Complex Dynamics of Bertrand Duopoly Games with Bounded Rationality

Jixiang Zhang and Guocheng Wang

Abstract**—**A dynamic of Bertrand duopoly game is analyzed, where players use different production methods and choose their prices with bounded rationality. The equilibriums of the corresponding discrete dynamical systems are investigated. The stability conditions of Nash equilibrium under a local adjustment process are studied. The stability conditions of Nash equilibrium under a local adjustment process are studied. The stability of Nash equilibrium, as some parameters of the model are varied, gives rise to complex dynamics such as cycles of higher order and chaos. On this basis, we discover that an increase of adjustment speed of bounded rational player can make Bertrand market sink into the chaotic state. Finally, the complex dynamics, bifurcations and chaos are displayed by numerical simulation.

Keywords**—**Bertrand duopoly model, Discrete dynamical system, Heterogeneous expectations, Nash equilibrium.

I. Introduction

BERTRAND, a French mathematician, was first to introduce the Bertrand model which was widely use mathematical representations of duopoly market [1]. Bertrand model is a model of price competition between duopoly firms which results in each charging the price that would be charged under perfect competition, known as marginal cost pricing. The Bertrand model, in which each duopoly firm sets its optimal product's price by competitors' price, is a fully rational game based on the following assumptions:

There are at least two competitive firms producing homogeneous products; each firm has a complete knowledge of the market demand function; all firms compete in price, and choose their respective prices simultaneously; consumers buy everything from the cheaper firm or half at each, if the price is equal.

Under these conditions of full information all firms will try to reduce their products price, until the product is selling at no profit. It is a Nash equilibrium. This surprising result is referred to as the Bertrand paradox. The Bertrand paradox rarely appears in practice because real products are almost always differentiated in some way other than price. One way of avoiding the paradox is to allow the firms to sell differentiated products [2]. Bertrand model assumed the each player's behavior is fully rational. But it is impossible that all players are naive. There, different players' expectations are proposed: naive player, bounded rational player and adaptive [3]-[5]. So each player adopts his expectations to adjust his product's price in order to maximize his profit.

In the past few years, several works of Bertrand model have been done. Robert [6] considered a model of Bertrand competition with fixed costs of entry or production. Walter and Elmar [7] analyzed a simple, repeated game of simultaneous entry and pricing. They report a surprising property of the symmetric equilibrium solution: If the number of potential competitors is increased above two, the market breaks down with higher probability, and the competitive outcome becomes less likely. More potential competition lowers welfare – another Bertrand paradox. Dufwenberg and Gneezy [8] studied the effects of the number of firms in a standard Bertrand competition framework with constant marginal cost and inelastic demand. In their experiments, price is above marginal cost for the case of two firms but equal to that cost for three and four firms. Carlos, Ana and Klaus [9] presented an evolutionary model of Bertrand competition in a market for a homogeneous good, where identical firms face a technology with decreasing returns to scale. They found that the dynamic process selects a strict subset of the Nash equilibrium of the underlying game, even under simple behavior. Lo and Kiang [10] applied "minimal" quantization rules to investigate the quantum version of the Bertrand duopoly with differentiated products. They found that while negative entanglement diminishes the profit of each firm below the classical limit, positive entanglement enhances the profit monotonically, reaching a maximum in the limit of maximal entanglement. Tao [11] considered a new type of share function in Bertrand model. Under his assumption, the firms can earn a positive profit in price competition no matter how many firms are present in the market. Yue, Samar and Zhu [12] considered a market where customers need to buy two complementary goods as mixed bundle, offered by two separate firms. They presented a profit maximization model to obtain optimal strategies for a firm making decisions under information asymmetry. Jason [13] studied a two-stage game with capacity precommitment followed by price competition where firms have incomplete information about their rival's marginal cost. The game has a Cournot outcome if and only if the lowest possible marginal cost is sufficiently high relative to the expected marginal cost. Andersson [14] analyzed the role of patience in a repeated Bertrand duopoly where firms bargain over which collusive price and market share to implement. He found that the least patient firm's market share is not monotone in its own discount factor. Fernanda and Flávio [15] considered an international

Jixiang Zhang is with the College of Economics & Management, Nanjing University of Aeronautics & Astronautics, 210016 Nanjing, China and with the Institute of Quantitative & Technical Economics, Chinese Academy of Social Sciences, 100732 Beijing, China. (corresponding author to provide phone: +86 026 6606 9019; fax: +86 025 84892751; e-mail: zhang_jixiang@nuaa.edu.cn)
Guocheng Wang is with the Institute of Quantitative & Technical Economics, Chinese Academy of Social Sciences, 100732 Beijing, China.

trade under Bertrand model with differentiated products and with unknown production costs.

In this paper, we set up a Bertrand duopoly model with heterogeneous players and apply dynamic methods to investigate the dynamic behaviors of this duopoly game. The paper is organized as follows. In Section II, we determine a duopoly Bertrand model with heterogeneous expectations by a two-dimensional map. In Section III, we study the model of duopoly game. Explicit parametric conditions of the existence, local stability and bifurcation of equilibrium points will be given. In Section IV, we show the complex dynamic phenomenon of bifurcation, chaos and chaotic attractor by numerical simulations.

II. Model

We consider that there are two firms which choose different prices for their products, respectively in an oligopoly market. Let $p_i(t), i=1,2$ represent the price of ith firm during period $t=0,1,2,\ldots$. The quantity each firm sells Q_i, a linear inverse demand function, is determined from the prices of their products in period t [16].

$$\begin{aligned} Q_1 &= a - bp_1 + dp_2, \\ Q_2 &= a - bp_2 + dp_1, \end{aligned} \tag{1}$$

where $a>0, b>0, d>0$. The parameter d reflects the extent to which the two products are substitutes for each other. We assume that the firms have constant marginal costs equal to c_i, and no fixed costs. Accordingly, the cost function is taken in the following form

$$C_i = c_i Q_i, \qquad i=1,2. \tag{2}$$

with above assumptions the profit of the ith firm in the single period is give by

$$\Pi_i(p_1,p_2) = (p_i - c_i)Q_i, \qquad i=1,2. \tag{3}$$

Hence ith firm's price for period t+1 is decided by solving the optimization problem

$$\begin{aligned} p_1(t+1) &= \arg\max \Pi_1(p_1(t), p_2^*(t+1)), \\ p_2(t+1) &= \arg\max \Pi_2(p_1^*(t+1), p_2(t)), \end{aligned} \tag{4}$$

where $p_i^*(t+1)$ represent the expectation of jth firm about jth firm's price of product during period $t+1$ $(i,j=1,2, i\neq j)$.

The marginal profit of the ith firm in period t is

$$\frac{\partial \Pi_i(p_i,p_j)}{\partial p_i} = (a+bc_i) - 2bp_i + dp_j, \quad i,j=1,2, i\neq j. \tag{5}$$

This optimization problem has unique solution:

$$p_i = \tfrac{1}{2b}(a + bc_i + dp_j), \tag{6}$$

Information in the market usually is incomplete, so players can use more complicated expectations such as bounded rationality and naive [3].

For the naive player, he determines price of production with his expectation of rival's price which is the same as that previous period. Then the naïve player decides his output according to (6).

$$p_i(t+1) = \tfrac{1}{2b}(a + bc_i + dp_j(t)), \tag{7}$$

The bounded rational player has no complete knowledge of market, hence he determines price of production with the information of local profit maximizers. He decides to increase (decrease) its price if it has a positive (negative) marginal profit. In [17], this adjustment mechanism has been called myopic by Dixit. Thus, the dynamic adjustment mechanism can be modelled as

$$p_i(t+1) = p_i(t) + \alpha p_i(t)\frac{\partial \Pi_i(p_i,p_j)}{\partial p_i}, \quad i=0,1,2,\ldots, \tag{8}$$

where α is a positive parameter which represents the speed of adjustment of ith firm.

We assume firm 1 is a bounded rational player and firm 2 is a naïve player. With above assumptions, the duopoly game with heterogeneous players if described by a two dimensional nonlinear map $T(p_1,p_2)\to(p_1',p_2')$ defined as

$$T:\begin{cases} p_1' = p_1 + \alpha p_1(a + bc_1 - 2bp_1 + dp_2), \\ p_2' = \tfrac{1}{2b}(a + bc_2 + dp_1), \end{cases} \tag{9}$$

where "′" denotes the unit-time advancement, that is, if the right-hand side variables are productions of period t, then the left-hand ones represent productions of period $(t+1)$.

III. Equilibrium Points and Local Stability

Because Bertrand model is a economic model, we only study the dynamic behavior of the nonnegative equilibrium points. The fixed points of the map (9) are obtained as nonnegative solution of the non-linear algebraic system

$$\begin{cases} p_1(a + bc_1 - 2bp_1 + dp_2) = 0, \\ \tfrac{1}{2b}(a + bc_2 + dp_1) = 0, \end{cases} \tag{10}$$

which is obtained by setting $p_i' = p_i, i=1,2$ in system (10). We have two fixed points of system (10) $E_0 = (0, (a+bc_2)/2b)$ and $E_1 = (p_1^*, p_2^*)$ where

$$p_1^* = \frac{ad + 2ab + bdc_2 + 2b^2c_1}{4b^2 - d^2},\; p_2^* = \frac{ad + 2ab + bdc_1 + 2b^2c_2}{4b^2 - d^2}.$$

Obviously, E_0 is a boundary equilibrium. The fixed point E_1 is a Nash equilibrium and has economic meaning when $2b > d$.

For studying the local stability of equilibrium, the eigenvalues of the Jacobian matrix of the system (1) on the complex plane must be considered. The Jacobian matrix of system (1) at the point (p_1, p_2) has the form

$$J(p_1,p_2)=\begin{bmatrix}1+\alpha(a+bc_1+dp_2-4bp_1) & \alpha dp_1\\ \frac{d}{2b} & 0\end{bmatrix} \quad (11)$$

Theorem 1 The boundary equilibrium E_0 of system (9) is a unstable equilibrium point.

Proof. In order to prove these results, we consider the eigenvalues of Jacobian matrix J at E_0 which take the form

$$J(E_0)=\begin{bmatrix}1+\alpha(a+\frac{d(a+bc_2)}{2b}+bc_1) & 0\\ \frac{d}{2b} & 0\end{bmatrix}. \quad (12)$$

whose eigenvalues are

$$\lambda_1 = 1+\alpha(a+\frac{d(a+bc_2)}{2b}+bc_1),\ \lambda_2 = 0.$$

From the condition that $a,b,c_i(i=1,2)$ are positive parameters, we have that $|\lambda_1|>1$. Then E_0 is unstable equilibrium point of system (9).

Now we investigate the local stability of Nash equilibrium E_1. At E_1, the Jacobian matrix is

$$J(E_1)=\begin{bmatrix}1-2\alpha bp_1^* & \alpha dp_1^*\\ \frac{d}{2b} & 0\end{bmatrix}. \quad (13)$$

The characteristic equation of the $J(E_1)$ has the form $f(\lambda)=\lambda^2-Tr(J)\lambda+Det(J)=0$, where $Tr(J)$ is the trace and $Det(J)$ is the determinant of the Jacobian matrix which are given by $Tr(J)=1-2\alpha bp_1^*$ and $Det(J)=-\alpha d^2p_1^*/2b$, since $Tr^2(J)-4Det(J)=(1-2\alpha bp_1^*)^2+2\alpha d^2p_1^*/b$.

With above assumption, it is clear that $Tr^2(J)-4Det(J)>0$, then the eigenvalues of Nash equilibrium are real.

If the eigenvalues of the Jacobian matrix of fixed point E_1 are inside the unit circle of complex plane, Nash equilibrium E_1 is local stability. Hence the local stability of Nash equilibrium is given by jury's condition which are the necessary and sufficient conditions for $|\lambda_i|<1, i=1,2$.

$$(1)\ 1-Tr(J)+Det(J)=\frac{\alpha(ad+2ab+bdc_2+2b^2c_1)}{2b}>0,$$

$$(2)\ 1+Tr(J)+Det(J)=2-\frac{\alpha(4b^2+d^2)(ad+2ab+bdc_2+2b^2c_1)}{2b(4b^2-d^2)},$$

$$(3)\ Det(J)-1=-\frac{\alpha d^2p_1^*}{2b}-1<0.$$

It is clear that the first condition and the third condition are always satisfied. The second condition becomes

$$\alpha<\frac{4b(2b+d)(2b-d)}{(4b^2+d^2)(ad+2ab+bdc_2+2b^2c_1)}. \quad (14)$$

This equation defines a region of stability of Nash equilibrium point E_1 about parameter α. If α is small, both players' price will reach the Nash equilibrium point E_1 after rounds of games. But once there is an increase of the speed of adjustment of bounded rational player pushing α beyond stable region

$$(0,\frac{4b(2b+d)(2b-d)}{(4b^2+d^2)(ad+2ab+bdc_2+2b^2c_1)}),$$

the Nash equilibrium point E_1 will lose stability.

Theorem 2 The Nash equilibrium E_1 of system (9) is a stable provided that

$$\alpha<\frac{4b(2b+d)(2b-d)}{(4b^2+d^2)(ad+2ab+bdc_2+2b^2c_1)}.$$

From Theorem 2, we can obtain the region of stability of the Nash equilibrium point E_1 about the model parameters. For example, an increase of the speed of adjustment of bounded rational player with the other parameters held fixed has a destabilizing effect. In factor, an increase of α, staring from a set of parameters which ensures the local stability of the Nash equilibrium can bring out the region of the stability of Nash equilibrium point, crossing the flip bifurcation surface

$$\alpha=\frac{4b(2b+d)(2b-d)}{(4b^2+d^2)(ad+2ab+bdc_2+2b^2c_1)}.$$

Obviously, the stability of Nash equilibrium point E_1 depends on the parameters of system. We also consider other case that the parameters α,b,d,c_1 and c_2 are fixed parameters and parameter a which represent the market capacity of the production. In this paper, the case parameter α increases. Complex behaviors such as period doubling and chaotic attractors are generated where the Lyapunov exponents of the system (9) become positive.

IV. Numerical Simulations

In this section, we use the bifurcation diagrams to illustration the above results and finding new dynamics of system (9) as the parameters varying. The complicated dynamic features of the dynamics of a Bertrand duopoly game with heterogeneous players will be shown.

In order to study the local stability properties of the equilibrium points conveniently, we will use certain value data to simulate the dynamic evaluation of the system. Taking $a=5, b=0.9, c_1=0.1, c_2=0.8$ and $d=0.3,$ we can get Fig. 1 that shows the bifurcation diagram with respect to α, while other parameters are fixed. From Fig. 1, we can see that the orbit with initial values $(0,0)$ approaches to the stabled fixed point $(3.45,3.75)$ for $\alpha < 0.313$. If α increases, the Nash equilibrium point E_1 becomes unstable and bifurcation scenario occurs. The first Hofp bifurcation occurs at $\alpha = 0.313$. As α increases, infinitely many period-doubling bifurcation of quantity behavior become chaotic.

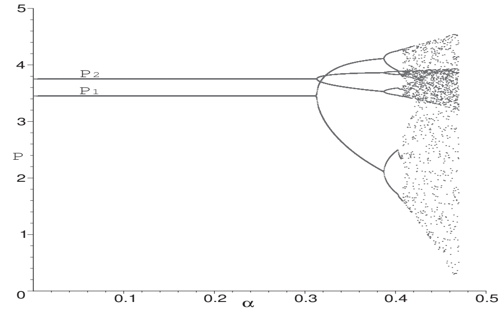


Fig. 1 The bifurcation diagram of the trajectories of the discrete dynamic system (9)

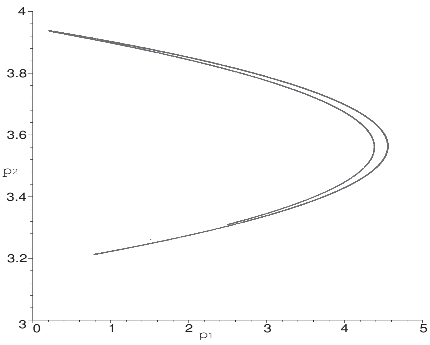


Fig. 2 The strange attractor of the discrete dynamic system (9)

Fig. 2 shows the strange attractor for the system (9) for $a = 5, b = 0.9, c_1 = 0.1, c_2 = 0.8, d = 0.3$ and $\alpha = 0.467$, which exhibits a fractal structure similar to Henon attractor [18].

In order to analyze the parameter sets for which aperiodic behavior occurs, we study the largest Lyapunov exponent, which depends on α. It is an evidence for chaos that the Largest Lyapunov exponent is positive. By the method of [19], we have Fig. 3 that displays the related maximal Lyapunov exponent as a function of α. From Fig. 3, we can easily get the degree of the local stability for different values α when the largest Lyapunov exponent is positive. We also determine the parameter sets for which the system (9) converges to cycles, aperiodic and chaotic behavior.

Strange attractors are typically characterized by fractal dimensions. We examine the important characteristic of neighboring chaotic orbits to see how rapidly they separate each other. The Lyapunov dimension is defined as follows [20]:

$$d_L = j + \frac{\sum_{i=1}^{i=j} \lambda_i}{|\lambda_j|}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$, where j is the largest integer such $\sum_{i=1}^{i=j} \lambda_i \geq 0$ and $\sum_{i=1}^{i=j+1} \lambda_i \leq 0$.

In our paper, two dimensional map (9) has a Lyapunov dimension

$$d_L = 1 + \frac{\lambda_1}{|\lambda_2|}, \lambda_1 > 0 > \lambda_2 .$$

By the definition of Lyapunov dimension [20] and simulation of the computer, we have the Lyapunov dimension of strange attractor of system (9). At the parameter values $(a, b, c_1, c_2, d, \alpha) = (5, 0.9, 0.1, 0.8, 0.3, 0.46)$, system (9) has two different Lyapunov exponents, $\lambda_1 = 0.51$ and $\lambda_2 = -1.81$. Therefore, the system (9) has a fractal dimension $d_L = 1 + (0.51/1.81) \approx 1.28$. Then the system (9) exhibits a fractal structure and its attractor has the fractal dimension $d_L \approx 1.28$.

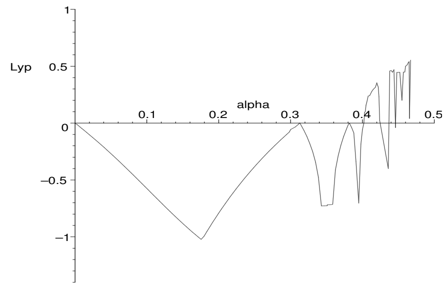
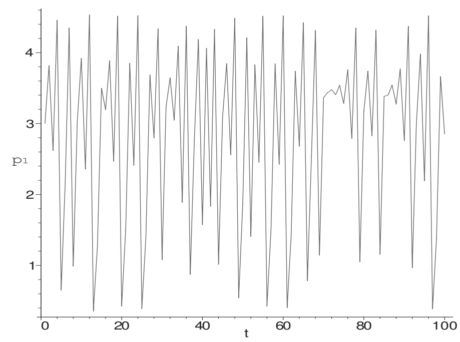
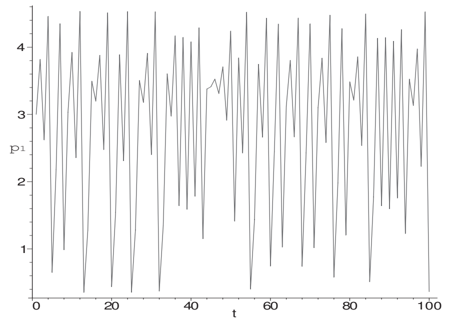


Fig. 3 Related the largest Lyapunov exponents as function of α

To demonstrate the sensitivity to initial conditions of system (9), we compute two orbits with initial points p_{10}, p_{20} and $p_{10} + 0.00001, p_{20}$ at the parameter values $a=5, b=0.9, c_1=0.1, c_2=0.8, d=0.3$ and $\alpha = 0.46$ respectively. We can get two firms' game results which are shown in Fig. 4. At the beginning the results are indistinguishable, but after a number of games, the difference between them builds up rapidly.



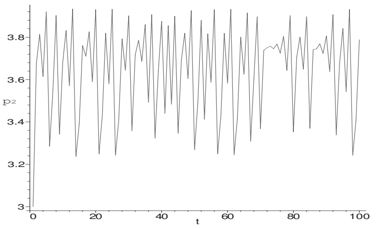
(a) $(p_{10}, p_{20}) = (3,3)$



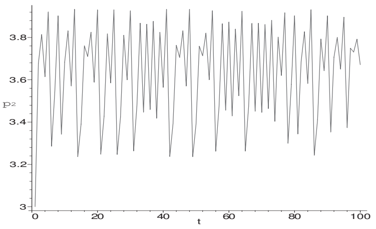
(b) $(p_{10}, p_{20}) = (3.00001,3)$

Fig. 4 Shows sensitive dependence on initial conditions, the two orbits of p_1 coordinates for $(a,b,c_1,c_2,d,\alpha)=(5,0.9,0.1,0.8,0.3,0.46)$

The same to variable p_2, Fig. 5 shows the sensitivity dependence on initial conditions, p_2 coordinates of the two orbits with the parameter values $(a,b,c_1,c_2,d,\alpha)=(5,0.9,0.1,0.8,0.3,0.46)$; the p_2 coordinates of initial conditions differ by 0.00001.



(a) $(p_{10}, p_{20}) = (3,3)$



(b) $(p_{10}, p_{20}) = (3,3.00001)$

Fig. 5 Shows sensitive dependence on initial conditions, the two orbits of p_2 coordinates for $(a,b,c_1,c_2,d,\alpha)=(5,0.9,0.1,0.8,0.3,0.46)$

V. CONCLUSIONS

In this paper, we have analyzed the complex dynamics of a repeated oligopoly Bertrand model with heterogeneous players: bounded rational player and naive player. The stability of two equilibrium points is investigated in this game. We found the parameter α (speed of adjustment of bounded rational player) may change the stability of Nash equilibrium point and cause bifurcation and chaos to occur. For the low values speeds of adjustment, the game has a stable Nash equilibrium. Increasing the values of speeds of adjustment, the Nash equilibrium becomes unstable, through period doubling bifurcation, more complex attractors obtained, which may be periodic cycles or chaotic sets. In the above discussion, we have given economic explanations to various dynamic phenomenons in the Bertrand market and provided theoretical reference for firms.

ACKNOWLEDGMENTS

The authors are supported by The National Natural Science Foundation of China (71101071), the National Basic Research Program of China (2012CB955802), social science foundation of Jiangsu (2012SJD630086) and foundation of NUAA (NR2013006).

REFERENCES

[1] Bertrand,J. Theories Mathematique de la Richesse Social.*Journal des Savants,* pp.499-508, 1883.
[2] H.Gravelle,R.Rees. *Microeconomics*. 2nd edition, Harlow: Longman,1992.
[3] Agiza,H.N., Elsadany,A.A. Chaotic dynamics in nonlinear duopoly game with heterogeneous players. *Applied Mathematics and Computation,* vol. 149, no. 3, pp. 843-860,2004.
[4] Bischi G I, Naimzada A. *Advances in Dynamic Games and Application*. Basel: Birkhauser,1999.
[5] Jixiang Zhang, Qingli Da, Yanhua Wang. The dynamics of Bertrand model with bounded rationality. *Chaos, Solitons and Fractals*, to be published.
[6] Robert Marquez. A note on Bertrand competition with asymmetric fixed costs. *Economics Letters,* vol. 57, no. 1, pp. 87-96,1997.
[7] Walter Elberfeld, Elmar Wolfstetter. A dynamic model of Bertrand competition with entry. *International Journal of Industrial Organization*, vol. 17, no. 4, pp.513-525, 1999.
[8] Dufwenberg.M, Gneezy.U. Price competition and market concentration: an experimental study. *International Journal of Industrial Organization*, vol. 18, no. 7, pp. 7-22, 2002..
[9] Alós–Ferrer, Ana.B.Ania and Klaus Reiner Schenk Hoppé. An Evolutionary Model of Bertrand Oligopoly[J]. *Games and Economic Behavior,* vol. 33, no. 1, pp. 1-19,2000.
[10] C.F.Lo, D.Kiang. Quantum Bertrand duopoly with differentiated products. *Physics Letters A*, vol. 321, no. 2, pp. 94-98,2004.
[11] Tao huang.Bertrand competition with incomplete share for lower price. *Economics Letters,* vol.83, no. 2, pp. 239-244,2004.
[12] Xiaohang Yue, Samar K. Mukhopadhyay, Xiaowei Zhu. A Bertrand model of pricing of complementary goods under information asymmetry. *Journal of Business Research,* vol.59, no.10, pp. 1182-1192,2006.
[13] Jason J. Lepore. Cournot and Bertrand–Edgeworth competition when rivals' costs are unknown. *Economics Letters*, vol. 103, no. 3, pp. 237-240,2007..
[14] Ola Andersson. On the role of patience in collusive Bertrand duopolies. *Economics Letters,* vol. 100, no. 1, pp. 60-63,2007.
[15] Fernanda A. Ferreira, Flávio Ferreiraa. Maximum revenue tariff under Bertrand duopoly with unknown costs[J]. *Communications in Nonlinear Science and Numerical Simulation,* to be published.
[16] H.S. Bierman, L. Fernandez. *Game Theory with Economic Applications, 2nd Edition,* Reading, Mass: Addison-Wesley, 1998.
[17] Dixit,A. Comparative statics for oligopoly. *International Economic Review*, vol.100, no.1, pp.107-122,1986.
[18] Henon,M. A two dimensional mapping with a strange attractor. *Communications in Mathematical Physics*, no. 50, pp.69-77,1976..
[19] Hubertus F. von Bremen, Firdaus E. Udwadia, Wlodek Proskurowski. An efficient QR based method for the computation of Lyapunov exponents. *Physica D,* vol. 101, no. 1, pp.1-16,1997.
[20] James L. Kaplan, James A. Yorke. A regime observed in a fluid flow model of Lorenz. *Communications in Mathematical Physics ,* vol. 67, no. 1, pp. 93-108,1979.