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A Projection Method Based on Extended Krylov Subspaces for Solving Sylvester Equations

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Abstract—In this paper we study numerical methods for solving Sylvester matrix equations of the form $AX + XB^T + CD^T = 0$. A new projection method is proposed. The union of Krylov subspaces in A and its inverse and the union of Krylov subspaces in B and its inverse are used as the right and left projection subspaces, respectively. The Arnoldi-like process for constructing the orthonormal basis of the projection subspaces is outlined. We show that the approximate solution is an exact solution of a perturbed Sylvester matrix equation. Moreover, exact expression for the norm of residual is derived and results on finite termination and convergence are presented. Some numerical examples are presented to illustrate the effectiveness of the proposed method.

Keywords—Arnoldi process; Krylov subspace; Iterative method; Sylvester equation; Dissipative matrix

I. Introduction

In this paper we will consider the numerical solution of large-scale Sylvester matrix equations of the form

$$AX + XB^T = -CD^T, \tag{1}$$

where the matrices $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{M\times M}$, $C \in \mathbb{R}^{N\times s}$, $D \in \mathbb{R}^{M\times s}$ with $s \ll \min\{N,\ M\}$, and $X \in \mathbb{R}^{N\times M}$ is the solution matrix sought. We assume that the coefficient matrices A and B are large and sparse, and the matrices C and D are of full column rank. If $B = A$ and $D = C$, the Lyapunov matrix equations result.

Let $X = [x_1, x_2, \cdots, x_M]$, where x_i is the i-th column of X. Define a linear operator vec : $\mathbb{R}^{N\times M} \to \mathbb{R}^{MN}$ by

$$\text{vec}(X) = [x_1^T, x_2^T, \cdots, x_M^T]^T.$$

Then, the Sylvester equation (1) can be written as a system of linear equations

$$\mathcal{A}\text{vec}(X) = -\text{vec}(CD^T),$$

where the coefficient matrix $\mathcal{A} = I_M \otimes A + B \otimes I_N \in \mathbb{R}^{MN\times MN}$. Here and in the following, $\otimes$ denotes the Kronecker product, see, for example, [12] for its definition and properties. Let $\Lambda(A)$ and $\Lambda(B)$ be the spectrum of A and B, respectively. Then, the spectrum of the matrix $\mathcal{A}$ is

$$\Lambda(\mathcal{A}) = \{\lambda_i + \mu_j :\ \lambda_i \in \Lambda(A), \mu_j \in \Lambda(B), \\ i = 1, 2, \cdots, N,\ j = 1, 2, \cdots, M\},$$

which shows that the Sylvester equation (1) has a unique solution if and only if $\lambda_i + \mu_j \neq 0$ for any $\lambda_i \in \Lambda(A)$, $\mu_j \in \Lambda(B)$, $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, M$, see [29].

For simplicity, we assume in this paper that A and B are dissipative, that is, $x^T(A + A^T)x < 0$ and $y^T(B + B^T)y < 0$ for all the nonzero real vectors $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. Under this assumption, the matrices A and B are stable, i.e., their spectrum are contained in the open left half of the complex plane, and therefore the Sylvester equation (1) has a unique solution.

Sylvester equations and Lyapunov equations play an important role in a number of applications such as control theory [11], model reduction of linear time invariant systems [1], [28], [42], image restoration [10], and block-diagonalization of matrices [18], [22], [28]. For some important theoretical results, the interested reader is referred to the survey paper [9] and references therein. The analytical solution of Sylvester equations has been considered by some authors, see, for example, [14], [21].

During the past four decades, a number of numerical solution methods have been proposed for Lyapunov and Sylvester equations. The classical direct methods are the Bartels-Stewart method [4], [43], the Hessenberg-Schur method [17], and the Hammarling method [20], [27]. These methods first compute the real Schur forms of the coefficient matrices by means of the QR algorithm [18], and then the original equation is transformed into a form that is easy to be solved by a forward substitution. The QR algorithm requires $\mathcal{O}(N^3)$ operations and $\mathcal{O}(N^2)$ memory. Therefore, direct methods are only practicable for problems of relatively small size. Besides direct methods, we mention, among several iterative methods, the Smith method [41], the alternating direction implicit iteration (ADI) method [44], [31], the Smith(l) method [33], the modified low-rank Smith method [19], the Cholesky factor-alternating direction implicit (CF-ADI) method [30], and the matrix sign function method [5], [7]. The matrix sign function method are appropriate for problems with the coefficient matrices dense and stable. For large-scale Lyapunov and Sylvester equations with sparse coefficient matrices, Krylov subspace methods [3], [16], [23], [24], [25], [26], [34] and ADI-type methods are commonly popular. In Krylov subspace methods, the original matrix equation is first projected onto some Krylov subspace to produce a low-dimensional Lyapunov or Sylvester equation,

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and then the solution is obtained via solving the reduced matrix equation by direct methods. Krylov subspace based methods have advantages over ADI-type methods in that no knowledge about the spectra of the coefficient matrices are needed. However, ADI-type methods often have better performance if good parameters are used and linear systems with shifted coefficient matrices are solved effectively at low cost; see, for example, [33] for some numerical experiments. Recently, a new projection method that does not require any estimation of parameters is proposed for solving large-scale Lyapunov equations in [38] by Simoncini. The projection subspace is a combination of Krylov subspaces in A and A^{-1}. Numerical results reported in that paper show that in general, the new method outperforms ADI-type methods for some benchmark problems.

The new iterative method presented in this paper for solving the large-scale Sylvester equation (1) is an extension of the work in [38]. The algorithm is developed also under the framework of projection. We use the union of Krylov subspaces in A and A^{-1} and the union of Krylov subspaces in B and B^{-1} as the right and left projection subspaces, respectively. The resulting low-dimensional Sylvester equation is solved by direct methods. The performance of the new method is compared with that of the ADI methd [8] and the projection method based on a global Arnoldi process in [3].

Throughout this paper, the following notation is used. The $l\times l$ identity matrix is denoted by I_l and the zero vector or zero matrix by 0. If the dimension of I_l is apparent from the context, we drop the index and simply use I. The actual dimension of 0 will always be apparent from the context. $\|\cdot\|$ indicates the 2-norm for vectors and the induced norm for matrices, while $\|\cdot\|_F$ denotes the Frobenius norm. The superscript T denotes the transpose of a vector or a matrix. $e_i^{(n)}$ denotes the i-th coordinate vector of $\mathbb{R}^n$. The notation $\mathrm{span}\{V\}$ denotes the space spanned by the column vectors of the matrix V and $\mathrm{span}\{V_1, V_2, \cdots, V_n\}$ denotes the space spanned by the matrix sequence $V_1, V_2, \cdots, V_n$. For a given square matrix F and a given rectangle matrix r_0, the Krylov subspace $\mathcal{K}_n(F, r_0)$ is defined by $\mathcal{K}_n(F, r_0) = \mathrm{span}\{r_0, Fr_0, \cdots, F^{n-1}r_0\}$. Finally, Matlab [32] notation is used whenever possible.

The remainder of the paper is organized as follows. In Section 2, we give a brief description of the extended Krylov subspace and the Arnoldi-like process for generating an orthonormal basis of the subspace. Section 3 gives low-rank solutions and residual error expressions for the Sylvester matrix equation. Section 4 is devoted to some numerical tests. Finally, some concluding remarks are given in Section 5.

II. Extended Krylov subspace

In this section, we introduce a class of extended Krylov subspaces, which will be employed to construct the projecting subspaces for solving the Sylvester equation (1).

Suppose that the matrix $F \in \mathbb{R}^{N\times N}$ is invertible and $r_0 \in \mathbb{R}^{N\times 1}$. The extended Krylov subspace $\mathbf{K}_n(F, r_0)$ is defined by

$$\mathbf{K}_n(F, r_0) = \mathrm{span}\{r_0, F^{-1}r_0, Fr_0, F^{-2}r_0, \cdots, F^{n-1}r_0, F^{-n}r_0\}.$$

Note that the extended subspace $\mathbf{K}_n(F, r_0)$ contain information on both A and A^{-1}. This class of subspaces has been used by Simoncini [38] for solving Lyapunov equations and by Druskin and Knizhnerman [13] for approximating matrix functions. Clearly, the extended Krylov subspace $\mathbf{K}_n(F, r_0)$ is the union of the Krylov subspace $\mathcal{K}_n(F, r_0)$ and $\mathcal{K}_n(F^{-1}, F^{-1}r_0)$, that is,

$$\mathbf{K}_n(F, r_0) = \mathcal{K}_n(F, r_0) \cup \mathcal{K}_n(F^{-1}, F^{-1}r_0).$$

An Arnoldi-like process for establishing an orthonormal basis of the subspace $\mathbf{K}_n(F, r_0)$ has been proposed in [38]. The algorithm is described as follows.

Algorithm 2.1: **Arnoldi-like process**

1. Compute V_1 by QR decomposition: $[r_0, F^{-1}r_0] = V_1 R$.
2. For $j = 1, 2, \cdots, n$
3. Set $V_j^{(1)}$: first column of V_j;
4. Set $V_j^{(2)}$: second column of V_j;
5. $\widetilde{V}_{j+1} = [FV_j^{(1)}, F^{-1}V_j^{(2)}]$;
6. For $i = 1, 2, \cdots, j$
7. $H_{ij} = V_i^T \widetilde{V}_{j+1}$;
8. $\widetilde{V}_{j+1} = \widetilde{V}_{j+1} - V_i H_{i,j}$;
9. End For
10. $\widetilde{V}_{j+1} = V_{j+1} H_{j+1,j}$ (QR decomposition);
11. End For

The columns of the matrix $\mathcal{V}_n = [V_1, V_2, \cdots, V_n]$ with $V_i \in \mathbb{R}^{N\times 2}$ are an orthonormal basis of the subspace $\mathbf{K}_n(F, r_0)$. The Matlab codes for the implementation of the Arnoldi-like process has been given in [38].

Algorithm 2.1 is known as an implementation of the Arnoldi-like process in the modified Gram-Schmidt orthogonalization form [12] for generating an orthonormal basis of $\mathbf{K}_n(F, r_0)$. It is well known that in the presence of finite precision arithmetic, a loss of orthogonality can occur when the orthogonalization algorithm progresses, see [12], [18], [35]. A remedy is the so-called reorthogonalization where the current vectors have to be orthogonalized against previously created vectors. One can choose between a selective reorthogonalization or a full reorthogonalization.

The product of F^{-1} with some matrix should be implemented by solving the linear systems of equations with the coefficient matrix F. To do it, the LU factorization [18] of F is employed for medium-size matrices, and the Cholesky factorization of $-F$ should be used for $-F$ symmetric definite. For large-scale matrices, a preconditioning iterative method could be employed to solve systems with F, where the preconditioner could be generated once for all. Iterative methods that are used nowadays are Krylov subspace methods such as GMRES [36]. There is possible stagnation for one or more iterations of GMRES in some instances. However, stagnation never happens if the coefficient matrix is dissipative; see [15], [40] for some conditions for non-stagnation of GMRES. For a comprehensive introduction of iterative methods for linear systems of equations, the interested reader is referred to [37].

Define the block upper Hessenberg matrix $\mathcal{H}_n \in \mathbb{R}^{2n\times 2n}$

with blocks 2 by 2 as

$$\mathcal{H}_n = \begin{bmatrix} H_{11} & H_{12} & \cdots & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & \cdots & \vdots \\ 0 & H_{32} & H_{33} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & H_{n,n-1} & H_{nn} \end{bmatrix}.$$

It is easy to verify that

$$V_{n+1}H_{n+1,n} = [AV_n^{(1)}, A^{-1}V_n^{(2)}] - \mathcal{V}_n\mathcal{H}_nE_n^T, \qquad (2)$$

where $E_n^T = [0, 0, \cdots, I] \in \mathbb{R}^{2\times 2n}$.

When the inexact iterative solvers are used for the linear systems of equations with the coefficient matrix F, the subspace $\mathrm{span}\{\mathcal{V}_n\}$ generated by Algorithm 2.1 is no longer the extended Krylov subspace $\mathbf{K}_n(F, r_0)$. In this case, the recursive relation (2) does not hold generally. We mention that the effect of inexact matrix-vector products on the solution of systems of linear equations has been studied by Simoncini and Szyld [39]. They have shown that the norm of the matrix-product perturbation can grow as the iteration progresses. However, we do not address the issue in the present paper.

The following proposition shows the relation between the subspace $\mathbf{K}_n(F, r_0)$ and the standard Krylov subspace $\mathcal{K}_{2n}(F, F^{-n}r_0)$.

Proposition 2.1: [38] For any $n \geq 1$, the space $\mathbf{K}_n(F, r_0)$ satisfies $\mathbf{K}_n(F, r_0) = \mathcal{K}_{2n}(F, F^{-n}r_0)$. In particular, it follows that

$$F\mathbf{K}_n(F, r_0) \subseteq \mathbf{K}_{n+1}(F, r_0).$$

Define $\bar{\mathcal{T}}_n := \mathcal{V}_{n+1}^T A\mathcal{V}_n$ and let $\mathcal{T}_n$ be the $2n \times 2n$ matrix obtained from $\bar{\mathcal{T}}_n$ by deleting the last 2 rows. We observe that the results of Proposition 2.1 ensure that $\bar{\mathcal{T}}_n$ is block upper Hessenberg, since $V_i^T F V_j = 0$ for $i > j + 1, j = 1, 2, \cdots$. We obtain the following relation:

$$F\mathcal{V}_n = \mathcal{V}_n\mathcal{T}_n + V_{n+1}T_{n+1,n}E_n^T.$$

Here and in the following, $T_{i,j}$ is the (i, j)-th block of $\bar{\mathcal{T}}_n$.

It seems that the computation of $\bar{\mathcal{T}}_n$ requires additional matrix-vector products with F and extra inner products of long vectors, but the following proposition shows that this additional cost can be avoided.

Proposition 2.2: [38] Consider the QR decomposition $\widetilde{V}_{k+1} = V_{k+1}H_{k+1,k}$ in Algorithm 2.1. Let $l^{(k)} = (l_{ij})$ be the 2×2 matrix such that $V_{k+1} = \widetilde{V}_{k+1}l^{(k)}$, $k = 1, 2, \cdots, n$. Let

$$\begin{aligned} \bar{\mathcal{T}}_n &= (t_{ij})_{i=1,\cdots,2n+2, j=1,\cdots,2n}, \\ \mathcal{H}_n &= (h_{ij})_{i=1,\cdots,2n, j=1,\cdots,2n}. \end{aligned}$$

Then (odd columns)

$$t_{:,2k-1} = h_{:,2k-1}, \quad k = 1, \cdots, n,$$

while (even columns)

$$\begin{aligned} (k=1) \quad & t_{:,2} = \frac{1}{l_{11}^{(1)}}(h_{:,1}l_{12}^{(1)} + e_1 l_{22}^{(1)}), \\ & t_{:,4} = (e_2 - \bar{\mathcal{T}}_1 h_{1:2,2})l_{22}^{(2)}, \\ & \rho^{(2)} = \frac{l_{12}^{(2)}}{l_{11}^{(2)}}, \\ (1 < k \leq n) \quad & t_{:,2k} = t_{:,2k} + t_{:,2k-1}\rho^{(k)}, \\ & t_{:,2k+2} = (e_{2k} - \bar{\mathcal{T}}_k h_{1:2k,2k})l_{22}^{(k+1)}, \\ & \rho^{(k+1)} = \frac{l_{12}^{(k+1)}}{l_{11}^{(k+1)}}. \end{aligned}$$

III. Projection method based on extended Krylov subspace

In this section, we will use the framework of a projection technique to derive a method for solving the Sylvester equation (1). To simply the presentation, we assume that the rank of C is 1. However, the generalization to larger rank is immediate.

By using the Arnoldi-like algorithm, we generate an orthonormal basis $\mathcal{V}_n$ of $\mathbf{K}_n(A, C)$ and an orthonormal basis $\mathcal{W}_m$ of $\mathbf{K}_m(B, D)$, respectively. With $\mathcal{H}_n^{(A)}$ and $\mathcal{H}_m^{(B)}$ defined by the block Arnoldi-like Algorithm, we can compute $\bar{\mathcal{T}}_n^{(A)}$ and $\bar{\mathcal{T}}_m^{(B)}$ according to the Proposition 2.2. Then, the following relations are satisfied:

$$\begin{cases} A\mathcal{V}_n &= \mathcal{V}_n\mathcal{T}_n^{(A)} + V_{n+1}T_{n+1,n}^{(A)}E_n^T, \\ B\mathcal{W}_m &= \mathcal{W}_m\mathcal{T}_m^{(B)} + W_{m+1}T_{m+1,m}^{(B)}E_m^T. \end{cases} \qquad (3)$$

The approximate solution to X is constructed as

$$X_{n,m} = \mathcal{V}_n Y_{n,m}\mathcal{W}_m^T.$$

Let $C_n := \mathcal{V}_n^T C$ and $D_m := \mathcal{W}_m^T D$. Since both C and D are vectors, we have $C_n = \|C\|e_1^{(2n)}$ and $D_m = \|D\|e_1^{(2m)}$. The residual matrix is then given by

$$\begin{aligned} R_{n,m} := A(\mathcal{V}_n Y_{n,m}\mathcal{W}_m^T) \\ + (\mathcal{V}_n Y_{n,m}\mathcal{W}_m^T)B^T + \mathcal{V}_n C_n D_m^T \mathcal{W}_m^T. \end{aligned} \qquad (4)$$

Substituting (3) into (4) gives (5), where

$$\mathcal{T}_n^{(A)} = \mathcal{V}_n^T A\mathcal{V}_n, \quad \mathcal{T}_m^{(B)} = \mathcal{W}_n^T A\mathcal{W}_n. \qquad (8)$$

According to the Galerkin condition, we want to find an approximate solution $X_{n,m} = \mathcal{V}_n Y_{n,m}\mathcal{W}_m^T$ that satisfies $\mathcal{V}_n^T R_{n,m}\mathcal{W}_m = 0$.

The following theorem is one main result of this section.

Theorem 3.1: Suppose that n steps of the Arnoldi-like process have been taken for $\mathbf{K}_n(A, C)$ and m steps of the Arnoldi-like process have been taken for $\mathbf{K}_m(B, D)$. Let the residual $R_{n,m}$ be defined by (4). Then,

(a) $\mathcal{V}_n^T R_{n,m}\mathcal{W}_m = 0$ if and only if $Y_{n,m} = Y$, where Y satisfies

$$\mathcal{T}_n^{(A)}Y + Y(\mathcal{T}_m^{(B)})^T + C_n D_m^T = 0; \qquad (9)$$

$$R_{n,m} = \mathcal{V}_{n+1} \begin{bmatrix} \mathcal{T}_n^{(A)} Y_{n,m} + Y_{n,m}(\mathcal{T}_m^{(B)})^T + C_n D_m^T & Y_{n,m} E_m (T_{m+1,m}^{(B)})^T \\ T_{n+1,n}^{(A)} E_n^T Y_{n,m} & 0 \end{bmatrix} \mathcal{W}_{m+1}^T \tag{5}$$

$$\begin{aligned} \mathcal{V}_n^T R_{n,m} \mathcal{W}_m &= \mathcal{V}_n^T \mathcal{V}_{n+1} \begin{bmatrix} \mathcal{T}_n^{(A)} Y_{n,m} + Y_{n,m}(\mathcal{T}_m^{(B)})^T + B_n D_m^T & Y_{n,m} E_m (T_{m+1,m}^{(B)})^T \\ T_{n+1,n}^{(A)} E_n^T Y_{n,m} & 0 \end{bmatrix} \mathcal{W}_{m+1}^T \mathcal{W}_m \\ &= [I \;\; 0] \begin{bmatrix} \mathcal{T}_n^{(A)} Y_{n,m} + Y_{n,m}(\mathcal{T}_m^{(B)})^T + B_n D_m^T & Y_{n,m} E_m (T_{m+1,m}^{(B)})^T \\ T_{n+1,n}^{(A)} E_n^T Y_{n,m} & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= \mathcal{T}_n^{(A)} Y_{n,m} + Y_{n,m}(\mathcal{T}_m^{(B)})^T + B_n D_m^T \end{aligned} \tag{6}$$

$$\begin{aligned} A(\mathcal{V}_n Y_{n,m} \mathcal{W}_m^T) + (\mathcal{V}_n Y_{n,m} \mathcal{W}_m^T) B^T + \mathcal{V}_n C_n D_m^T \mathcal{W}_m^T &= \mathcal{V}_{n+1} \begin{bmatrix} 0 & Y_{n,m} E_m (T_{m+1,m}^{(B)})^T \\ T_{n+1,n}^{(A)} E_n^T Y_{n,m} & 0 \end{bmatrix} \mathcal{W}_{m+1}^T \\ &= V_{n+1} T_{n+1,n}^{(A)} E_n^T Y_{n,m} \mathcal{W}_m^T + \mathcal{V}_n Y_{n,m} E_m (T_{m+1,m}^{(B)})^T W_{m+1}^T \end{aligned} \tag{7}$$

(b) If the conditions of (a) are met, then the residual norm is given by

$$\begin{aligned} &\|R_{n,m}\|_F \\ &= \sqrt{\|T_{n+1,n}^{(A)} E_n^T Y_{n,m}\|_F^2 + \|Y_{n,m} E_m (T_{m+1,m}^{(B)})^T\|_F^2}. \end{aligned} \tag{10}$$

Proof: Since $\mathcal{V}_{n+1} = [\mathcal{V}_n, V_{n+1}]$ and $\mathcal{W}_{m+1} = [\mathcal{W}_m, W_{m+1}]$ are orthonormal matrices, pre- and post-multiplying (5) by $\mathcal{V}_n^T$ and $\mathcal{W}_m$, respectively, gives (6).

The result follows immediately, establishing the proof of part (a). Substituting (9) into the (1,1) block of (5) gives

$$R_{n,m} := \mathcal{V}_{n+1} \begin{bmatrix} 0 & Y_{n,m} E_m (T_{m+1,m}^{(B)})^T \\ T_{n+1,n}^{(A)} E_n^T Y_{n,m} & 0 \end{bmatrix} \mathcal{W}_{m+1}^T.$$

Since $\mathcal{V}_{n+1}$ and $\mathcal{W}_{m+1}$ are orthonormal matrices, it is not difficult to verity that

$$\|R_{n,m}\|_F^2 = \|T_{n+1,n}^{(A)} E_n^T Y_{n,m}\|_F^2 + \|Y_{n,m} E_m (T_{m+1,m}^{(B)})^T\|_F^2.$$

■

Since the matrices A and B are dissipative, $\mathcal{T}_n^{(A)}$ and $\mathcal{T}_m^{(B)}$ defined by (8) are stable. This ensures that the reduced Sylvester equation (9) admits a unique solution. The expression for the norm of residual $R_{n,m}$ given by (10) can be used to stop the iterations in the extended Krylov subspace based algorithm. The approximate solution $X_{n,m}$ is computed only when convergence is achieved and this reduces the cost of the method.

The extended Krylov subspace based method for solving Sylvester matrix equation (1) is summarized as follows.

Algorithm 3.1: **Extended Krylov subspace based method**

1. Choose a tolerance $\epsilon > 0$, set two integer parameters $k_1,\ l_1$ and set $k = 0,\ l = 0,\ n = k_1,\ m = l_1$.
2. For $j = k+1, k+2, \cdots, k+k_1$, construct the orthonormal basis $V_{k+1}, \cdots, V_{k+k_1}$ of the subspace $\mathbf{K}(A, C)$ and the matrix $\mathcal{H}^{(A)}$ by Algorithm 2.1.
3. For $j = l+1, l+2, \cdots, l+l_1$, construct the orthonormal basis $W_{l+1}, \cdots, W_{l+l_1}$ of the subspace $\mathbf{K}(B, D)$ and the matrix $\mathcal{H}^{(B)}$ by Algorithm 2.1.
4. Compute $\bar{\mathcal{T}}^{(A)}$ and $\bar{\mathcal{T}}^{(B)}$ according to Proposition 2.2.
5. Solve the low-dimensional problem: $\mathcal{T}_n^{(A)} Y_{n,m} + Y_{n,m}(\mathcal{T}_m^{(B)})^T + C_n D_m^T = 0$ using direct methods.
6. Compute the residual norm:
$$\begin{aligned} &\|R_{n,m}\|_F = \\ &\sqrt{\|T_{n+1,n}^{(A)} E_n^T Y_{n,m}\|_F^2 + \|Y_{n,m} E_m (T_{m+1,m}^{(B)})^T\|_F^2}. \end{aligned}$$
7. If $\|R_{n,m}\|_F > \epsilon$, set $k := k + k_1$, $n = k + k_1$, $l := l + l_1$, $m = l + l_1$ and go to step 2.
8. Form the approximate solution: $X_{n,m} = \mathcal{V}_n Y_{n,m} \mathcal{W}_m^T$.

We remark that the iterative method proposed here for solving the Sylvester equation (1) is different from the one in [3] in that the projection subspaces used in these two methods are different. The projection subspaces used in [3] are global Krylov subspaces while the subspaces used in this paper are extended Krylov subspaces.

The following result shows that $X_{n,m}$ is an exact solution of a perturbed Sylvester matrix equation.

Theorem 3.2: Suppose that n steps of the Arnoldi-like process have been taken for $\mathbf{K}_n(A, C)$ and m steps of the Arnoldi-like process have been taken for $\mathbf{K}_m(B, D)$. Let $X_{n,m} = \mathcal{V}_n Y_{n,m} \mathcal{W}_m^T$ be the low-rank approximate solution of (1), where $Y_{n,m}$ satisfies (9). Then

$$(A - \Delta_n) X_{n,m} + X_{n,m}(B - \Delta_m)^T + C D^T = 0, \tag{11}$$

where $\Delta_n = V_{n+1} T_{n+1,n}^{(A)} V_n^T$, $\Delta_m = W_{m+1} T_{m+1,m}^{(B)} W_m^T$ and $\|\Delta_n\|_F = \|T_{n+1,n}^{(A)}\|_F$, $\|\Delta_m\|_F = \|T_{m+1,m}^{(B)}\|_F$.

Proof: We have (7). Equation (11) follows by rearranging (7) and noting that $E_n^T = V_n^T \mathcal{V}_n$ and $E_m^T = W_m^T \mathcal{W}_m$. The expression for $\|\Delta_n\|_F$ and $\|\Delta_m\|_F$ follows from the fact that V_{n+1} and W_{m+1} are orthonormal matrices. Finally, observe that both Δ_n and Δ_m are at most a rank 2 perturbation. ■

In exact arithmetic, the procedure just described has finite termination, since for n and m such that $2n \geq N$ and $2m \geq$

M the generated vectors span the whole space. However, since two vectors at the time are added to the current basis, loss of rank may occur during the orthogonalization with respect to the old basis vectors, so that the next basis pair using V_{n+1} or W_{m+1} cannot be built. In the remainder of this section, we will show that this implies convergence. Before presenting the main results, we need the following lemma, which is given in [38].

Lemma 3.3: Suppose that $n-1$ steps of the Arnoldi-like process have been taken for $\mathbf{K}(A,C)$. At the nth step, assume that $\widetilde{V}_{n+1}$ has rank less than two. Then $\mathcal{V}_n \cup \{\widetilde{V}_{n+1}\}$ is an invariant subspace of A respect to C. That means

$$A\mathcal{V}_n = \mathcal{V}_n\mathcal{T}_n,$$

or

$$A[\mathcal{V}_n, V_n^{(1)}] = [\mathcal{V}_n, V_n^{(1)}]\hat{\mathcal{T}}_n,$$

where $\hat{\mathcal{T}}_n$ is the restriction of $\mathcal{T}_{n+1}$ to the first $(2n+1)$ columns and rows.

The following theorem provides some results concerning the convergence of Algorithm 3.1 for solving the Sylvester equation (1).

Theorem 3.4: Suppose that $n-1$ steps of the Arnoldi-like process have been taken for $\mathbf{K}_n(A,C)$ and $m-1$ steps of the Arnoldi-like process have been taken for $\mathbf{K}_m(B,D)$. At the nth and mth step, assume that both $\widetilde{V}_{n+1}$ and $\widetilde{W}_{m+1}$ have rank less than two. Then we can find the exact solution of (1).

Proof: From Lemma 3.3, we have

$$A\tilde{\mathcal{V}} = \tilde{\mathcal{V}}\tilde{\mathcal{T}}^{(A)} \quad \text{and} \quad B\tilde{\mathcal{W}} = \tilde{\mathcal{W}}\tilde{\mathcal{T}}^{(B)},$$

where $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ are orthonormal matrices. Let

$$\tilde{X} = \tilde{\mathcal{V}}\tilde{Y}\tilde{\mathcal{W}}^T,$$

where $\tilde{Y}$ satisfies $\tilde{\mathcal{T}}^{(A)}Y + Y(\tilde{\mathcal{T}}^{(B)})^T + \tilde{C}\tilde{D}^T = 0$. We obtain

$$\begin{aligned} &AX + XB^T + CD^T \\ &\quad = \tilde{\mathcal{V}}(\tilde{\mathcal{T}}^{(A)}Y + Y(\tilde{\mathcal{T}}^{(B)})^T + \tilde{C}\tilde{D}^T)\tilde{\mathcal{W}}^T = 0, \end{aligned}$$

where $\tilde{C} = \tilde{\mathcal{V}}^T C$ and $\tilde{D} = \tilde{\mathcal{W}}^T D$. This completes the proof. ■

IV. Numerical experiments

In this section, we use two examples to illustrate the numerical effectiveness of the extended Krylov subspace based method (Algorithm 3.1) for the Sylvester equation (1). Algorithm 3.1 with $k_1 = l_1 = 1$ is denoted by EKS-SYL. Hence, $n = m$ in EKS-SYL. For the purpose of comparison, we also present the test results obtained by the Sylvester global Arnoldi method (denoted by G-Arnoldi) proposed in [3] and the low-rank alternating direction implicit method (denoted by LR-ADI) proposed in [8]. In the following examples, we compare the numerical behavior of these methods with respect to the number of iterations (ITs), the CPU time (in seconds) and the relative residuals (RES). Here the relative residuals are defined by

$$\text{RES} = \frac{\|AX_n + X_nB^T + CD^T\|_F}{\|CD^T\|_F},$$

where X_n denotes the nth iterate of EKS-SYL, G-Arnoldi, or LR-ADI. The entries of the matrices C and D in all the examples are random values uniformly distributed on $[0,1]$ with $s = 2$.

The stopping criterion for three methods is

$$\text{RES} < 10^{-10}.$$

The relative residuals for EKS-SYL can be obtained by using the computationally inexpensive relation (10). However, since such a similar relation does not exist for G-Arnoldi or LR-ADI, these two methods need more cost to calculate the relative residuals. Hence, for the sake of fairness, this cost for computing the relative residuals will not be included in the total CPU time of these methods.

All numerical experiments were run on an Intel Pentiem Dual E2160 with CPU 1.8GHz and RAM 1GB under the Window XP operating system using Matlab 7.1 with the machine precision 2.22×10^{-16}.

TABLE I
Computational Results for Example 1

Method	ITs	CPU	RES
EKS-SYL	28	4.7031	3.5639e-011
G-Arnoldi	100	40.6875	0.1792
LR-ADI	59	27.3750	5.8921e-011

A. Example 1

For the first experiment, we use the test matrix obtained from the finite difference discretization of the following two-dimensional partial differential operator

$$\begin{aligned} \mathcal{L}u = &-\frac{\partial}{\partial x}(e^{-xy}\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(e^{xy}\frac{\partial u}{\partial y}) \\ &+ \beta(x+y)\frac{\partial u}{\partial x} + \beta\frac{\partial}{\partial x}((x+y)u) \\ &+ \gamma(x+y)\frac{\partial u}{\partial y} + \gamma\frac{\partial}{\partial y}((x+y)u) + \frac{1}{1+x+y}u \end{aligned}$$

on the unit square $(0,1) \times (0,1)$ with Dirichlet boundary conditions, where β, γ are parameters used to control the degree of nonormality of the matrices generated. We discretize the operator using central differences to obtain an 2961×2961 nonsymmetric matrix PDE2961 of NEP Collection [2].

Let A be negative PDE2961 of Set NEP and set $B^T = A$. Note that since A is nonsymmetric at this moment, (1) is not a Lyapunov equation.

Table I indicates that EKS-SYL and LR-ADI can converge to the solution of Example 1 while G-Arnoldi does not converge after 100 iteration steps. According to the number of iterations and the CPU time, EKS-SYL works better than LR-ADI for this example. For a plot of the convergence, see Figure 1.

B. Example 2

For the second experiment, we consider a model of heat flow with convection in the domain $\Omega = (0,1)^2$ [33]. The

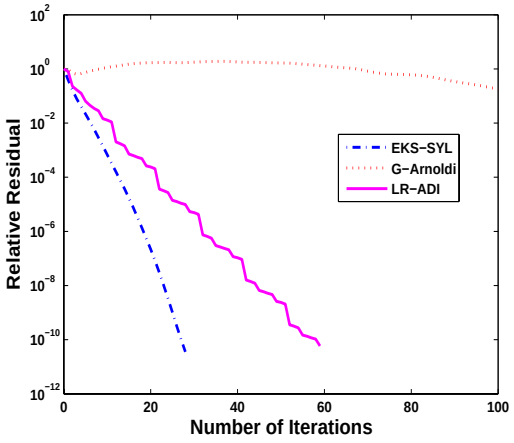

Fig. 1. Comparison for Example 1.

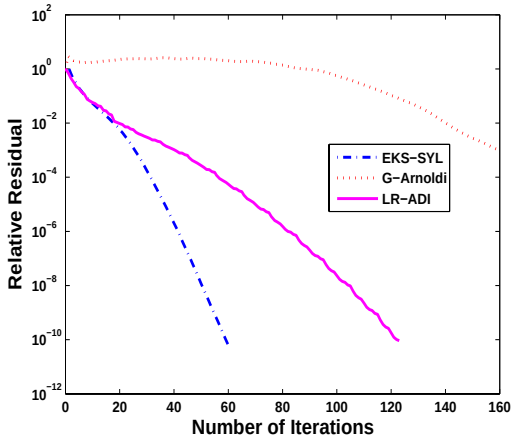

Fig. 2. Comparison for Example 2.

TABLE II
COMPUTATIONAL RESULTS FOR EXAMPLE 2

Method	ITs	CPU	RES
EKS-SYL	60	20.3594	6.2680e-011
G-Arnoldi	160	83.3594	9.6014e-004
LR-ADI	123	53.5625	9.2183e-011

associated parabolic equation is given by

$$\dot{\mathbf{x}} = \Delta\mathbf{x} - \mathbf{f}_1(\xi)\frac{\partial x}{\partial \xi_1} - \mathbf{f}_2(\xi)\frac{\partial x}{\partial \xi_2} + \mathbf{b}(\xi)u(\tau)$$

with $\mathbf{x} = \mathbf{x}(\xi, \tau)$, $\xi = [\xi_1 \quad \xi_2]^T \in \Omega$, $\tau \in [0, \infty]$. The coefficient functions in the convection term are defined as $\mathbf{f}_1(\xi) = 10\xi_1$ and $\mathbf{f}_2(\xi) = 1000\xi_1$. The differential equation is discretized by centered finite difference discretization using a grid with equidistant spacing and 50×50 grid points. The resulting stiffness matrix $A \in \mathbb{R}^{2500\times 2500}$ is sparse and stable and we let $B = A^T$. Such Sylvester equations arise in the problem of balanced reduced order systems based on the computation and approximation of the cross gramian. The cross gramian is the solution of such Sylvester equations [6], [42].

The computational results were reported in Table II. We note that the CPU time of the EKS-SYL method is 20.3594 while the CPU time of the LR-ADI method is 53.5625. The G-Arnoldi method does not converge after 160 iterations. It is clearly seen that the performance of EKS-SYL is much better than the other methods for this example. The convergence curves are depicted in Figure 2.

V. CONCLUSIONS

We propose a new method for solving the Sylvester matrix equations. This method is based on modified Arnoldi algorithm. We derive some results, such as the norm of the residual and perturbation results. The numerical tests presented in this paper show the effectiveness of the proposed method.

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REFERENCES

[1] A. C. Antoulas, *Approximation of Large-scale Dynamical Systems*, SIAM, Philadelphia, PA, 2005.
[2] Z. Bai, D. Day, J. Demmel and J. Dongarra, *A Test Matrix Collection for Non-Hermitian Eigenvalue Problems (Release 1.0)*, Available at http://www.cs.ucdavis.edu/ bai/NEP/document/collection.ps, 1996.
[3] L. Bao, Y. Lin and Y. Wei, *A new projection method for solving large Sylvester equations*, Appl. Numer. Math., 57(2007), 521-532.
[4] R. H. Bartels and G. W. Stewart, *Algorithm 432: Solution of the matrix equation $AX + XB = C$*, Comm. ACM, 15(1972), 820-826.
[5] U. Baur and P. Benner, *Factorized solution of Lyapunov equations based on hierarchical matrix arithmetic*, Computing, 78(2006), 211–234.
[6] U. Baur and P. Benner, *Cross-gramian based model reduction for data-sparse systems*, Tech. rep., Fakultät für mathematik, TU Chemnitz, 09107 Chemnitz, FRG, submitted for publication, 2007.
[7] P. Benner, E. S. Quintana-Ortí and G. Quintana-Ortí, *Solving stable Sylvester equations via rational iterative schemes*, J. Sci. Comput., 28(2006), 51-83.
[8] P. Benner, R.-C. Li, and N. Truhar, *On the ADI method for Sylvester equations*, preprint.
[9] R. Bhatia and P. Rosenthal, *How and why to solve the operator equation $AX - XB = Y$*, Bull. London Math. Soc., 29(1997), 1-21.
[10] A. Bouhamidi, K. Jbilou, *Sylvester Tikhonov-regularization methods in image restoration*, J. Comput. Appl. Math., 206(2007), 86-98.
[11] B. N. Datta, *Numerical Methods for Linear Control Systems: Design and Analysis*, Elsevier Academic Press, Amsterdam, 2004.

[12] J. W. Demmel, *Applied Numerical Linear Algebra*, SIAM, Philadelphia, 1997.
[13] V. Druskin and L. Knizhnerman, *Extended Krylov subspaces: Approximation of the matrix square root and related functions*, SIAM J. Matrix Anal. Appl., 19(1998), 755-771.
[14] E. de Souza and S. P. Bhattacharyya, *Contollability, observability and solution of* $AX - XB = C$, Linear Algebra Appl., 39(1981), 167-188.
[15] S. C. Eisenstat, H. C. Elman, and M. H. Schultz, *Variational iterative methods for nonsymmetric systems of linear equations*, SIAM J. Numer. Anal., 20(1983), 345-357.
[16] A. El Guennouni, K. Jbilou and J. Riquet, *Krylov subspace methods for solving large Sylvester equations*, Numer. Algorithms, 29(2002), 75-96.
[17] G. H. Golub, S. Nash and C. Van Loan, *A Hessenberg-Schur method for the problem* $AX + XB = C$, IEEE Trans. Automat. Control, 24(1979), 909-913.
[18] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd edition, John Hopkins University Press, Baltimore, 1996.
[19] S. Gugercin, D. C. Sorensen, and A. C. Antoulas, *A modified low-rank Smith method for large-scale Lyapunov equations*, Numer. Algorithms, 32(2003), 27-55.
[20] S. J. Hammarling, *Numerical solution of the stable non-negative definite Lyapunov equation*, IMA J. Numer. Anal., 2(1982), 303-323.
[21] J. Z. Hearon, *Nonsingular solutions of* $TA - BT = C$, Linear Algebra Appl., 16(1997), 57-63.
[22] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, SIAM, Second Edition, 2002.
[23] D. Y. Hu and L. Reichel, *Krylov-subspace methods for the Sylvester equation*, Linear Algebra Appl., 172(1992), 283-313.
[24] I. M. Jaimoukha and E. M. Kasenally, *Krylov subspace methods for solving large Lyapunov equations*, SIAM J. Numer. Anal., 31(1994), 227-251.
[25] K. Jbilou, A. Messaoudi and H. Sadok, *Global FOM and GMRES algorithms for matrix equations*, Appl. Numer. Math., 31(1999), 49-63.
[26] K. Jbilou and A. J. Riquet, *Projection methods for large Lyapunov matrix equations*, Linear Algebra Appl., 415(2006), 344-358.
[27] D. Kressner, *Block variants of Hammarling's method for solving Lyapunov equations*, ACM Trans. Math. Software, 34(2008), 1–15.
[28] A. J. Laub, M. T. Heath, C. C. Paige and R. C. Ward, *Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms*, IEEE Trans. Automat. Control, 32(1987), 115-122.
[29] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd edition, Academic Press, Orlando, 1985.
[30] J.-R. Li and J. White, *Low rank solution of Lyapunov equations*, SIAM J. Matrix Anal. Appl., 24(2002), 260-280.
[31] A. Lu and E. Wachspress, *Solution of Lyapunov equations by alternating direction implicit iteration*, Comput. Math. Appl., 21(1991), 43-58.
[32] The MathWorks, Inc., *MATLAB 7*, September 2004.
[33] T. Penzl, *A cyclic low-rank Smith method for large sparse Lyapunov equations*, SIAM J. Sci. Comput., 21(2000), 1401-1418.
[34] M. Robbe and M. Sadkane, *A convergence analysis of GMRES and FOM methods for Sylvester equations*, Numer. Algorithms, 30(2002), 71-84.
[35] A. Ruhe, *Numerical aspects of Gramm-Schmidt orthogonalization of vectors*, Linear Algebra Appl., 52(1983), 591-601.
[36] Y. Saad and M. H. Schultz, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput., 7(1986), 856-869.
[37] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd edition., SIAM, Philadelphia, 2003.
[38] V. Simoncini, *A new iterative method for solving large-scale Lyapunov matrix equations*, SIAM J. Sci. Comput., 29(2007), 1268-1288.
[39] V. Simoncini and D. B. Szyld, *Theory of inexact Krylov subspace methods and applications to scientific computing*, SIAM J. Sci. Comput., 25(2003), 454-477.
[40] V. Simoncini and D. B. Szyld, *New conditions for non-stagnation of minimal residual methods*, Numer. Math., 109(2008), 477-487.
[41] R. Smith, *Matrix equation* $XA + BX = C$, SIAM J. Appl. Math., 16(1968), pp. 198-201.
[42] D. C. Sorensen and A. C. Antoulas, *The Sylvester equation and approximate balanced reduction*, Linear Algebra Appl., 352(2002), 671-700.
[43] D. C. Sorensen and Y. Zhou, *Direct methods for matrix Sylvester and Lyapunov equations*, J. Appl. Math., 6(2003), 277-303.
[44] E. Wachspress, *Iterative solution of the Lyapunov matrix equation*, Appl. Math. Lett., 107(1988), 87-90.