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Some New Upper Bounds for the Spectral Radius of Iterative Matrices

Guangbin Wang [+], Xue Li, and Fuping Tan

***Abstract*—**In this paper, we present some new upper bounds for the spectral radius of iterative matrices based on the concept of doubly α diagonally dominant matrix. And subsequently, we give two examples to show that our results are better than the earlier ones.

***Keywords*—**doubly α diagonally dominant matrix, eigenvalue, iterative matrix, spectral radius, upper bound.

I. Introduction

WE consider the linear system

$$Ax = b, \quad (1)$$

where $A \in C^{n,n}$ is a nonsingular square matrix, and $x, b \in C^n$ with x unknown and b known.

To solve (1), we often split A into $A = M - N$, where M is nonsingular, and apply iterative schemes

$$X^{k+1} = M^{-1}NX^k + d,\ k = 0,1,\cdots.$$

For example, let

$$A = D - C,$$

Where $D = diag(A)$, then the associated Jacobi iteration matrix J can be expressed as $J = D^{-1}C$.

Let us denote the class of all complex matrices by $C^{n\times n}$, any eigenvalue by $\lambda(A)$ and the spectral radius of matrix A by $\rho(A)$. And we denote

$$<n> = \{1,2,\cdots,n\},\ R_i(A) = \sum_{j\neq i}|a_{ij}|,$$

$$S_i(A) = \sum_{i\neq j}|a_{ji}|,\ P_{i,\alpha}(A) = \alpha P_i(A) + (1-\alpha)Q_i(A),$$

$i \in <n>$.

Definition 1 $[1]$ $A = (a_{ij}) \in C^{n,n}$, then A is called a strictly diagonally dominant matrix and denoted by $A \in D$ if $|a_{ii}| > R_i(A)$, for any $i \in \langle n\rangle$; if $|a_{ii}||a_{jj}| > R_i(A)R_j(A)$, for $i, j \in \langle n\rangle, i \neq j$, then A is said to be a doubly diagonally dominant matrix and denoted by $A \in DD$.

Definition 2 [2] if exists an $\alpha \in [0,1]$, and such that $|a_{ii}||a_{jj}| > P_{i,\alpha}(A)P_{j,\alpha}(A)$, for $i, j \in \langle n\rangle, i \neq j$, then A is said to be a doubly α diagonally dominant matrix and denoted by $A \in DD(\alpha)$.

As is well known, it is interesting to estimate upper bounds for moduli of eigenvalues or the spectral radius $\rho(M^{-1}N)$ of iteration matrix $M^{-1}N$, and the bound play an important role in many of theoretic analysis $(cf.[3,5-10])$. In Refs.[3,4,5,6], the following results are presented.

Theorem 1 [3,4] Let $M = (m_{ij}) \in D, N = (n_{ij}) \in C^{n\times n}$. Then

$$\left|\lambda(M^{-1}N)\right| \le \max_{i\in<n>} \frac{|n_{ii}| + \sum_{j\neq i}|n_{ij}|}{|m_{ii}| - \sum_{j\neq i}|m_{ij}|}.$$

Theorem 2 [5] Let $M = (m_{ij}) \in DD, N = (n_{ij}) \in C^{n\times n}$. Then

$$\left|\lambda(M^{-1}N)\right| \le \max_{\substack{i\in<n>\\ i\neq j}} \frac{B + \sqrt{B^2 - 4AC}}{2A},$$

where

$$A = |m_{ii}m_{jj}| - R_i(M)R_j(M),$$

$$B = |m_{ii}n_{jj}| + |n_{ii}m_{jj}| + R_i(M)R_j(N) + R_i(N)R_j(M),$$

$$C = |n_{ii}n_{jj}| - R_i(N)R_j(N).$$

Theorem 3 [6] Let $M = (m_{ij}) \in DD, N = (n_{ij}) \in C^{n\times n}$. Then

$$\left|\lambda(M^{-1}N)\right| \le \max_{\substack{i\in<n>\\ i\neq j}} \frac{B' + \sqrt{B'^2 - 4AC'}}{2A},$$

where

$$A = |m_{ii}m_{jj}| - R_i(M)R_j(M),$$

Guangbin Wang and Xue Li, Department of Mathematics, Qingdao University of Science and Technology, Qingdao, 266061, China.

Fuping Tan, Department of Mathematics, Shanghai University, Shanghai, 200444, China.

[+] Corresponding author. E-mail address: wguangbin750828@sina.com.
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$$B' = \left|m_{ii}n_{jj} + n_{ii}m_{jj}\right| + R_i(M)R_j(N) + R_i(N)R_j(M),$$
$$C' = -\left[\left|n_{ii}n_{jj}\right| + R_i(N)R_j(N)\right].$$

In the following section, we present some new bounds of iteration matrix $M^{-1}N$ when $M \in DD(\alpha)$.

II. Main results

Lemma 1 [2] Let $A = (a_{ij}) \in C^{n\times n}$, if $A \in DD(\alpha)$, then A is a nonsingular matrix.

Lemma 2 [7] Let $A = (a_{ij}) \in C^{n\times n}$, then each eigenvalue of A is included in

$$\bigcup_{i\neq j}\left\{\lambda \in C : \left|\lambda - a_{ii}\right|\left|\lambda - a_{jj}\right| \le R_i(A)R_j(A)\right\}.$$

Theorem 4 Let $M = (m_{ij}) \in DD(\alpha), N = (n_{ij}) \in C^{n\times n}$. Then

$$\left|\rho(M^{-1}N)\right| \le \max_{\substack{i\in<n>\\ i\neq j}} \frac{P_2 + \sqrt{P_2^{\,2} - 4P_1P_3}}{2P_1},$$

where

$$P_1 = \left|m_{ii}m_{jj}\right| - P_{i,\alpha}(M)P_{j,\alpha}(M),$$
$$P_2 = \left|m_{ii}n_{jj}\right| + \left|n_{ii}m_{jj}\right| + P_{i,\alpha}(M)P_{j,\alpha}(N) + P_{i,\alpha}(N)P_{j,\alpha}(M),$$
$$P_3 = \left|n_{ii}n_{jj}\right| - P_{i,\alpha}(N)P_{j,\alpha}(N).$$

Proof. Since $M \in DD(\alpha)$, by Lemma 1, we know that M is nonsingular. Without loss of generality, let λ be an arbitrary eigenvalue of $M^{-1}N$, then

$$\det(\lambda I - M^{-1}N) = 0, \text{ i.e.,}$$
$$\det(\lambda M - N) = 0.$$

Moreover if $\lambda M - N \in DD(\alpha)$, by Definition 1 and Lemma 1, we have that

$$\left|\lambda m_{ii} - n_{ii}\right|\left|\lambda m_{jj} - n_{jj}\right| > P_{i,\alpha}(\lambda M - N)P_{j,\alpha}(\lambda M - N),$$
$$i, j \in \langle n\rangle, i \neq j,$$

then λ is not an eigenvalue of $M^{-1}N$. Especially, if

$$\left|\left(|\lambda||m_{ii}| - |n_{ii}|\right)\left(|\lambda||m_{jj}| - |n_{jj}|\right)\right| > P_{i,\alpha}(\lambda M - N)P_{j,\alpha}(\lambda M - N)$$
$$, \ i, j \in \langle n\rangle, i \neq j,$$

λ is not an eigenvalue of $M^{-1}N$. Hence by Lemma 2, if λ is an eigenvalue of $M^{-1}N$, then there exists at least a couple of $i, j \in \langle n\rangle (i \neq j)$, such that

$$\left|\left(|\lambda||m_{ii}| - |n_{ii}|\right)\left(|\lambda||m_{jj}| - |n_{jj}|\right)\right| \le \left[|\lambda|P_{i,\alpha}(M) + P_{i,\alpha}(N)\right]\left[|\lambda|P_{j,\alpha}(M) + P_{j,\alpha}(N)\right],$$

i.e.,

$$\left(\left|m_{ii}m_{jj}\right| - P_{i,\alpha}(M)P_{j,\alpha}(M)\right)|\lambda|^2 - \left(\left|m_{ii}n_{jj}\right| + \left|n_{ii}m_{jj}\right| + P_{i,\alpha}(M)P_{j,\alpha}(N) + P_{i,\alpha}(N)P_{j,\alpha}(M)\right)|\lambda| + \left|n_{ii}n_{jj}\right| - P_{i,\alpha}(N)P_{j,\alpha}(N) \le 0 \qquad (2)$$

and

$$\left(\left|m_{ii}m_{jj}\right| + P_{i,\alpha}(M)P_{j,\alpha}(M)\right)|\lambda|^2 - \left(\left|m_{ii}n_{jj}\right| + \left|n_{ii}m_{jj}\right| - P_{i,\alpha}(M)P_{j,\alpha}(N) - P_{i,\alpha}(N)P_{j,\alpha}(M)\right)|\lambda| + \left[\left|n_{ii}n_{jj}\right| + P_{i,\alpha}(N)P_{j,\alpha}(N)\right] \ge 0 \qquad (3)$$

For inequality (2): we have

$$P_1|\lambda|^2 - P_2|\lambda| + P_3 \le 0. \qquad (4)$$

Since $M \in DD(\alpha)$, $P_1 = \left|m_{ii}m_{jj}\right| - P_{i,\alpha}(M)P_{j,\alpha}(M) > 0$ and $P_2 > 0$, $P_3 > 0$, and the discriminant of a curve of second order $\Delta = P_2^{\,2} - 4P_1P_3 \ge 0$, the solution of (4) satisfies

$$\frac{P_2 - \sqrt{P_2^{\,2} - 4P_1P_3}}{2P_1} \le |\lambda| \le \frac{P_2 + \sqrt{P_2^{\,2} - 4P_1P_3}}{2P_1},$$
$$i, j \in \langle n\rangle, i \neq j,$$

Thus,

$$|\lambda| \le \frac{P_2 + \sqrt{P_2^{\,2} - 4P_1P_3}}{2P_1}.$$

For inequality (3): Obviously, $\left|m_{ii}m_{jj}\right| + P_{i,\alpha}(M)P_{j,\alpha}(M) > 0$, so we have that the discriminant of a curve of second order $\Delta \le 0$, solutions of (3) are all complex numbers, i.e., $\lambda \in C$.

Summarizing the above analysis, we get

$$|\lambda| \le \frac{P_2 + \sqrt{P_2^{\,2} - 4P_1P_3}}{2P_1}.$$

Because λ is an arbitrary eigenvalue of $M^{-1}N$, then

$$\rho(M^{-1}N) \le \max_{\substack{i,j\in<n>\\ i\neq j}} \frac{P_2 + \sqrt{P_2^{\,2} - 4P_1P_3}}{2P_1}.$$

Theorem 5 Let $M = (m_{ij}) \in DD(\alpha), N = (n_{ij}) \in C^{n\times n}$. Then

$$\left|\rho(M^{-1}N)\right| \le \max_{\substack{i\in<n>\\ i\neq j}} \frac{P_4 + \sqrt{P_4^{\,2} - 4P_1P_5}}{2P_1},$$

where P_1 is the same as in Theorem 4 and

$$P_4 = \left|m_{ii}n_{jj} + n_{ii}m_{jj}\right| + P_{i,\alpha}(M)P_{j,\alpha}(N) + P_{i,\alpha}(N)P_{j,\alpha}(M),$$

$$P_5 = -\left[\left|n_{ii}n_{jj}\right| + P_{i,\alpha}(N)P_{j,\alpha}(N)\right].$$

Proof. Since $M \in DD(\alpha)$, by Lemma 1, we have that M is nonsingular. Without loss of generality, let λ be an arbitrary eigenvalue of $M^{-1}N$, then

$\det(\lambda I - M^{-1}N) = 0$, i.e.,

$\det(\lambda M - N) = 0$.

Moreover, if $\lambda M - N \in DD(\alpha)$, by Definition 1 and Lemma 1, we have that

$\left|\lambda m_{ii} - n_{ii}\right|\left|\lambda m_{jj} - n_{jj}\right| > P_{i,\alpha}(\lambda M - N)P_{j,\alpha}(\lambda M - N)$,

$i, j \in \langle n \rangle, i \neq j$,

then λ is not an eigenvalue of $M^{-1}N$. Especially, when $i \neq j$

$$\left(\left|m_{ii}m_{jj}\right|\right)|\lambda|^2 - \left(\left|m_{ii}n_{jj} + n_{ii}m_{jj}\right|\right)|\lambda| - \left|n_{ii}n_{jj}\right| > \left[|\lambda| P_{i,\alpha}(M) + P_{i,\alpha}(N)\right]\left[|\lambda| P_{j,\alpha}(M) + P_{j,\alpha}(N)\right],$$

λ is not an eigenvalue of $M^{-1}N$. When λ is an eigenvalue of $M^{-1}N$, then there exists at least a couple of $i, j \in \langle n \rangle (i \neq j)$, such that

$$\left(\left|m_{ii}m_{jj}\right|\right)|\lambda|^2 - \left(\left|m_{ii}n_{jj} + n_{ii}m_{jj}\right|\right)|\lambda| - \left|n_{ii}n_{jj}\right| \le \left[|\lambda| P_{i,\alpha}(M) + P_{i,\alpha}(N)\right]\left[|\lambda| P_{j,\alpha}(M) + P_{j,\alpha}(N)\right]$$

i.e.,

$$\left(\left|m_{ii}m_{jj}\right| - P_{i,\alpha}(M)P_{j,\alpha}(M)\right)|\lambda|^2 - \left(\left|m_{ii}n_{jj} + n_{ii}m_{jj}\right| + P_{i,\alpha}(M)P_{j,\alpha}(N) + P_{i,\alpha}(N)P_{j,\alpha}(M)\right)|\lambda| - \left|n_{ii}n_{jj}\right| - P_{i,\alpha}(N)P_{j,\alpha}(N) \le 0,$$

that is,

$$P_1|\lambda|^2 - P_4|\lambda| + P_5 \le 0. \quad (5)$$

Since $P_1 > 0, P_4 \ge 0$ and $P_5 \le 0$. and the discriminant of a curve of second order $\Delta = P_4^{\,2} - 4P_1P_5 \ge 0$, the solution of (5) satisfies

$$\frac{P_4 - \sqrt{P_4^{\,2} - 4P_1P_5}}{2P_1} \le |\lambda| \le \frac{P_4 + \sqrt{P_4^{\,2} - 4P_1P_5}}{2P_1},$$

$i, j \in \langle n \rangle, i \neq j$.

Thus,

$$\rho(M^{-1}N) \le \max_{\substack{i \in \langle n \rangle \\ i \neq j}} \frac{P_4 + \sqrt{P_4^{\,2} - 4P_1P_5}}{2P_1}.$$

From Theorem 4 and Theorem 5 we can obtain the following corollary.

Corollary Let $M = (m_{ij}) \in DD(\alpha), N = (n_{ij}) \in C^{n \times n}$. Then

$$\left|\rho(M^{-1}N)\right| \le \min\left\{\max_{\substack{i \in <n> \\ i \neq j}} \frac{P_2 + \sqrt{P_2^{\,2} - 4P_1P_3}}{2P_1}, \max_{\substack{i \in <n> \\ i \neq j}} \frac{P_4 + \sqrt{P_4^{\,2} - 4P_1P_5}}{2P_1}\right\},$$

Where P_1, P_2, P_3, P_4 and P_5 are the same as in Theorem 4 and Theorem 5.

III. Examples

Examples 1 Let $M = \begin{pmatrix} -3 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 4 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$.

Obviously, $M \in DD\left(\frac{1}{2}\right)$, but $M \notin DD$, so we can't apply Theorem 1, 2 and 3 here. By Theorem 4, we have

$\left|\lambda(M^{-1}N)\right| \le 10.77$.

By Theorem 5, we have

$\left|\lambda(M^{-1}N)\right| \le 7.115$.

So

$\left|\lambda(M^{-1}N)\right| \le 7.115$.

Examples 2 Let $M = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 4 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$.

Obviously, $M \in DD\left(\frac{1}{2}\right)$, but $M \notin DD$, so we can't apply Theorem 1, 2 and 3 here. By Theorem 4, we have

$\left|\lambda(M^{-1}N)\right| \le 10.77$.

By Theorem 5, we have

$\left|\lambda(M^{-1}N)\right| \le 11.01$.

So

$\left|\lambda(M^{-1}N)\right| \le 10.77$.

Remark: Example 1 shows that Theorem 5 is better than Theorem 4; however, Example 2 shows that Theorem 4 is better than Theorem 5. That is Theorem 5 is better than Theorem 4 sometimes and Theorem 4 is better than Theorem 5 sometimes.

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