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Asymmetric Tukey's Control Chart Robust to Skew and Non-Skew Process Observation

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Abstract**—**In reality, the process observations are away from the assumption that are normal distributed. The observations could be skew distributions which should use an asymmetric chart rather than symmetric chart. Consequently, this research aim to study the robustness of the asymmetric Tukey's control chart for skew and non-skew distributions as Lognormal and Laplace distributions. Furthermore, the performances in detecting of a change in parameter of asymmetric and symmetric Tukey's control charts are compared by Average ARL (AARL). The results found that the asymmetric performs better than symmetric Tukey's control chart for both cases of skew and non-skew process observation.

Keywords**—**Asymmetric control limit, average of average run length, Tukey's control chart and skew distributions.

I. Introduction

SATISTICAL Process Control (SPC) charts are widely used for monitoring, measuring, controlling and improving quality of production in many areas of application, for example, in industry and manufacturing, finance and economics, epidemiology and health care, environmental sciences and other fields. Control charts are usually designed and evaluated under the assumption that the observations from a process are independent and identically distributed (i.i.d.) and from a normal distribution. In real applications, there are many situations in which the process data come from a non-normal distribution, for example, an Exponential, Laplace, Student-t or Gamma distribution (see, e.g., Borror et al. [1]; Stoumbos and Reynolds [2]; Mititelu et al. [3]; Sukparungsee and Novikov [4]). Processes with data from a non-normal distribution need to be monitored by appropriate control charts.

Recently, many types of control charts are proposed which an appropriate control chart must be selected under many assumptions and several factors. Specially, for this kind of process monitoring, individual control charts take only one sample to measure due to economic issue for a company. Consequently, Tukey's control chart has been popular used for individual process which Alemi [5] who was first proposed. Torng and Lee [6] and Torng et al. [7] have been investigated the average run length of Tukey's control chart. There are many advantage of using Tukey's control chart which it is easy to use and simple control limits setup. It can be used with not only non-Normal observations but also when distribution of process is unknown. Furthermore, Tukey's control chart does not sensitive to unusual data such as an outlier.

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Consequently, this paper aim to study the performance of Tukey's control chart robust to the skew distribution processes such exponential and Laplace distributions. They are usually represented as lifetime of products and growth rate of a company, respectively.

II. Tukey's Control Chart and Their Properties

A. Tukey's Control Chart with Symmetric Control Limit

In 2004, Alemi [5] who first proposed the Tukey's control chart which applied the principle of Box-plot to obtain the control limits. The control limits of Tukey's control chart are presented by Torng and Lee [7] under assumption a known population. They used the symmetric control limits so-called SCL-Tukey's control chart which control limits as follows:

$$UCL = F^{-1}(0.75) + L\left(IQR\right)$$
$$LCL = F^{-1}(0.25) - L\left(IQR\right)$$

where UCL and LCL are upper and lower control limits, respectively. The $F^{-1}(0.75)$ and $F^{-1}(0.25)$ are the third quartile (Q3) and first quartile (Q1) and IQR is the Inter-Quartile Range (IQR = Q3-Q1). The value of L is a coefficient of control limit which this value of L is usually set as 1.5 for the case of a normal distribution assumption.

B. Tukey's Control Chart with Asymmetric Control Limit

In 2011, Lee [8] extended the Tukey's control chart with asymmetrical control limits so called ACL-Tukey's control chart to detect a change in parameter of skew population. The upper and lower control limit of ACL-Tukey's control chart can be written as following:

$$UCL = F^{-1}(0.75) + L_1\left(IQR\right)$$
$$LCL = F^{-1}(0.25) - L_2\left(IQR\right)$$

where L_1 and L_2 are the upper and lower control limit coefficients, respectively. The value of $F^{-1}(0.75)$ and $F^{-1}(0.25)$ are the third quartile (Q3) and first quartile (Q1) and IQR is the Inter-Quartile Range (IQR = Q3-Q1).

C. The Properties of Average Run Length

One of the common characteristics of control charts is in-control Average Run Length (ARL_0) or a mean of false alarm. Ideally, an acceptable ARL_0 of in-control process should be enough large. Otherwise it should be small when the process is out-of-control, so-called out-of-control Average Run Length (ARL_1) or a mean of true alarm. In general, the ARL of Shewhart control chart can be calculated as follows:

$$ARL_0 = \frac{1}{P_I} \quad \text{and} \quad ARL_0 = \frac{1}{P_{II}}$$

where P_I and P_{II} are a probability of type I and type II error.

We assume that mean and variance of in-control process are α_0 and σ^2 and the process mean has been changed to be $\alpha_1 = \alpha_0 + \delta\sigma,$ where $\delta = \frac{(\alpha - \alpha_0)}{\sigma}$ is the magnitudes of shift. Let x be the sample observation, $P(\delta)$ be the probability that an observation falls outside control limits for a specific $\delta,$ $f(x)$ be the probability density function (pdf.) of population, and then $P(\delta)$ is:

$$P(\delta) = 1 - \int_{LCL-\delta\sigma}^{UCL-\delta\sigma} f(x)dx. \tag{1}$$

The ARL of Tukey's control chart for a specific δ is:

$$ARL(\delta) = \frac{1}{P(\delta)}. \tag{2}$$

However, the ARL is not always good measurement of performance when δ is uncertain (Wu et al. [9] and Ryu et al. [10]). To overcome this problem, the Average ARL (AARL) is an effective alternative to the ARL. Then, the AARL for shift range $[-\tau, \tau]$ is

$$AARL(\delta) = \int_{-\tau}^{\tau} w(\delta) ARL(\delta)\, d\delta, \tag{3}$$

where $w(\delta)$ is the weight of $ARL(\delta)$ which $w(\delta)$ is defined as δ^2. In this paper, however, we study the AARL as this following form

$$AARL = \frac{\sum_{i=-3}^{3} ARL(\delta_i)}{\sum_{i=-3}^{3} (\delta_i)}.$$

According to the asymmetrical control limits, they can be found by using linear programming to obtain L_1 and L_2 by given the $ARL_0 = T$, where T is usually equal to 370.4 for standard Shewhart control chart. The linear programming can be written as

$$Min \quad AARL$$

Subject to

$$\begin{aligned} &ARL(\delta = 0) = T, \\ &L_1,\ L_2 \ \geq\ 0. \end{aligned} \tag{4}$$

If $L_1 = L_2$, then the ACL-Tukey's control chart coincides the SCL-Tukey's control chart.

III. Process Observation

In this study, the process observations are selected to be lognormal distribution as skew population and Laplace distribution as non-skew distribution.

A. Lognormal Distribution

The probability density function of lognormal distribution is following:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \alpha)^2}{2\sigma^2}} \; ; x > 0$$

and mean is $e^{\alpha + \frac{\sigma^2}{2}}$ and variance is $e^{2\alpha + \frac{\sigma^2}{2}}(e^{\sigma^2} - 1).$

In this research, the parameter of lognormal distribution is selected as LN(0,0.1), LN(0,0.125), LN(0,0.25) and LN(0,0.4) for in-control process.

B. Laplace Distribution

The probability density function of Laplace distribution is following:

$$f(x) = \frac{1}{2\beta} \exp\left(-\frac{|x - \alpha|}{\beta} \right) \; ; x > 0$$

and mean is $\alpha, \beta > 0$ and variance is $2\beta^2.$

In this research, the parameter of lognormal distribution is selected as Lp(0,1), Lp(0,1), Lp(0,4) and Lp(2,1) for in-control process.

IV. Numerical Results

In this section, we show the numerical results of ARL and AARL of ACL and SCL Tukey's control chart when the process observations are from lognormal and Laplace distributions on Tables I and II, respectively. Table I provides the control limit coefficient of SCL and ACL Tukey's control chart (L_1, L_2) as optimal design parameter. The lognormal acts as right-skew distribution, the performances of SCL and ACL in detecting of positive mean shifts are in good agreement. For detecting the negative mean shifts, however, the performance of ACL is superior to SCL and also the performance of SCL is not sensitive for detection negative side obviously. Therefore, the values of AARL is an alternative measurement of the performance which the AARL of ACL for all cases LN(0,0.1), LN(0,0.125), LN(0,0.25) and LN(0,0.4) has better ability in detecting process mean shifts than SCL for skew population. Otherwise, when the process observations are closed to symmetric as Laplace distribution (i.e., Lp(0,1), Lp(0,2), Lp(0,4) and Lp(2,1)), the performance of SCL and ACL are in good agreement for detection of negative and positive process mean shift and also the AARL of this case have similar ability as shown on Table II. The performance of ACL is robust to both skew and non-skew population.

TABLE I
VALUES OF THE ARL AND AARL OF TUKEY'S CONTROL CHART FOR THE CASE OF LOGNORMAL DISTRIBUTION

	LN(0,0.1)		LN(0,0.125)		LN(0,0.25)		LN(0,0.4)	
Chart	SCL	ACL	SCL	ACL	SCL	ACL	SCL	ACL
L_2	1.958	1.974	1.944	1.955	2.423	2.427	3.172	4.036
L_1	1.958	1.780	1.944	1.722	2.423	1.424	3.172	0.824
UCL	1.334	1.334	1.416	1.418	2.005	2.006	3.068	3.514
LCL	0.670	0.670	0.591	0.628	0.0237	0.362	0.985	0.313
δ								
-3.0	1	1	1	1.004	1.048	1.006	2.346	1.024
-2.0	1.003	1.001	1.019	1.010	1.861	1.121	661.560	1.145
-1.5	1.078	1.038	1.216	1.136	6.542	1.503	3694.11	1.389
-1.0	2.016	1.550	3.085	2.263	196.253	3.608	1750.99	2.247
-0.75	4.948	2.964	9.324	5.436	2551.53	8.932	1195.73	3.626
-0.50	23.980	9.961	57.292	24.006	1739.18	39.043	812.094	8.048
-0.25	272.88	74.026	644.111	226.571	798.823	303.37	548.54	32.49
0	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0.25	67.015	80.017	85.428	88.014	172.044	173.59	246.31	804.3
0.50	15.415	17.859	23.196	23.808	81.387	82.102	163.80	544.0
0.75	4.7982	5.367	7.662	7.829	39.300	39.634	108.43	365.5
1.0	2.093	2.254	3.187	3.239	19.526	19.684	71.49	244.2
1.5	1.052	1.064	1.199	1.206	5.485	5.522	30.80	107.5
2.0	1.	1	1.003	1.003	2.03	2.039	13.25	46.56
3.0	1	1	1	1	1.003	1.003	2.716	8.663
AARL	**80.68**	**50.53**	**120.11**	**48.42**	**399.09**	**70.17**	**644.84**	**169.4**

TABLE II
VALUES OF THE ARL AND AARL OF TUKEY'S CONTROL CHART FOR THE CASE OF LAPLACE DISTRIBUTION

	Lp(0,1)		Lp(0,2)		Lp(0,4)		Lp(2,1)	
Chart	SCL	ACL	SCL	ACL	SCL	ACL	SCL	ACL
L_2	3.983	4.004	3.322	3.337	2.822	2.836	3.767	3.791
L_1	3.983	3.963	3.322	3.308	2.822	2.808	3.767	3.743
UCL	6.2146	6.244	10.597	10.637	18.421	18.498	7.915	7.948
LCL	6.2146	6.186	-10.597	-10.557	-18.421	-18.345	-3.915	-3.8822
δ								
-3.0	117.530	114.35	123.15	121.243	87.12	86.32	87.11	84.40
-2.0	228.29	222.64	157.99	156.09	93.74	93.14	169.20	164.385
-1.5	307.47	300.71	174.1	172.42	96.26	95.79	227.88	222.128
-1.0	394.46	387.71	187.33	186.06	98.13	97.80	292.35	286.600
-0.75	434.68	428.72	192.36	191.36	98.79	98.55	322.16	317.082
-0.50	467.71	463.21	196.09	195.4	99.27	99.11	346.63	342.800
-0.25	489.57	487.13	198.38	198.03	99.57	99.48	363.56	361.473
0	500	500	200	200	100	100	370.4	370.4
0.25	489.55	492.02	198.38	198.73	99.57	99.65	362.83	364.930
0.50	467.68	472.27	196.08	196.77	99.27	99.44	346.62	350.536
0.75	434.64	440.77	192.35	193.35	98.79	99.04	322.13	327.367
1.0	394.42	401.42	187.32	188.6	98.13	98.45	292.32	298.882
1.5	307.43	314.51	174.09	175.8	96.26	96.74	227.8	233.906
2.0	228.26	234.22	157.98	159.92	93.74	94.35	169.17	174.273
3.0	117.51	120.87	123.14	125.10	87.12	87.94	87.09	89.97
AARL	**358.62**	**358.71**	**177.25**	**177.26**	**96.38**	**96.39**	**265.94**	**265.819**

V. CONCLUSION

This study objects to study of robustness of asymmetric Tukey's control chart with skew population as lognormal distribution and non-skew population as Laplace distribution. The comparative study of asymmetric and symmetric control limits of Tukey's control chart are investigated by ARL and AARL. We found that the ACL is robust to both skew and non-skew distributions while the SCL is seriously insensitive to detect the negative shifts. Therefore, the ACL Tukey's control chart can use for skew and non-skew population.

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