..., n, \\ -a_n k^3\cos\theta + b_n k^3\sin\theta + 6c_n, & j = n+1. \end{cases} \quad (34)$$

The values of $u^{(i)}(x), i = 1, 2, 3$ have been computed by our second order method (i). To compare with the method in [1] the maximum absolute errors are listed in tables 6.

Example 2: We consider the system of differential equation solved by Al-Said and Noor [1].

$$u^{(4)} = \begin{cases} 0, & -1 \le x \le \frac{-1}{2},\ \frac{1}{2} \le x \le 1, \\ 1-4u, & \frac{-1}{2} \le x \le \frac{1}{2}, \end{cases} \tag{35}$$

with the boundary conditions:

$$u(-1) = u(\frac{-1}{2}) = u(\frac{1}{2}) = u(1) = 0,$$
$$u''(-1) = -u''(\frac{-1}{2}) = u''(\frac{1}{2})$$
$$= -u''(1) = \epsilon, \tag{36}$$

where $\epsilon \to 0$. The analytical solution for this boundary value problem is

$$u(x) = \begin{cases} \Lambda_1(x), & \text{for } -1 \le x \le \frac{-1}{2}, \\ \Lambda_2(x), & \text{for } \frac{-1}{2} \le x \le \frac{1}{2}, \\ \Lambda_3(x) =, & \text{for } \frac{1}{2} \le x \le 1, \end{cases} \tag{37}$$

where
$\Lambda_1(x) = (\frac{-2}{3}x^3 - \frac{3}{2}x^2 - \frac{13}{12}x - \frac{1}{4})\epsilon$,
$\Lambda_2(x) = 0.25 - \frac{1}{2\varphi_1}[\varphi_2 \sin x \sinh x + \varphi_3 \cos x \cosh x]$,
$\Lambda_3(x) = (\frac{-2}{3}x^3 + \frac{3}{2}x^2 - \frac{13}{12}x + \frac{1}{4})\epsilon$,
$\varphi_1 = \cos(1) + \cosh(1)$, $\varphi_2 = \sin(\frac{1}{2})\sinh(\frac{1}{2})$,
$\varphi_3 = \cos(\frac{1}{2})\cosh(\frac{1}{2})$.
We solved this example over the whole interval [-1,1] by using our second order methods (i),(ii), with step lengths $h = 2^{-m}, m = 3, 4, 5$. The maximum absolute error in solution are listed in table 7 and 8, our results compared with the results obtained in [1,2,11]. The results shows superiority of our second orders methods.

TABLE I
MAXIMUM ABSOLUTE ERRORS IN SOLUTION OF EXAMPLE 1

m	$O(h^2)(i)$	$O(h^2)$(ii)	$O(h^4)$	$O(h^6)$
3	8.19×10^{-6}	9.04×10^{-7}	1.69×10^{-8}	7.65×10^{-11}
4	2.73×10^{-6}	2.26×10^{-7}	4.08×10^{-10}	3.31×10^{-13}
5	7.44×10^{-7}	7.19×10^{-8}	1.30×10^{-11}	6.41×10^{-15}

TABLE II
MAXIMUM ABSOLUTE ERRORS IN SOLUTION OF EXAMPLE 1 IN MIDDLE POINTS

m	$O(h^2)$(i)	$O(h^2)$(ii)	$O(h^4)$	$O(h^6)$
3	1.25×10^{-6}	2.26×10^{-7}	4.40×10^{-10}	5.95×10^{-13}
4	1.56×10^{-7}	7.19×10^{-8}	1.30×10^{-11}	5.49×10^{-15}
5	1.95×10^{-8}	1.84×10^{-8}	4.06×10^{-13}	–

TABLE III
MAXIMUM ABSOLUTE ERRORS IN SOLUTION OF EXAMPLE 1

m	Our fourth-order	Fourth-order[4]	Finite difference[14]
3	1.7×10^{-8}	2.4×10^{-8}	1.3×10^{-6}
4	4.1×10^{-10}	1.5×10^{-9}	8.7×10^{-8}
5	1.3×10^{-11}	9.5×10^{-11}	6.8×10^{-9}

TABLE IV
MAXIMUM ABSOLUTE ERRORS IN SOLUTION OF EXAMPLE 1

h	Second-order(i)	Second-order(ii)	Second-order[1]
1/12	4.5×10^{-6}	1.2×10^{-7}	1.2×10^{-5}
1/24	1.2×10^{-6}	2.3×10^{-7}	2.8×10^{-6}
1/48	3.3×10^{-7}	3.2×10^{-8}	6.9×10^{-7}

VI. CONCLUSION

We have developed a new non-polynomial quintic spline for solving a system of fourth-order boundary-value problems. This approach has the advantages over finite difference methods that it provides continuous approximations to not only for $u(x)$ but also for $u^{(i)}(x), i = 1, 2, 3$, at every point of the range of integration. Our numerical results are better than those produced by collocation and finite difference methods for solution of equation(1).

REFERENCES

[1] AL-SAID, E.A. and NOOR, M.A., *Quartic Spline Method for Solving Fourth-Order Obstacle Boundary Value Problems*, Journal of Computational and Applied Mathematics Vol.143,pp.107-116,2002.
[2] AL-SAID,E.A. NOOR,M.A. *Computational Methods for Fourth-Order Obstacle Boundary Value Problems*, Comm. Appl. Nonlinear. Anal. Vol.2,pp.73-83,1995.
[3] AL-SAID, E.A. NOOR,M.A. and RASSIAS,T.M., *Cubic Splines Method for Solving Fourth-Order Obstacle Problems*, Appl. Math. Comput.,Vol.174,
pp.180-187,2006.
[4] AL-SAID,E.A., NOOR,M.A., KAYA,D., Al-KHALED,K., *Finite difference method for solving fourth-order obstacle problems*, Int. J. Comput. Math. Vol.81,pp.741-748,2004.
[5] BAIOCCHI,C.C. and CALEO,A., *Variational and quasi-variational Inequalities*, John Wiley and Sons, New York, 1984.
[6] CHAWLA,M.M. and SUBRAMANIAN, *High accuracy quintic spline solution of fourth-order two-Point boundary value problems*, Int. J. Computer. Math. Vol.31,pp.87-94,1989.
[7] HENRICI,P., *Discrete variable method in ordinary differential equations*, John Wiley, New York, 1961.
[8] JAIN,M.K., *Numerical solution of differential equations*, Second Editions,Wiley Eastern Limited, 1984.
[9] JAIN,M.K., IYANGER,S.R.K. and SOLDHANHA,J.S.V., *Numerical solution of a fourth-order ordinary differential equation*, J. Engg. Math. Vol.11,pp.373-380,1977.
[10] KIKUCHI,N., and ODEN,J.T., *Contact problem in elasticity*, SIAM, Publishing Co. Philadelphia,1988.

TABLE V
MAXIMUM ABSOLUTE ERRORS IN SOLUTION OF EXAMPLE 1

h	Second-order In [2]	Second-order[3]
1/12	6.2×10^{-5}	7.8×10^{-6}
1/24	1.6×10^{-5}	1.9×10^{-6}
1/48	3.9×10^{-6}	4.9×10^{-7}

TABLE VI
MAXIMUM ABSOLUTE ERRORS OF u', u'' AND u''' FOR EXAMPLE 1

m	$\|\|u'(x_i)-u'_i\|\|_\infty$	$\|\|u''(x_i)-u''_i\|\|_\infty$	$\|\|u'''(x_i)-u'''_i\|\|_\infty$
		Our $O(h^2)$(i)	
3	6.28×10^{-5}	1.94×10^{-4}	2.83×10^{-4}
4	1.77×10^{-5}	5.51×10^{-5}	8.35×10^{-5}
5	4.71×10^{-6}	1.46×10^{-5}	2.25×10^{-5}
		$O(h^2)$ in[1]	
3	8.81×10^{-5}	1.30×10^{-3}	2.08×10^{-2}
4	3.17×10^{-5}	3.66×10^{-4}	5.76×10^{-3}
5	1.06×10^{-5}	9.66×10^{-5}	1.72×10^{-3}

[11] KHALIFA,A.K. and NOOR,M.A., *Quintic spline solutions of a class of contact problems*, Math. Comput. Modlelling Vol.13,pp.51-58,1990.

[12] LEWY,H. and STAMPACCHIA,G., *On the regularity of the solution of the variational inequalities*, Comm. pure and Appl. Math. Vol.22,pp.153-188,1969.

[13] NOOR,M.A. and Al-SAID,E.A., *Fourth order obstacle problems*.In: Th.M.Rassias and H.M.Srivastava(Eds), Analytic and geometric inequalities and applications, kluwer Academic Publishers, Dordrecht, Netherlands,pp.277-300,1999.

[14] NOOR,M.A. and AL-SAID,E.A., *Numerical solution of fourth order variational inequalities*, Inter. J. Comput. Math. Vol.75,pp.107-116,2000.

[15] PAPAMICHEL,N. and WORSEY,E.A., *A cubic spline method for the solution of linear fourth-order two-point boundary value problem*, J. Comput. Appl. Math. Vol.7,pp.187-189,1981.

[16] RASHIDINIA, J., *Applications of splines to the numerical solution of differential equations*, Ph.D. thesis, Aligarh Muslim University, Aligarh, India, 1994.

[17] SIRAJ-UL-ISLAM, TIRMIZI,S.I.A. and SAADAT ASHRAF, *A class of method based on non-polynomial spline function for the solution of a special fourth-order boundary-value problems with engineering applications*, Appl. Math. Comput. Vol.174,pp.1169-1180,2006.

[18] USMANI,R.A. and WARSI,S.A., *Smooth spline solutions for boundary value problems in plate deflection thoery*, Comput. Maths. with Appls. Vol.6, pp.205-211,1980.

[19] USMANI,R.A., *Discrete variable method for a boundary value problem with engineering applications*, Math. Comput. Vol.32,1087-1096,1978.

[20] Van DAELE,M., VANDEN BERGHE,G. and De MEYER,H., *A smooth approximation for the solution of a fourth-order boundary value problem based on non-polynomial splines*, J. Comput. Appl. Math. Vol.pp.51,383-394,1994.

TABLE VII
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 2 WHEN $\epsilon = 10^{-6}$

m	Second-order(i)	Second-order(ii)	Second-order In [1]
3	4.66×10^{-6}	4.45×10^{-7}	1.3×10^{-5}
4	1.39×10^{-6}	1.32×10^{-7}	3.2×10^{-6}
5	3.74×10^{-7}	4.25×10^{-8}	8.1×10^{-7}

TABLE VIII
MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 2 WHEN $\epsilon = 10^{-6}$

m	Second-order In [11]	Second-order In [2]
3	3.0×10^{-4}	1.4×10^{-4}
4	7.0×10^{-5}	3.6×10^{-5}
5	1.4×10^{-5}	8.9×10^{-6}

Jalil Rashidinia, Associate professor, School of Mathematics, Iran University of Science & Technology Narmak, Tehran 16844, Iran
e-mail: rashidinia@iust.ac.ir
Nationality: Iranian
Research Interests: Applied mathematics and computational sciences
Numerical solution of ODE, PDE and IE
Spline approximations
Numerical linear algebra

Reza Jalilian Assistant professor, Department of Mathematics, Ilam University, PO Box 69315516, Ilam, Iran
e-mail: rezajalilian@iust.ac.ir
Research Interests: Spline approximations