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Recovery of Missing Samples in Multi-channel Oversampling of Multi-banded Signals

J. M. Kim, and K. H. Kwon

***Abstract*—** We show that in a two-channel sampling series expansion of band-pass signals, any finitely many missing samples can always be recovered via oversampling in a larger band-pass region. We also obtain an analogous result for multi-channel oversampling of harmonic signals.

***Keywords*—** oversampling, multi-channel sampling, recovery of missing samples, band-pass signal, harmonic signal

I. Introduction

FOR a bounded and closed band-region B, let PW_B be the Paley-Wiener space of finite energy (i.e. square integrable) signals of which frequencies are confined in B. That is,

$$PW_B := \{f(t) \in L^2(\mathbb{R}) : \text{supp } \hat{f}(\xi) \subset B\},$$

where $\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt$ is the Fourier transform of $f(t)$ with inverse Fourier transform $f(t) = \mathcal{F}^{-1}(\hat{f})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{it\xi}d\xi$.
If a signal $f(t)$ is single-banded with band-region $B = [-\pi\omega, \pi\omega]$ $(\omega > 0)$, then $f(t)$ can be expanded as a Shannon sampling series:

$$f(t) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{\omega}\Big) \frac{\sin \pi(t-n)}{\pi(t-n)} ,$$

in which all samples $\{f(\frac{n}{\omega}) : n \in \mathbb{Z}\}$ are independent. However, if we oversample $f(t)$ with higher rate than the optimal Nyquist rate ω, then the resulting samples are dependent. Using this observation, we may recover finitely many missing samples([2,3,5,8]). When we join oversampling and multi-channeling, we may or may not able to recover finitely missing samples depending on the nature of the band-region B and pre-filters used in channeling ([6,10]). In this work, we show that in case of band-pass and harmonic signals, any finitely many missing samples can be always recovered through a multi-channel oversampling in a larger band-region of the same type.

II. Oversampling of band-pass signals

Consider a band-pass region $B = B_- \cup B_+$, where $w_0, w > 0$ and

$$B_- = [-\pi(\omega_0 + \omega), -\pi\omega_0] \text{ and } B_+ = [\pi\omega_0, \pi(\omega_0 + \omega)].$$

Then the optimal Nyquist rate for signals in PW_B is ω samples per second. For τ with $0 < \tau \le w_0$, let $\tilde{B} = \tilde{B}_- \cup \tilde{B}_+$ be another band-pass region, where

$$\tilde{B}_- = [-\pi(\omega_0 + \omega + \tau), -\pi(\omega_0 - \tau)]$$

and

$$\tilde{B}_+ = [\pi(\omega_0 - \tau), \pi(\omega_0 + \omega + \tau)].$$

We take τ so that $r := \frac{2\omega_0+\omega}{2\tau+\omega}$ is a positive integer. Then $\tilde{B}_+ = \tilde{B}_- + r\pi(2\tau + \omega)$ so that $\tilde{B}$ becomes a so-called selectively tiled band-region([4]) of length $2\pi\tilde{\omega}$ with $\tilde{\omega} = \omega + 2\tau$. Note that the smallest such τ is obtained when we take r to be the largest integer less than $1 + \frac{2\omega_0}{\omega}$. We now take two pre-filters of bounded measurable functions $A_j(\xi)$ $(j = 1, 2)$ on $\tilde{B}$. We set

$$A(\xi) = \begin{bmatrix} A_1(\xi) & A_1(\xi + r\pi\tilde{\omega}) \\ A_2(\xi) & A_2(\xi + r\pi\tilde{\omega}) \end{bmatrix} \text{ on } \tilde{B}_-$$

and assume for some constant $\alpha > 0$, $|\det A(\xi)| \ge \alpha$ a.e. on $\tilde{B}_-$.

For any band-pass signal $f(t)$ in $PW_{\tilde{B}}$, let

$$c_j(f)(t) := \mathcal{F}^{-1}(A_j(\xi)\hat{f}(\xi))(t) = \frac{1}{\sqrt{2\pi}} \int_{\tilde{B}} A_j(\xi)\hat{f}(\xi)e^{it\xi}d\xi \tag{1}$$

be the channeled output signals of the input signal $f(t)$. Then ([4,7,8,9])

$$f(t) = \sum_{j=1}^{2} \sum_{n} c_j(f)\Big(\frac{2n}{\tilde{\omega}}\Big) S_{j,n}(t), \tag{2}$$

which converges in $PW_{\tilde{B}}$ and also converges uniformly on $\mathbb{R}$. By taking Fourier transform on (2), we obtain

$$\hat{f}(\xi) = \sum_{j=1}^{2} \sum_{n} c_j(f)\Big(\frac{2n}{\tilde{\omega}}\Big) \phi_{j,n}(\xi),$$

which converges in $L^2(B)$, where

$$\phi_{j,n}(\xi) = \frac{1}{\tilde{\omega}} \sqrt{\frac{2}{\pi}} U_j(\xi) e^{-i\frac{2n}{\tilde{\omega}}\xi}$$

and

$$A(\xi)^{-1} = \begin{bmatrix} U_1(\xi) & U_2(\xi) \\ U_1(\xi + r\pi\tilde{\omega}) & U_2(\xi + r\pi\tilde{\omega}) \end{bmatrix} \text{ on } \tilde{B}_-. \tag{3}$$

If $f(t)$ is in PW_B, i.e., supp $\hat{f} \subset B$, then

$$\hat{f}(\xi) = \sum_{j=1}^{2} \sum_{n} c_j(f)\Big(\frac{2n}{\tilde{\omega}}\Big) \phi_{j,n}(\xi)\chi_B(\xi) \text{ in } L^2(B), \tag{4}$$

where $\chi_B(\xi)$ is the characteristic function of B. By taking inverse Fourier transform on (4), we have

$$f(t) = \sum_{j=1}^{2} \sum_{n} c_j(f)\Big(\frac{2n}{\tilde{\omega}}\Big) T_{j,n}(t) \tag{5}$$

where $T_{j,n}(t) = \frac{1}{\sqrt{2\pi}} \int_B \phi_{j,n} e^{it\xi}d\xi$. We may call (5) a two-channel oversampling series expansion of $f(t)$ in PW_B.

The authors are with the Division of Applied Mathematics KAIST, Daejeon, Korea (e-mail: franzkim@amath.kaist.ac.kr, khkwon@amath.kaist.ac.kr)

III. Recovering missing samples

For a band-pass signal $f(t)$ in PW_B, consider its oversampled expansion (5).

Lemma 1. *We have for any integer* m

$$c_k(f)\Big(\frac{2m}{\tilde\omega}\Big) = \frac{1}{\pi\tilde\omega}\sum_n c_k(f)\Big(\frac{2n}{\tilde\omega}\Big)\int_{B_-} e^{i\frac{2}{\tilde\omega}(m-n)\xi} \qquad (6)$$

for $k = 1, 2$.

Proof: By (1) and (4), we have

$$c_k(f)(t) = \frac{1}{\sqrt{2\pi}}\int_{\tilde B} A_k(\xi)\hat f(\xi)e^{it\xi}d\xi$$

$$= \frac{1}{\pi\tilde\omega}\sum_{j=1}^{2}\sum_n c_j\Big(\frac{2n}{\tilde\omega}\Big)\int_{\tilde B} A_k(\xi)U_j(\xi)\chi_B(\xi)e^{i(t-\frac{2n}{\tilde\omega})\xi}d\xi.$$

Hence for any integer m we have

$$\begin{aligned}
&c_k(f)\Big(\frac{2m}{\tilde\omega}\Big) = \\
&\frac{1}{\pi\tilde\omega}\sum_{j=1}^{2}\sum_n c_j\Big(\frac{2n}{\tilde\omega}\Big)\Big[\int_{\tilde B_-} A_k(\xi)U_j(\xi)\chi_B(\xi)e^{i\frac{2}{\tilde\omega}(m-n)}d\xi \\
&+\int_{\tilde B_+} A_k(\xi)U_j(\xi)\chi_B(\xi)e^{i\frac{2}{\tilde\omega}(m-n)}d\xi\Big] \\
&= \frac{1}{\pi\tilde\omega}\sum_{j=1}^{2}\sum_n c_j\Big(\frac{2n}{\tilde\omega}\Big)\int_{\tilde B_-}\Big[A_k(\xi)U_j(\xi) \\
&\qquad + A_k(\xi + r\pi\tilde\omega)U_j(\xi + r\pi\tilde\omega)\Big]\chi_{B_-}(\xi)e^{i\frac{2}{\tilde\omega}(m-n)}d\xi,
\end{aligned}$$

from which (6) comes since $A_k(\xi)U_j(\xi) + A_k(\xi+\pi\tilde\omega)U_j(\xi+\pi\tilde\omega) = \delta_{jk}$ by (3). □

Theorem 1. *For any finite index sets of integers* I_1 *and* I_2*, any finite missing samples* $\{c_1(f)(\frac{2m}{\tilde\omega}) : m \in I_1\} \cup \{c_2(f)(\frac{2n}{\tilde\omega}) : n \in I_2\}$ *can be uniquely recovered.*

Proof: Set $I_1 = \{m_1, m_2, \cdots, m_M\}$ if $I_1 \neq \phi$ and $I_2 = \{n_1, n_2, \cdots, n_N\}$ if $I_2 \neq \phi$. Then we have from (6)

$$c_1(f)\Big(\frac{2m_j}{\tilde\omega}\Big) = \frac{1}{\pi\tilde\omega}\sum_{k=1}^{M} r(m_j, m_k)c_1(f)\Big(\frac{2m_k}{\tilde\omega}\Big) + g_{1j} \qquad (7)$$

for $1 \le j \le M$ and

$$c_2(f)\Big(\frac{2n_j}{\tilde\omega}\Big) = \frac{1}{\pi\tilde\omega}\sum_{k=1}^{N} r(n_j, n_k)c_2(f)\Big(\frac{2n_k}{\tilde\omega}\Big) + g_{2j} \qquad (8)$$

for $1 \le j \le N$ where g_{1j}'s and g_{2j}'s are known quantities and

$$r(m,n) := \int_{B_-} e^{i\frac{2}{\tilde\omega}(m-n)\xi}d\xi \text{ for } m, n \in \mathbb{Z}.$$

We may write (7-8) in a vector form as :

$$\begin{cases} (I - S_1)\mathbf{c}_1 = \mathbf{g}_1 \\ (I - S_2)\mathbf{c}_2 = \mathbf{g}_2 \end{cases} \qquad (9)$$

where

$$\begin{aligned}
\mathbf{c}_1 &:= \Big(c_1(f)\Big(\frac{2m_1}{\tilde\omega}\Big), \cdots, c_1(f)\Big(\frac{2m_M}{\tilde\omega}\Big)\Big)^T, \\
\mathbf{c}_2 &:= \Big(c_2(f)\Big(\frac{2n_1}{\tilde\omega}\Big), \cdots, c_2(f)\Big(\frac{2n_N}{\tilde\omega}\Big)\Big)^T, \\
\mathbf{g}_1 &:= (g_{11}, \cdots, g_{1M})^T, \\
\mathbf{g}_2 &:= (g_{21}, \cdots, g_{2N})^T,
\end{aligned}$$

and

$$S_1 = \Big[\frac{1}{\pi\tilde\omega}r(m_j, m_k)\Big]_{j,k=1}^{M}, \; S_2 = \Big[\frac{1}{\pi\tilde\omega}r(n_j, n_k)\Big]_{j,k=1}^{N}.$$

Note that S_1 and S_2 are self-adjoint. Now for any $u = (u_1, \cdots, u_M) \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\langle S_1 u, u\rangle &= \frac{1}{\pi\tilde\omega}\sum_{j,k=1}^{M} r(m_j, m_k)u_k\overline{u_j} \\
&= \int_{\tilde B_-}\Big|\sum_{j=1}^{M}\overline{u_j}\frac{1}{\sqrt{\pi\tilde\omega}}e^{i\frac{2}{\tilde\omega}m_j\xi}\Big|^2\chi_{B_-}(\xi)d\xi \\
&< \int_{\tilde B_-}\Big|\sum_{j=1}^{M}\overline{u_j}\frac{1}{\sqrt{\pi\tilde\omega}}e^{i\frac{2}{\tilde\omega}m_j\xi}\Big|^2 d\xi = \sum_{j=1}^{M}|u_j|^2
\end{aligned}$$

since $\{\frac{1}{\sqrt{\pi\tilde\omega}}e^{i\frac{2}{\tilde\omega}m\xi}\}_{m\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\tilde B_-)$. Hence, 1 cannot be an eigenvalue of S_1. Similarly, 1 cannot be an eigenvalue of S_2. Therefore, both equations in (9) have unique solutions $\mathbf{c}_1$ and $\mathbf{c}_2$. □

Above process can be readily extended to multi-channel oversampling of harmonic signals (see [1] and Chaper 13 in [4]). Let $f(t)$ be a harmonic signal in PW_B, where

$$B := \bigcup_{i=1}^{N}[a_i, b_i]$$

is a harmonic band-region and

$$\begin{aligned}
b_i - a_i &= \pi\omega \; (1 \le i \le N) \\
a_{i+1} - b_i &= 2\pi\omega_0 \; (1 \le i < N) \text{ for } \omega, \; \omega_0 > 0.
\end{aligned}$$

For $0 < \tau \le \omega_0$, let $\tilde B := \cup_{i=1}^{N}\tilde B_i$ be another harmonic band-region, where

$$\tilde B_i = [a_i - \pi\tau, b_i + \pi\tau] \text{ for } 1 \le i \le N.$$

We take τ so that $r := \frac{2\omega_0+\omega}{2\tau+\omega}$ is a positive integer. Then $\tilde B_j = \tilde B_i + (j-i)r\pi(2\tau+\omega)$ for $1 \le i < j \le N$ so that $\tilde B$ becomes a so-called selectively tiled band-region of total length $N\pi\tilde\omega$, where $\tilde\omega = \omega + 2\tau$. We now take N pre-filters $A_j(\xi)$ $(j = 1, 2, \cdots, N)$ of bounded measurable functions on $\tilde B$. We set $A(\xi)$ be the $N \times N$ matrix whose (j,k)th component is given by

$$A_{jk}(\xi) = A_j(\xi + (k-1)r\pi\tilde\omega)$$

and assume $|\det A(\xi)| \ge \alpha > 0$ a.e. on $\tilde B_1$. Let

$$c_j(f)(t) := \mathcal{F}^{-1}(A_j(\xi)\hat f(\xi))(t) = \frac{1}{\sqrt{2\pi}}\int_{\tilde B} A_j(\xi)\hat f(\xi)e^{it\xi}d\xi$$

be the channeled output signals. Proceeding as in Section 2, we can obtain an oversampling formula for any harmonic signal $f(t)$ in PW_B (but viewed as a signal in $PW_{\tilde{B}}$) as

$$f(t) = \sum_{j=1}^{N} \sum_{n} c_j(f)\Big(\frac{2n}{\tilde{\omega}}\Big) T_{j,n}(t). \tag{10}$$

Then, we have the following multi-channel analog of Theorem 3.2.

Theorem 2. *For any finite index sets of integers $I_i (i = 1, 2, \cdots, N)$, any finite missing samples $\cup_{i=1}^{N} \{c_i(f)(\frac{2m}{\tilde{\omega}}) : m \in I_i\}$ from the oversampling (10) can be uniquely recovered.*

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