

# The Strict Stability of Impulsive Stochastic Functional Differential Equations with Markovian Switching

Dezhi Liu Guiyuan Yang Wei Zhang

**Abstract**—Strict stability can present the rate of decay of the solution, so more and more investigators are beginning to study the topic and some results have been obtained. However, there are few results about strict stability of stochastic differential equations. In this paper, using Lyapunov functions and Razumikhin technique, we have gotten some criteria for the strict stability of impulsive stochastic functional differential equations with markovian switching.

**Keywords**—Impulsive; Stochastic functional differential equation; Strict stability; Razumikhin technique.

## I. INTRODUCTION

**I**N many applications, one often assumes that the future states of the systems considered are independent of the past states and decided solely by the present states. However, in most cases, the future states are concerned with the past states. Functional differential equations give a mathematical formulation of such systems [1]. Since functional differential equations play an important role in a large number of fields, qualitative properties of functional differential equations are interesting to many researchers. Satisfyingly, some results are derived ([11]-[13] and references therein). Especially, the conditions which ensure the stability of the equilibrium solution of functional differential equations are provided ([14],[15] and references therein).

Impulsive effects exist in many evolution processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics (see [4] and reference therein). However, in addition to impulsive effects, stochastic effects likewise exist in real systems. It is well known that a lot of dynamical systems have variable structures subject to stochastic abrupt changes. The abrupt changes in their structure and parameters usually result from phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. ([6]-[7] and references therein).

As we know, most of the definitions of stability are one-sided estimation. However, strict stability is an strict concept which presents much information about the rate of the decay of the solutions. It not only give an estimation of upper bound for the rate at which solutions approach to the trivial

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solution but also of lower bound [3]. V.Lakshmikantham and S.Leea [2] introduced the concepts of strict stability in tube-like domain. V.Lakshmikantham [3] employed strict stability to prove uniformly asymptotical stability result. Zhang [5] investigated the strict stability of impulsive functional differential equations. Motivated by the above description, taking into account of impulsive effects and stochastic effects, we want to consider the strict stability of impulsive stochastic functional differential equations (ISFDE). To the best of our knowledge to date, there are few literatures with respect to strict stability of ISFDE. Therefore, we want to close he gap.

## II. IMPULSIVE STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MARKOVIAN SWITCHING

Let  $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P}\}$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (i.e. the filtration is continuous on the right and contains all  $\mathbf{P}$ -zero sets). Let  $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$  be an  $m$ -dimensional Brownian motion defined on the probability space. Let  $PC(I; R^n)$  denote the family of functions  $\varphi$  from  $I$  to  $R^n$  with  $\varphi(t^+) = \varphi(t)$  for  $t \in I$  and  $\varphi(t^-)$  existing for  $t \in [0, \infty]$  as well as  $\varphi(t^+) = \varphi(t)$  for all but points  $t_k \in [0, \infty]$ . If  $A$  is a vector or matrix, its trace norm is denoted by  $|A| = \sqrt{\text{trace}(A^T A)}$ , while its operator norm is denoted by  $\|A\| = \sup\{|Ax| : x = 1\}$ .  $PC_{\mathcal{F}_0}^b(I; R^n)$  denotes the family of all bounded,  $\mathcal{F}_0$ -measurable,  $PC([- \tau, 0]; R^n)$ -valued random variable. Let  $PC_{\mathcal{F}_0}^b(\delta) = \{\varphi | \varphi \in PC([- \tau, 0]; R^n) \text{ and } E\|\varphi\| \leq \delta\}$ ,  $PC_{\mathcal{F}_0}^b(\theta) = \{\varphi | \varphi \in PC([- \tau, 0]; R^n) \text{ and } E\|\varphi\|_1 > \theta\}$ , where the norm  $\|\varphi\| = \sup_{-\tau \leq s \leq 0} |\varphi(s)|$ ,  $\|\varphi\|_1 = \inf_{-\tau \leq s \leq 0} |\varphi(s)|$ , respectively, and  $|\cdot|$  is the Euclidean norm in  $R^n$ , i.e.,  $|x| = \sqrt{x^T x}$  ( $x \in R^n$ ).

Let  $\{r(t), t \in R_+ = [0, \infty)\}$  be a right continuous Markov chain on the probability space  $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P}\}$  taking values in a finite state space  $S = \{1, 2, \dots, N\}$  with generator  $\Gamma = (\gamma_{ij})_{N \times N}$  given by

$$P(r(t+\Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij} \Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii} \Delta + o(\Delta), & \text{if } i = j. \end{cases}$$

where  $\Delta > 0$ . Here  $\gamma_{ij} \geq 0$  is the transition rate from  $i$  to  $j$ , if  $i \neq j$ ; while  $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$ . We assume that Markov chain  $r(\cdot)$  is independent of the Brownian motion  $B(\cdot)$ . It is known that almost every sample path of  $r(t)$  is right continuous step function with a finite number of simple jumps in any finite

sub-interval of  $R_+$ .

In this article, we consider a class of impulsive stochastic functional differential equation in which the state variables on the impulsive are related to the time delay

$$\begin{cases} dx(t) = f(t, x_t, r(t))dt + g(t, x_t, r(t))dW(t), \\ t \geq t_0, \quad t \neq t_k, \\ x(t_k) = I(t_k, x(t_k^-)) + J(t_k - \tau, x(t_k^- - \tau)), \\ k = 1, 2, \dots \quad t = t_k, \end{cases} \quad (1)$$

on the entire  $t \geq 0$  with initial data  $x_0 = \varphi \in PC_{\mathcal{F}_0}^b(\delta)$ , where  $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$  which is regarded as a  $PC([-\tau, 0]; R^n)$ -valued stochastic process. Furthermore, we assume that the moments of impulsive time  $t_k$  satisfy  $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$ ,  $\lim_{k \rightarrow \infty} t_k = \infty$ . Under some assumptions, we assume that there exists a unique stochastic process which satisfies Eq.(1) and all solutions which are denoted by  $x(t; t_0, \varphi)$  on the entire  $t \geq -\tau$  are continuous on the right and limitable on the left. We also assume that  $f(t, 0) \equiv 0$  and  $g(t, 0) \equiv 0$ , together with  $I(t_k, 0) \equiv 0, J(t_k - \tau, 0) \equiv 0$ . Therefore, Eq.(1) admits an equilibrium solution  $x(t) \equiv 0$ .

Presently, we introduce and recall some notations and definitions. Let  $K_1$  denote the family of strictly increasing continuous convex functions  $\omega_1 : R^+ \rightarrow R^+$  such that  $\omega_1(0) = 0$ ;  $K_2$  denote the family of strictly increasing continuous concave functions  $\omega_2 : R^+ \rightarrow R^+$  such that  $\omega_2(0) = 0$ . Let  $K$  denote the family of increasing continuous functions  $\varphi : R^+ \rightarrow R^+$  such that  $\varphi(s) < s$  for  $s > 0$ .

**Definition 2.1** The function  $V : [t_0, \infty) \times R^n \times S \rightarrow R^+$  belongs to the family  $C^{1,2}$ , if

- (1) the function  $V(\cdot)$  is continuously once differential in  $t$  and twice differentiable in  $x$  on each of the sets  $[t_{k-1}, t_k) \times R^n \times S (k = 1, 2, \dots)$  and for all  $t \geq t_0$ ,  $V(t, 0, 0) \equiv 0$ .
- (2)  $V(t, x, i)$  is locally Lipschitzian in  $x$ .
- (3) for each  $k = 1, 2, \dots$ , there exist finite limits

$$\begin{aligned} \lim_{(t,y) \rightarrow (t_k^-, x)} V(t, y, i) &= V(t_k^-, y, i), \\ \lim_{(t,y) \rightarrow (t_k^+, x)} V(t, y, i) &= V(t_k^+, y, i), \\ \text{and } V(t_k^+, x, i) &= (t_k, x, i), \end{aligned}$$

where  $i \in S$ .

**Definition 2.2** The equilibrium solution of Eq.(1) is called to be strictly stable, if for any  $\varepsilon_1 > 0$ , there exists a  $\delta_1 = \delta_1(t_0, \varepsilon_1) > 0$  such that  $\varphi \in PC_{\mathcal{F}_0}^b(\delta_1)$  implies  $E|x(t; t_0, \varphi)| < \varepsilon_1, t \geq t_0$ , and for every  $0 < \delta_2 \leq \delta_1$ , there exists an  $0 < \varepsilon_2 < \delta_2$  such that

$$\varphi \in PC_{\mathcal{F}_0}^b(\delta_2), \quad \text{means } \varepsilon_2 < E|x(t; t_0, \varphi)|, \quad t \geq t_0.$$

**Definition 2.3** The equilibrium solution of Eq.(1) is called to be strictly uniformly stable, if  $\delta_1, \delta_2$  and  $\varepsilon_2$  is independent of  $t_0$ .

### III. MAIN RESULTS

In this section, we present our main results and complete their proofs.

**Theorem 3.1** Assume that

(H1) There exist functions  $a_1, b_1 \in K_1, V_1 \in C^{1,2}$  such that

$$\begin{aligned} a_1(|x(t)|) &\leq V_1(t, x, i) \leq b_1(|x(t)|), \\ \text{for } (t, x, i) &\in [t_0 - \tau, \infty) \times R^n \times S, \end{aligned}$$

(H2) For any solution  $x(t)$  of Eq.(1),  $EV_1(t+s, x(t+s), r(t+s)) \leq EV_1(t, x(t), r(t))$  with  $s \in [-\tau, 0]$ , implies that  $D^+EV_1(t, x(t), r(t)) < 0$ ;

(H3)

$$\begin{aligned} &V_1(t_k, I(t_k, x(t_k^-)) + J(t_k - \tau, x(t_k^- - \tau)), r(t_k)) \\ &\leq \frac{1+d_k}{2}[V_1(t_k^-, x(t_k^-), r(t_k^-)) \\ &\quad + V_1(t_k^- - \tau, x(t_k^- - \tau), r(t_k^- - \tau))], \end{aligned}$$

where  $d_k \geq 0$  and  $\sum_{k=1}^{\infty} d_k < \infty$ ;

(H4) There are functions  $a_2, b_2 \in K_2, V_2 \in C^{1,2}$  satisfying

$$\begin{aligned} a_2(|x(t)|) &\leq V_2(t, x, i) \leq b_2(|x(t)|), \\ \text{for } (t, x) &\in [t_0 - \tau, \infty) \times R^n; \end{aligned}$$

(H5) For any solution  $x(t)$  of Eq.(1),  $EV_2(t+s, x(t+s), r(t+s)) \geq EV_2(t, x(t), r(t))$  with  $s \in [-\tau, 0]$ , implies that  $D^+EV_2(t, x(t), r(t)) > 0$ .

(H6)

$$\begin{aligned} &V_2(t_k, I(t_k, x(t_k^-)) + J(t_k - \tau, x(t_k^- - \tau)), r(t_k)) \\ &\geq \frac{1-c_k}{2}[V_2(t_k^-, x(t_k^-), r(t_k^-)) \\ &\quad + V_2(t_k^- - \tau, x(t_k^- - \tau), r(t_k^- - \tau))], \end{aligned}$$

where  $0 \leq c_k < 1$  and  $\sum_{k=1}^{\infty} c_k < \infty$ .

Thus the equilibrium solution of Eq.(1) is strictly uniformly stable.

**Proof** For simplicity, we set  $t_0 = 0$ . By virtue of  $a_1, b_1 \in K_1$ , for given  $\varepsilon_1 > 0, M > 0$ , we can find  $\delta_1 > 0$  satisfying  $b_1(\delta_1) < a_1(\varepsilon_1)/M$ . Note that for  $t \in [-\tau, 0]$  and  $\varphi \in PC_{\mathcal{F}_0}^b(\delta_1)$

$$EV_1(t, x(t), i_0) = EV_1(t, \varphi(t), i_0) \leq b_1(\delta_1).$$

In what follows, we claim that inequality

$$EV_1(t, x(t), r(t)) \leq b_1(\delta_1), \quad t \in [0, t_1], \quad (2)$$

is true. However, if the inequality (2) is not true, then there exists  $s_1 \in [0, t_1)$  satisfying

$$EV_1(s_1, x(s_1), r(s_1)) > b_1(\delta_1) \geq EV_1(t_0, x(t_0), i_0).$$

Define

$$s_2 = \inf\{s_1 \in [0, t_1) | EV_1(s_1, x(s_1), r(s_1)) > b_1(\delta_1)\},$$

then

$$\begin{aligned} EV_1(s_2, x(s_2), r(s_2)) &= b_1(\delta_1), \\ EV_1(s_2 + s, x(s_2 + s), r(s_2 + s)) &\leq EV_1(s_2, x(s_2), r(s_2)), \\ s &\in [-\tau, 0], \\ D^+EV_1(s_2, x(s_2), r(s_2)) &\geq 0, \end{aligned}$$

which contradicts the assumption (H2). Therefore inequality (2) must hold. Now we estimate  $EV_1(t, x(t), r(t))$  at the moment of impulsive time  $t_1$ . From (H3) and together with (2), yield that

$$\begin{aligned} &EV_1(t_1, x(t_1), r(t_1)) \\ &= EV_1(t_1, I(t_1, x(t_1^-)) + J(t_1 - \tau, x(\tau_1^- - \tau)), r(t_1)) \\ &\leq \frac{1+d_1}{2}[EV_1(t_1^-, x(t_1^-), r(t_1^-)) \\ &+ EV_1(t_1^- - \tau, x(t_1^- - \tau), r(t_1^- - \tau))] \\ &\leq (1+d_1)b_1(\delta_1). \end{aligned}$$

We assume that, for  $m = 1, 2, \dots, k$ , the following inequalities hold

$$\begin{aligned} &EV_1(t, x(t), r(t)) \leq (1+d_1) \cdots (1+d_{m-1})b_1(\delta_1), \\ &t \in [t_{m-1}, t_m), \\ &EV_1(t_m, x(t_m), r(t_m)) \leq (1+d_1) \cdots (1+d_m)b_1(\delta_1). \end{aligned} \quad (3)$$

For  $m = k + 1$ , we will show that

$$EV_1(t, x(t), r(t)) \leq (1+d_1) \cdots (1+d_k)b_1(\delta_1), \quad t \in [t_k, t_{k+1}), \quad (4)$$

Assume that the inequality (4) is not true. Set

$$\begin{aligned} s_1 &= \inf\{s \in [t_k, t_{k+1}) | EV_1(s, x(s), r(s)) \\ &> (1+d_1) \cdots (1+d_k)b_1(\delta_1)\}, \end{aligned}$$

then

$$\begin{aligned} &EV_1(s_1, x(s_1), r(s_1)) = (1+d_1) \cdots (1+d_k)b_1(\delta_1), \\ &EV_1(s_1 + t, x(s_1 + t), r(s_1 + t)) \\ &\leq (1+d_1) \cdots (1+d_k)b_1(\delta_1), \quad t \in [-\tau, 0], \\ &D^+ EV_1(s_1, x(s_1), r(s_1)) \geq 0, \end{aligned}$$

together with (3), which contradicts the assumption (H2). Therefore, the inequality (4) holds. We estimate  $EV_1(t, x(t), r(t))$  at the moment of impulsive time  $t_{k+1}$ . With the assumption (H3), obtain that

$$\begin{aligned} &EV_1(t_{k+1}, x(t_{k+1}), r(t_{k+1})) \\ &= EV_1(t_{k+1}, I(t_{k+1}, x(t_{k+1}^-)) \\ &+ J(t_{k+1} - \tau, x(t_{k+1}^- - \tau)), r(t_{k+1})) \\ &\leq \frac{1+d_{k+1}}{2}[EV_1(t_{k+1}^-, x(t_{k+1}^-), r(t_{k+1}^-)) \\ &+ EV_1(t_{k+1}^- - \tau, x(t_{k+1}^- - \tau), r(t_{k+1}^- - \tau))] \\ &\leq (1+d_1) \cdots (1+d_k)(1+d_{k+1})b_1(\delta_1). \end{aligned}$$

Consequently, the following inequalities hold for any  $k \geq 0$ ,

$$\begin{aligned} &EV_1(t, x(t), r(t)) \leq (1+d_1) \cdots (1+d_{k-1})b_1(\delta_1), \\ &t \in [t_{k-1}, t_k), \\ &EV_1(t_k, x(t_k), r(t_k)) \leq (1+d_1) \cdots (1+d_k)b_1(\delta_1). \end{aligned} \quad (5)$$

In view of assumption of (H1), for any  $t \geq 0$ , we can choose  $k \geq 0$  such that  $t \in [-\tau, +\infty)$  and show that

$$Ea_1(|x(t)|) \leq EV_1(t, x(t), r(t)) \leq Mb_1(\delta_1) < a_1(\varepsilon_1).$$

Therefore,

$$E|x(t; 0, \varphi)| < \varepsilon_1.$$

Now, let  $0 < \delta_2 \leq \delta_1$  and choose  $0 < \varepsilon_2 \leq \delta_2$ , such that  $a_2(\delta_2) > b_2(\varepsilon_2)/N$ . Note that for  $t \in [-\tau, 0]$  and  $\varphi \in PC_{\mathcal{F}_0}^b(\delta_2)$

$$EV_2(t, x(t), i_0) = EV_2(t, \varphi(t), i_0) \geq a_2(\delta_2).$$

In what follows, we claim that inequality

$$EV_2(t, x(t), r(t)) \geq a_2(\delta_2), \quad t \in [0, t_1), \quad (6)$$

is true. However, if the inequality (6) is not true, then there exists  $s_1 \in [0, t_1)$  satisfying

$$EV_2(s_1, x(s_1), r(s_1)) < a_2(\delta_2) \leq EV_2(t_0, x(t_0), i_0).$$

Define

$$s_2 = \inf\{s_1 \in [0, t_1) | EV_2(s_1, x(s_1), r(s_1)) < a_2(\delta_2)\},$$

then

$$\begin{aligned} &EV_2(s_2, x(s_2), r(s_2)) = a_2(\delta_2), \\ &EV_2(s_2 + s, x(s_2 + s), r(s_2 + s)) \geq EV_2(s_2, x(s_2), r(s_2)), \\ &s \in [-\tau, 0], \\ &D^+ EV_2(s_2, x(s_2), r(s_2)) \leq 0, \end{aligned}$$

which contradicts the assumption (H5). Therefore inequality (6) must hold. Now we estimate  $EV_2(t, x(t), r(t))$  at the moment of impulsive time  $t_1$ . From (H6) and together with (6), yield that

$$\begin{aligned} &EV_2(t_1, x(t_1), r(t_1)) \\ &= EV_2(t_1, I(t_1, x(t_1^-)) + J(t_1 - \tau, x(\tau_1^- - \tau)), r(t_1)) \\ &\geq \frac{1-c_1}{2}[EV_2(t_1^-, x(t_1^-), r(t_1^-)) \\ &+ EV_2(t_1^- - \tau, x(t_1^- - \tau), r(t_1^- - \tau))] \\ &\geq (1-c_1)a_2(\delta_2). \end{aligned}$$

We assume that, for  $m = 1, 2, \dots, k$ , the following inequalities hold

$$\begin{aligned} &EV_2(t, x(t), r(t)) \geq (1-c_1) \cdots (1-c_{m-1})a_2(\delta_2), \\ &t \in [t_{m-1}, t_m), \\ &EV_2(t_m, x(t_m), r(t_m)) \geq (1-c_1) \cdots (1-c_m)a_2(\delta_2). \end{aligned} \quad (7)$$

For  $m = k + 1$ , we will show that

$$EV_2(t, x(t), r(t)) \geq (1-c_1) \cdots (1-c_k)a_2(\delta_2), \quad t \in [t_k, t_{k+1}), \quad (8)$$

Assume that the inequality (8) is not true. Set

$$\begin{aligned} s_1 &= \inf\{s \in [t_k, t_{k+1}) | EV(s, x(s), r(s)) \\ &< (1-c_1) \cdots (1-c_k)a_2(\delta_2)\}, \end{aligned}$$

then

$$\begin{aligned} &EV_2(s_1, x(s_1), r(s_1)) = (1-c_1) \cdots (1-c_k)a_2(\delta_2), \\ &EV_2(s_1 + t, x(s_1 + t), r(s_1 + t)) \geq (1-c_1) \cdots (1-c_k)a_2(\delta_2), \\ &t \in [-\tau, 0], \\ &D^+ EV_2(s_1, x(s_1), r(s_1)) \leq 0, \end{aligned}$$

together with (7), which contradicts the assumption (H5). Therefore, the inequality (8) holds. We estimate

$EV_2(t, x(t), r(t))$  at the moment of impulsive time  $t_{k+1}$ . With the assumption (H6), obtain that

$$\begin{aligned} & EV_2(t_{k+1}, x(t_{k+1}), r(t_{k+1})) \\ &= EV_2(t_{k+1}, I(t_{k+1}, x(t_{k+1}^-)), r(t_{k+1})) \\ &+ J(t_{k+1} - \tau, x(t_{k+1}^- - \tau), r(t_{k+1})) \\ &\geq \frac{1 - c_{k+1}}{2} [EV_2(t_{k+1}, x(t_{k+1}^-), r(t_{k+1}^-))] \\ &+ EV_2(t_{k+1} - \tau, x(t_{k+1}^- - \tau), r(t_{k+1}^- - \tau)) \\ &\geq (1 - c_1) \cdots (1 - c_k)(1 - c_{k+1})a_2(\delta_2). \end{aligned}$$

Consequently, the following inequalities hold for any  $k \geq 0$ ,

$$\begin{aligned} EV_2(t, x(t), r(t)) &\geq (1 - c_1) \cdots (1 - c_{k-1})a_2(\delta_2), \\ t &\in [t_{k-1}, t_k], \end{aligned} \quad (9)$$

$$EV_2(t_k, x(t_k), r(t_k)) \geq (1 - c_1) \cdots (1 - c_k)a_2(\delta_2).$$

In view of assumption of (H4), for any  $t \geq 0$ , we can choose  $k \geq 0$  such that  $t \in [t_k, t_{k+1})$  and show that

$$Eb_2(|x(t)|) \geq EV_2(t, x(t), r(t)) \geq Na_2(\delta_2) > b_2(\varepsilon_2).$$

By virtue of  $a_2, b_2 \in K_2$ , therefore,

$$E|x(t; 0, \varphi)| > \varepsilon_2.$$

Thus, the zero solution of (1) is strictly uniformly stable, and the proof is completed.

**Theorem 3.2** Assume that

(A1) There exist functions  $a_1, b_1 \in K_1$ ,  $V_1 \in C^{1,2}$  such that

$$\begin{aligned} a_1(|x(t)|) &\leq V_1(t, x, i) \leq b_1(|x(t)|), \\ \text{for } (t, x) &\in [t_0 - \tau, \infty) \times R^n \times S; \end{aligned}$$

(A2) For any solution  $x(t)$  of Eq.(1), there exists a  $\omega_1 \in K$ , such that  $EV_1(t + s, x(t + s), r(t + s)) \leq \omega_1^{-1}(EV_1(t, x(t), r(t)))$  with  $s \in [-\tau, 0]$ , implies that  $D^+EV_1(t, x(t), r(t)) < \phi_1(t)\psi_1(EV_1(t, x(t), r(t)))$ ,  $\phi_1, \psi_1 : C[R^+, R^+]$ , locally integrable, and for all  $k \in Z^+$ ,  $V_1(t_k, x(t_k), r(t_k)) \leq \omega_1(V_1(t_k^-, x(t_k^-), r(t_k^-)))$ ;

(A3) There exists a constant  $C_1 > 0$  such that

$$\int_{t_{k-1}}^{t_k} \phi_1(s)ds < C_1, \quad k \in Z^+;$$

Also for any  $u > 0$ ,

$$\int_u^{\omega_1^{-1}(u)} \frac{ds}{\psi_1(s)} ds \geq C_1;$$

(A4) There are functions  $a_2, b_2 \in K_2$ ,  $V_2 \in C^{2,1}$  satisfying

$$\begin{aligned} a_2(|x(t)|) &\leq V_2(t, x, i) \leq b_2(|x(t)|), \\ \text{for } (t, x) &\in [t_0 - \tau, \infty) \times R^n \times S; \end{aligned}$$

(A5) For any solution  $x(t)$  of Eq.(1), there exists a  $\omega_2 \in K$ , such that  $EV_2(t + s, x(t + s), r(t + s)) \geq \omega_2(EV_2(t, x(t), r(t)))$  with  $s \in [-\tau, 0]$ , implies that  $D^+EV_2(t, x(t), r(t)) < \phi_2(t)\psi_2(EV_2(t, x(t), r(t)))$ ,  $\phi_2, \psi_2 : C[R^+, R^+]$ , locally integrable, and for all  $k \in Z^+$ ,  $V_2(t_k, x(t_k), r(t_k)) \geq \omega_2^{-1}(V_2(t_k^-, x(t_k^-), r(t_k^-)))$ ;

(A6) There exists a constant  $C_2 > 0$  such that

$$\int_{t_{k-1}}^{t_k} \phi_2(s)ds < C_2, \quad k \in Z^+;$$

Also for any  $u > 0$ ,

$$\int_u^{\omega_2(u)} \frac{ds}{\psi_2(s)} ds \geq C_2.$$

Thus the equilibrium solution of Eq.(1) is strictly uniformly stable.

**Proof** For simplicity, we set  $t_0 = 0$ . For given  $0 < \varepsilon_1$ . Choose  $\delta_1 = \delta_1(\varepsilon_1) > 0$ , such that  $\omega_1^{-1}(b_1(\delta_1)) < a_1(\varepsilon_1)$ . Note that for  $t \in [-\tau, 0]$  and  $\varphi \in PC_{\mathcal{F}_0}^b(\delta_1)$ ,

$$EV_1(t, x(t), i_0) = EV_1(t, \varphi(t), i_0) \leq \omega_1^{-1}(b_1(\delta_1)).$$

In what follows, we claim that inequality

$$EV_1(t, x(t), r(t)) \leq \omega_1^{-1}(b_1(\delta_1)), \quad t \in [0, t_1], \quad (10)$$

is true. However, if the inequality (10) is not true, then there exists  $s \in [0, t_1)$  satisfying

$$EV_1(s, x(s), r(s)) > \omega_1^{-1}(b_1(\delta_1)) > b_1(\delta_1) \geq EV_1(t_0, x(t_0), i_0).$$

Define

$$s_1 = \inf\{s \in [0, t_1] | EV_1(s, x(s), r(s)) > \omega_1^{-1}(b_1(\delta_1))\},$$

then

$$\begin{aligned} EV_1(s_1, x(s_1), r(s_1)) &= \omega_1^{-1}(b_1(\delta_1)), \\ EV_1(t, x(t), r(t)) &\leq \omega_1^{-1}(b_1(\delta_1)), \quad t \in [t_0 - \tau, s_1], \end{aligned}$$

and also, there exists an  $s_2 \in [t_0 - \tau, s_1)$ , such that

$$\begin{aligned} EV_1(s_2, x(s_2), r(s_2)) &= b_1(\delta_1), \\ EV_1(t, x(t), r(t)) &\geq b_1(\delta_1), \quad t \in [s_2, s_1], \end{aligned}$$

Therefore, for  $t \in [s_2, s_1]$  and  $-\tau \leq s \leq 0$ , we have

$$\begin{aligned} EV_1(t + s, x(t + s), r(t + s)) &\leq \omega_1^{-1}(b_1(\delta_1)) \\ &\leq \omega_1^{-1}(EV_1(t, x(t), r(t))) \end{aligned}$$

In view of condition(A2), we get

$$\begin{aligned} D^+EV_1(t, x(t), r(t)) & \\ &< \phi_1(t)\psi_1(EV_1(t, x(t), r(t))), \quad t \in [s_2, s_1]. \end{aligned} \quad (11)$$

And integrate the inequality (11) over  $[s_2, s_1]$ , we have by condition (A3),

$$\int_{V_1(s_2, x(s_2), r(s_2))}^{V_1(s_1, x(s_1), r(s_1))} \frac{du}{\psi_1(u)} \leq \int_{s_2}^{s_1} \phi_1(s)ds \leq \int_0^{t_1} \phi_1(s)ds < C_1.$$

On the other hand,

$$\int_{V_1(s_2, x(s_2), r(s_2))}^{V_1(s_1, x(s_1), r(s_1))} \frac{du}{\psi_1(u)} = \int_{b_1(\delta_1)}^{\omega_1^{-1}(b_1(\delta_1))} \frac{du}{\psi_1(u)} \geq C_1,$$

which is a contradiction. So inequality (10) holds. Now, we estimate  $EV_1(t, x(t), r(t))$  at the moment of impulsive time  $t_1$ . With the assumption (A2), obtain that

$$EV_1(t_1, x(t_1), r(t_1)) \leq \omega_1(EV_1(t_1^-, x(t_1^-), r(t_1^-))) \leq b_1(\delta_1).$$

We assume that, for  $m = 1, 2, \dots, k$ , the following inequalities hold

$$\begin{aligned} EV_1(t, x(t), r(t)) &\leq \omega_1^{-1}(b_1(\delta_1)), \quad t \in [t_{m-1}, t_m], \\ EV_1(t_m, x(t_m), r(t_m)) &\leq b_1(\delta_1) \leq \omega_1^{-1}(b_1(\delta_1)). \end{aligned} \quad (12)$$

For  $m = k + 1$ , we will show that

$$EV_1(t, x(t), r(t)) \leq \omega_1^{-1}(b_1(\delta_1)), \quad t \in [t_k, t_{k+1}], \quad (13)$$

In view of the similar proof for above, then the inequality (13) holds. Now, we estimate  $EV_1(t, x(t), r(t))$  at the moment of impulsive time  $t_{k+1}$ . With the assumption (A2), obtain that

$$\begin{aligned} EV_1(t_{k+1}, x(t_{k+1}), r(t_{k+1})) \\ \leq \omega_1(EV_1(t_{k+1}^-, x(t_{k+1}^-), r(t_{k+1}^-))) \\ \leq b_1(\delta_1) \leq \omega_1^{-1}(b_1(\delta_1)). \end{aligned}$$

In view of assumption of (A1), for any  $t \geq 0$ , we can choose  $k \geq 0$  such that  $t \in [-\tau, +\infty)$  and show that

$$Ea_1(|x(t)|) \leq EV_1(t, x(t), r(t)) \leq \omega_1^{-1}(b_1(\delta_1)) < a_1(\varepsilon_1).$$

Therefore,

$$E|x(t; 0, \varphi)| < \varepsilon_1.$$

Now, let  $0 < \delta_2 \leq \delta_1$ . Choose  $0 < \varepsilon_2 \leq \delta_2$ , such that  $b_2(\varepsilon_2) < \omega_2(a_2(\delta_2))$ . Note that for  $t \in [-\tau, 0]$  and  $\varphi \in PC_{\mathcal{F}_0}^b(\delta_2)$ ,

$$EV_2(t, x(t), i_0) = EV_2(t, \varphi(t), i_0) \geq \omega_2(a_2(\delta_2)).$$

In what follows, we claim that inequality

$$EV_2(t, x(t), r(t)) \geq \omega_2(a_2(\delta_2)), \quad t \in [0, t_1], \quad (14)$$

is true. However, if the inequality (14) is not true, then there exists  $r' \in [0, t_1]$  satisfying

$$\begin{aligned} EV_2(r', x(r'), r(r')) &< \omega_2(a_2(\delta_2)) \\ &< a_2(\delta_2) \leq EV_2(t_0, x(t_0), i_0). \end{aligned}$$

Define

$$r_1 = \inf\{r' \in [0, t_1] | EV_2(r', x(r'), r(r')) < \omega_2(a_2(\delta_2))\},$$

then

$$\begin{aligned} EV_2(r_1, x(r_1), r(r_1)) &= \omega_2(a_2(\delta_2)), \\ EV_2(t, x(t), r(t)) &\geq \omega_2(a_2(\delta_2)), \quad t \in [t_0 - \tau, r_1], \end{aligned}$$

and also, there exists an  $r_2 \in [t_0 - \tau, r_1]$ , such that

$$\begin{aligned} EV_2(r_2, x(r_2), r(r_2)) &= a_2(\delta_2), \\ EV_2(t, x(t), r(t)) &\leq a_2(\delta_2), \quad t \in [r_2, r_1], \end{aligned}$$

Therefore, for  $t \in [r_2, r_1]$  and  $-\tau \leq s \leq 0$ , we have

$$\begin{aligned} EV_2(t+s, x(t+s), r(t+s)) &\geq \omega_2(a_2(\delta_2)) \\ &\geq \omega_2(EV_2(t, x(t), r(t))) \end{aligned}$$

In view of condition(A5), we get

$$\begin{aligned} D^+ EV_2(t, x(t), r(t)) \\ < \phi_2(t)\psi_2(EV_2(t, x(t), r(t))), \quad t \in [r_2, r_1]. \end{aligned} \quad (15)$$

And integrate the inequality (15) over  $[r_2, r_1]$ , we have by condition (A3),

$$\int_{V_2(r_2, x(r_2), r(r_2))}^{V_2(r_1, x(r_1), r(r_1))} \frac{du}{\psi_2(u)} \leq \int_{r_2}^{r_1} \phi_2(s)ds \leq \int_0^{t_1} \phi_2(s)ds < C_2.$$

On the other hand,

$$\int_{V_2(r_2, x(r_2), r(r_2))}^{V_2(r_1, x(r_1), r(r_1))} \frac{du}{\psi_2(u)} = \int_{a_2(\delta_2)}^{\omega_2(a_2(\delta_2))} \frac{du}{\psi_2(u)} \geq C_2,$$

which is a contradiction. So inequality (14) holds. Now, we estimate  $EV_2(t, x(t), r(t))$  at the moment of impulsive time  $t_1$ . With the assumption (A5), obtain that

$$EV_2(t_1, x(t_1), r(t_1)) \geq \omega_2^{-1}(EV_2(t_1^-, x(t_1^-), r(t_1^-))) \geq a_2(\delta_2).$$

We assume that, for  $m = 1, 2, \dots, k$ , the following inequalities hold

$$\begin{aligned} EV_2(t, x(t), r(t)) &\geq \omega_2(a_2(\delta_2)), \quad t \in [t_{m-1}, t_m], \\ EV_2(t_m, x(t_m), r(t_m)) &\geq a_2(\delta_2) \geq \omega_2(a_2(\delta_2)). \end{aligned} \quad (16)$$

For  $m = k + 1$ , we will show that

$$EV_2(t, x(t), r(t)) \geq \omega_2(a_2(\delta_2)), \quad t \in [t_k, t_{k+1}], \quad (17)$$

In view of the similar proof for above, then the inequality (17) holds. Now, we estimate  $EV_2(t, x(t), r(t))$  at the moment of impulsive time  $t_{k+1}$ . With the assumption (A5), obtain that

$$\begin{aligned} EV_2(t_{k+1}, x(t_{k+1}), r(t_{k+1})) \\ \geq \omega_2^{-1}(EV_2(t_{k+1}^-, x(t_{k+1}^-), r(t_{k+1}^-))) \\ \geq a_2(\delta_2) \geq \omega_2(a_2(\delta_2)). \end{aligned}$$

In view of assumption of (A4), for any  $t \geq 0$ , we can choose  $k \geq 0$  such that  $t \in [-\tau, +\infty)$  and show that

$$Eb_2(|x(t)|) \geq EV_2(t, x(t), r(t)) \geq \omega_2(a_2(\delta_2)) > b_2(\varepsilon_2).$$

Therefore,

$$E|x(t; 0, \varphi)| > \varepsilon_2.$$

Thus, the zero solution of (1) is strictly uniformly stable, and the proof is completed.

**Remark 3.1** In Eq.(1), let  $g(t, x_t, r(t)) \equiv 0$  and  $J(t_k - \tau, x(t_k^- - \tau)) \equiv 0$ , then our result is the result derived in paper [5], that is to say, our result is the generation of paper [5].

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