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Optimal Parameters of Double Moving Average Control Chart

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Abstract**—**The objective of this paper is to present explicit analytical formulas for evaluating important characteristics of Double Moving Average control chart (DMA) for Poisson distribution. The most popular characteristics of a control chart are Average Run Length (ARL_0) - the mean of observations that are taken before a system is signaled to be out-of control when it is actually still in-control, and Average Delay time (ARL_1) - mean delay of true alarm times. An important property required of ARL_0 is that it should be sufficiently large when the process is in-control to reduce a number of false alarms. On the other side, if the process is actually out-of-control then ARL_1 should be as small as possible. In particular, the explicit analytical formulas for evaluating ARL_0 and ARL_1 be able to get a set of optimal parameters which depend on a width of the moving average (w) and width of control limit (H) for designing DMA chart with minimum of ARL_1.

Keywords**—**Optimal parameters, Average Run Length, Average Delay time, Double Moving Average chart.

I. Introduction

CONTROL chart is an effective tool in statistical process control for detecting changes in a processes, and uses for measuring, controlling and improving quality in many areas of interest including finance and economics, medicine, sociology, engineering, and others. Attribute control charts are important technique in SPC to monitor the discrete data. When the quality characteristic cannot be measured on a continuous scale, for instance, in counting the number of defective products or the number of nonconformities in a production process, an attribute control chart must be used. Commonly-used attribute control charts are p, np, c, and u charts. Additionally, Exponentially Weighted Moving Average (EWMA) and Cumulative Sum (CUSUM) charts for attribute data have also been applied to discrete processes (see, e.g., Page [1], Alwan [2]). Recently, the Moving Average control chart (MA) first has studied for monitoring the non-conforming or defective fraction in discrete processes by Khoo [3]). Later, Double Moving Average chart (DMA) was extended by Khoo and Wong [4] with moving average of the MA statistic one more time. They proposed this chart with normal observations and also showed the numerical simulations of ARL. According to Khoo and Wong [4], the performance of the DMA chart is superior to the MA, EWMA and CUSUM charts for monitoring small and moderate shifts for process mean. Furthermore, the explicit formulas for computing the ARL_0 and ARL_1 when the weighted moving average (w) equal to 1 and 2 were proposed by Areepong and Sukparungsee [5]. Consequently, the explicit formulas of ARL_0 and ARL_1 for DMA chart with arbitrary the values of *w* when observations are binomial distribution also submitted in Sukparungsee and Areepong [6]. In this paper, the explicit analytical formulas for evaluating ARL_0 and ARL_1 DMA chart for Poisson distribution and a set of optimal parameters which depend on a width of the moving average (w) and width of control limit (H) for designing DMA chart with minimum of ARL_1 are presented.

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II. Control Charts and Their Properties

We consider SPC charts under the assumption that sequential observations $X_1, X_2, \ldots$ of some process are identical independently distributed random variables with a distribution function $F(x,\alpha)$, where α is a control parameter. It is assumed that $\alpha = \alpha_0$ while the process is in-control and $\alpha = \alpha_1 > \alpha_0$ when the process goes out-of-control. It is assumed that there is a change-point time $\theta \le \infty$ at which the parameter changes from $\alpha = \alpha_0$ to $\alpha = \alpha_1$. Note that $\theta = \infty$ means that the process always remains in the in-control state.

All popular charts, such as Shewhart, Cumulative Sum (CUSUM) and EWMA charts are based on some function of parameter values that is used as a criterion for a process to go "out-of-control" if this function value goes above an upper control limit (UCL) or below a lower control limit (LCL). The minimum time required for a chart to signal out-of-control is defined as the stopping (alarm) time τ.

Let $E_\theta(.)$ denote the expectation that the change-point from $\alpha = \alpha_0$ to $\alpha = \alpha_1$ for a distribution function $F(x,\alpha)$ occurs at time θ, where $\theta \le \infty$. In the literature on quality control the quantity $E_\infty(\tau)$ is called the Average Run Length (ARL_0) of the chart for the given process.

A typical condition imposed on an ARL_0 is that:

$$ARL_0 = E_\infty(\tau) = T\,, \qquad (1)$$

where T is given (usually large). For given distribution function and chart, this condition then determines choices for the UCL and LCL.

A typical definition of the ARL_1 is that

$$ARL_1 = E_1(\tau \mid \tau \ge 1)\,, \qquad (2)$$

for the change point occurs at $\theta = 1$. One could expect that a sequential control chart has a near optimal performance if ARL_1 is close to a minimal value.

There are many other criteria for optimality of SPC (see Lorden [7]; Shiryaev [8]); however, in practice, ARL_0 and ARL_1 remain the most popular characteristics which are convenient to use for comparisons of charts.

A Moving Average control chart, let observations $X_1, X_2, ..., X_m$ be i.i.d random variables with Poisson distribution, where X_i number of nonconforming is items in sample *i* of *m* samples of size *n*.

A Moving Average control chart is defined by the following statistics:

$$M_i = \frac{X_i + X_{i-1} + ... + X_{i-w+1}}{w} = \frac{\sum_{j=i-w+1}^{i} X_j}{w}, \quad i \ge w$$

$$M_i = \frac{X_i + X_{i-1} + ... + X_1}{i} = \frac{\sum_{j=1}^{i} X_j}{i}, \quad 1 \le i < w.$$

where *w* is the width of the moving average chart. The Mean and variance of moving average chart, MA_i are

$$E(MA_i) = \alpha_0$$

$$Var(MA_i) = \begin{cases} \dfrac{\alpha_0}{i}, & i < w \\ \dfrac{\alpha_0}{w}, & i \ge w \end{cases}$$

The upper and lower control limits are:

$$UCL/LCL = \begin{cases} \alpha_0 \pm B\sqrt{\dfrac{\alpha_0}{i}}, & i < w \\ \alpha_0 \pm B\sqrt{\dfrac{\alpha_0}{w}}, & i \ge w \end{cases}$$

where B is a constant to be chosen.

The alarm time for the MA procedure is given by

$$\tau = \inf\{i > 0 : M_i > UCL \text{ or } M_i < LCL\}.$$

A Double Moving Average (DMA) control chart was initially proposed by Khoo and Wong [4] which defined by the following statistics:

$$DMA_i = \begin{cases} \dfrac{MA_i + MA_{i-1} + MA_{i-2} + ...}{i}; & i \le w \\ \dfrac{MA_i + MA_{i-1} + ... + MA_{i-w+1}}{w}; & w < i < 2w-1 \\ \dfrac{MA_i + MA_{i-1} + ... + MA_{i-w+1}}{w}; & w \ge 2w-1 \end{cases}$$

The Mean and variance of Double Moving Average chart, DMA_i are

$$E(DMA_i) = \alpha_0$$

$$V(DMA_i) = \begin{cases} \dfrac{\alpha_0}{i^2}, i \le w \\ \dfrac{\alpha_0 + (i-w+1)(1/w)}{w^2}, w < i < 2w-1 \\ \dfrac{\alpha_0}{w^2}, i \ge 2w-1 \end{cases}$$

The upper and lower control limits are:

$$UCL/LCL = \begin{cases} \alpha_0 \pm H\sqrt{\dfrac{\alpha_0 \sum_{j=1}^{i}(1/j)}{i^2}}, & i \le w \\ \alpha_0 \pm H\sqrt{\dfrac{\alpha_0 \sum_{j=i-w+1}^{i}\left(\frac{1}{j}\right) + (i-w+1)(1/w)}{w^2}}, & w+1 < i < 2w-1 \\ \alpha_0 \pm H\sqrt{\dfrac{\alpha_0}{w^2}}, & i \ge 2w-1 \end{cases}$$

where H is a constant to be chosen.

The alarm time for the DMA procedure is given by

$$\tau = \inf\{i > 0 : DMA_i > UCL \text{ or } DMA_i < LCL\}.$$

The ARL values of a Double Moving Average control chart can be derived as follows:

Let $ARL = n$, then

$$\frac{1}{N} = \frac{1}{N}P\left[o.o.c\ signal\ at\ time\ i \le w\right] + \frac{1}{N}P[o.o.c\ signal\ at\ time\ w < i < 2w-1] + \left[\frac{N-(2w-2)}{N}\right]P\left[o.o.c\ signal\ at\ time\ i \ge 2w-1\right]$$

The solution can be obtained by central limit theorem, then the explicit formula of ARL_0 for DMA chart is

$$ARL_0 = \Bigg[1-\Bigg[\Bigg[\sum_{i=1}^{w}\Bigg\{P\Bigg[z>\frac{\alpha_0+H\sqrt{\frac{\alpha_0\sum_{j=1}^{i}(1/j)}{i^2}}-\alpha_0}{\sqrt{\frac{\alpha_0\sum_{j=1}^{i}(1/j)}{i^2}}}\Bigg]+P\Bigg[z<\frac{\alpha_0-H\sqrt{\frac{\alpha_0\sum_{j=1}^{i}(1/j)}{i^2}}-\alpha_0}{\sqrt{\frac{\alpha_0\sum_{j=1}^{i}(1/j)}{i^2}}}\Bigg]\Bigg\}\Bigg]\Bigg]$$
$$-\Bigg[\sum_{i=w+1}^{2w-2}\Bigg\{P\Bigg[z>\frac{\alpha_0+H\sqrt{\frac{\alpha_0\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}-\alpha_0}{\sqrt{\frac{\alpha_0\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}}\Bigg]$$
$$+P\Bigg[z<\frac{\alpha_0-H\sqrt{\frac{\alpha_0\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}-\alpha_0}{\sqrt{\frac{\alpha_0\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}}\Bigg]\Bigg\}\Bigg]\Bigg]\Bigg]\Bigg]$$
$$\times\Bigg[P\Bigg[z>\frac{\alpha_0+H\sqrt{\frac{\alpha_0}{w^2}}-\alpha_0}{\sqrt{\frac{\alpha_0}{w^2}}}\Bigg]+P\Bigg[z<\frac{\alpha_0-H\sqrt{\frac{\alpha_0}{w^2}}-\alpha_0}{\sqrt{\frac{\alpha_0}{w^2}}}\Bigg]\Bigg]^{-1}+(2w-2) \qquad (3)$$

and the explicit formula of ARL_1 for a width of control limit *H*, can therefore be written as follows:

$$ARL_1 = \Bigg[1-\Bigg[\Bigg[\sum_{i=1}^{w}\Bigg\{P\Bigg[z>\frac{\alpha_0+H\sqrt{\frac{\alpha_0\sum_{j=1}^{i}(1/j)}{i^2}}-\alpha_1}{\sqrt{\frac{\alpha_1\sum_{j=1}^{i}(1/j)}{i^2}}}\Bigg]+P\Bigg[z<\frac{\alpha_0-H\sqrt{\frac{\alpha_0\sum_{j=1}^{i}(1/j)}{i^2}}-\alpha_1}{\sqrt{\frac{\alpha_1\sum_{j=1}^{i}(1/j)}{i^2}}}\Bigg]\Bigg\}\Bigg]\Bigg]$$
$$-\Bigg[\sum_{i=w+1}^{2w-2}\Bigg\{P\Bigg[z>\frac{\alpha_0+H\sqrt{\frac{\alpha_0\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}-\alpha_1}{\sqrt{\frac{\alpha_1\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}}\Bigg]$$
$$+P\Bigg[z<\frac{\alpha_0-H\sqrt{\frac{\alpha_0\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}-\alpha_1}{\sqrt{\frac{\alpha_1\sum_{j=i-w+1}^{i}(1/j)+(i-w+1)(1/w)}{w^2}}}\Bigg]\Bigg\}\Bigg]\Bigg]\Bigg]\Bigg]$$
$$\times\Bigg[P\Bigg[z>\frac{\alpha_0+H\sqrt{\frac{\alpha_0}{w^2}}-\alpha_1}{\sqrt{\frac{\alpha_1}{w^2}}}\Bigg]+P\Bigg[z<\frac{\alpha_0-H\sqrt{\frac{\alpha_0}{w^2}}-\alpha_1}{\sqrt{\frac{\alpha_1}{w^2}}}\Bigg]\Bigg]^{-1}+(2w-2) \qquad (4)$$

We first describe a procedure for obtaining optimal designs for DMA chart. The criterion used for choosing optimal values for is the width of the double moving average chart (*w*) and boundary parameter (H) is minimization of ARL_1 for a given in-control parameter value $\alpha_0 = 5$, 10 and $ARL_0 = T$ and a given out-of-control parameter value ($\alpha = \alpha_1$). We compute optimal (w, H) values for T = 370.4 and 500 and magnitudes of change. Table of the optimal parameters values are shown in Tables II and III.

The numerical procedure for obtaining optimal parameters for DMA designs

1. Select an acceptable in-control value of ARL_0 and decide on the change parameter value (α_0) for an out-of-control state.
2. For given α_0 and T, find optimal values of *w* and H to minimize the ARL_1 values given by (4) subject to the constraint that $ARL_0 = T$ in (3), i.e. w and H are solutions of the optimality problem.

III. Numerical Results

In this section, the numerical results for ARL_0 and ARL_1 for a DMA chart were calculated from Equation (3) and Equation (4) as shown in Table I.The parameter values for DMA chart was chosen by given desired ARL_0 = 370, in-control parameter α_0 = 4 and out-of-control parameter values α_1 from 4.01 to 5 (or shift parameters (δ) = 0.01, 0.03,…,1). The results shows that when small shifts ($\delta \le 0.05$) the DMA chart have the best performance when $w = 20$ but for a moderate and large shifts ($\delta > 0.05$) the performance of DMA chart was shown that for the shift increasing DMA performs better as the value of (w) decreases. For example, when $\delta = 0.05$ DMA chart with $w = 20$ shows the best performance because of given minimum ARL_1. Note that, calculations with explicit formula from (3) and (4) is simple and very fast to calculate which the computational times takes less than 1 second.

TABLE I
COMPARISON OF ARL1 FROM PROPOSED FORMULAS WITH C CHART FOR GIVEN $ARL_0 = 370$ AND $\alpha_0 = 4$

δ	c chart	DMA chart w =2	w=3	w=4	w=5	w=10	w=15	w=20
0.00	370.4	370.40	370.40	370.40	370.40	370.40	370.40	370.40
0.01	352.0	367.41	363.78	358.87	352.78	309.41	257.27	**209.12**
0.03	315.5	345.49	318.50	286.76	253.70	125.65	69.08	**48.61**
0.05	280.2	308.15	253.09	200.70	156.85	52.49	31.08	**29.66**
0.07	247.0	264.0	190.5	134.0	94.78	28.11	**22.61**	26.15
0.09	216.6	220.2	140.4	89.36	58.88	**19.25**	20.07	24.33
0.10	202.5	199.8	120.2	73.41	47.19	**17.09**	19.39	23.35
0.20	102.3	71.67	29.29	15.38	**10.83**	11.57	13.16	13.82
0.30	54.22	28.24	10.64	7.12	**6.83**	8.52	8.87	8.91
0.40	31.10	13.04	5.86	**5.29**	5.71	6.39	6.44	6.44
0.50	19.31	7.14	**4.32**	4.54	4.83	5.02	5.02	5.02
1.00	4.17	**2.22**	2.38	2.38	2.38	2.38	2.38	2.38

Bold : Minimum ARL_1

TABLE II
OPTIMAL DESIGN PARAMETERS AND ARL_1^* FOR DMA CHART $\alpha_0 = 5$

α_1	w	H	ARL_1^*	α_1	w	H	ARL_1^*
5.01	133	3	224.573	5.01	153	3.0903	257.7000
5.03	73	3	120.813	5.03	80	3.0903	131.871
5.05	53	3	86.0989	5.05	57	3.0903	92.6761
5.07	42	3	68.0758	5.07	45	3.0903	72.7566
5.10	33	3	52.6570	5.10	35	3.0903	55.9196
5.30	15	3	22.9279	5.30	15	3.0903	24.0094
5.50	10	3	15.2816	5.50	10	3.0903	15.9750

TABLE III
OPTIMAL DESIGN PARAMETERS AND ARL_1^* FOR DMA CHART $\alpha_0 = 10$

α_1	w	H	ARL_1^*	α_1	w	H	ARL_1^*
5.01	115	3	183.8480	5.01	130	3.0903	205.9990
5.03	60	3	92.9922	5.03	65	3.0903	100.143
5.05	43	3	65.1759	5.05	45	3.0903	69.4076
5.07	34	3	51.1230	5.07	36	3.0903	54.1271
5.10	26	3	39.2729	5.10	27	3.0903	41.3889
5.30	11	3	16.9832	5.30	12	3.0903	17.6538
5.50	8	3	11.3161	5.50	8	3.0903	11.7342

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