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New Recursive Representations for the Favard Constants with Application to the Summation of Series

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***Abstract*—**In this study integral form and new recursive formulas for Favard constants and some connected with them numeric and Fourier series are obtained. The method is based on preliminary integration of Fourier series which allows for establishing finite recursive representations for the summation. It is shown that the derived recursive representations are numerically more effective than known representations of the considered objects.

***Keywords*—**Effective summation of series, Favard constants, finite recursive representations, Fourier series.

I. INTRODUCTION

It is well known that the Fourier series and Favard constants have significant theoretical and practical role in many areas [1], [2]. Different methods for their calculations are given, for instance, in [2, Ch. 5.2]. In general these methods are based on the properties of the well-known special functions and constants as gamma function $\Gamma(z)$, generalized Riemann zeta function $\zeta(z,a)$, the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n, the Euler polynomials $E_n(x)$ and the Euler numbers E_n, given by the following expressions [2], [3]:

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,\quad (\operatorname{Re} z>0) \tag{1}$$

$$\zeta(z,a)=\sum_{k=0}^{\infty}(k+a)^{-z}=\left(1/\Gamma(z)\right)\int_0^\infty t^{z-1}e^{-at}/\left(1-e^{-t}\right)dt, \quad (\operatorname{Re} z>1) \tag{2}$$

$$B_n(x):te^{xt}/\left(e^t-1\right)=\sum_{k=0}^{\infty}\left(B_k(x)t^k/k!\right),\quad (|t|<2\pi)$$
$$B_n=B_n(0),\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k} \tag{3}$$

$$E_n(x):2e^{xt}/\left(e^t+1\right)=\sum_{k=0}^{\infty}E_k(x)t^k/k!,\quad (|t|<\pi);$$
$$\frac{1}{\operatorname{ch}(t)}=\sum_{k=0}^{\infty}E_kt^k/k!,\quad (|t|<\frac{\pi}{2}) \tag{4}$$

Further on there will be investigated some properties and obtained applications of the Favard constants K_r [1], [4], [5]:

$$K_r=(4/\pi)\sum_{\nu=0}^{\infty}\left((-1)^{\nu(r+1)}(2\nu+1)^{-r-1}\right),\quad (r=0,1,2,...) \tag{5}$$

These constants find wide applications in establishing many exact and asymptotic results on the approximation of functions [1], [5]-[8], including approximation by Euler splines and other type of splines [9]-[11]. The Favard constants play also an important role in estimating optimal quadrature and cubature formulas, calculation of singular integrals, differential, integro-differential and integral equations [12]-[16], and in other areas.

Nevertheless widely used, as a whole, the Favard constants have not been investigated well enough [17], except for some particular cases.

For further theory let us consider the following notations:

$$D_n(x)=x^n/n! \tag{6}$$

$$T_r=\sum_{n=1}^{\infty}n^{-r},\ (r=2,3,...);\quad Q_r=\sum_{n=1}^{\infty}(-1)^nn^{-r},\ (r=1,2,3,...) \tag{7}$$

It is clear that T_r is a particular case of (2).

The main purpose of this study is to establish some new integral representations and recursive formulas for the above stated objects and some trigonometric series, which can be used for various aims.

II. FINITE RECURSIVE REPRESENTATIONS FOR FAVARD CONSTANTS K_r

One of the main results is the following

Theorem 1. For the Favard constants K_r the following recursive finite representations hold:

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$$K_{2s+1}=\left[(-1)^s D_{2s+1}(\pi)+\sum_{p=1}^{s}K_{2s+1-2p}\frac{(-1)^{p-1}\pi^{2p}}{(2p)!}\right]/2 \quad (8)$$

$$K_{2s}=(-1)^s D_{2s}(\pi/2)+\sum_{p=1}^{s}K_{2s+1-2p}\frac{(-1)^{p-1}}{(2p-1)!}(\pi/2)^{2p-1} \quad (9)$$

$$K_0=1,\quad K_1=\pi/2,\quad (s=1,2,3,...)$$

Proof. It is based on the method of induction and preliminary integration of appropriate Fourier series. Consider the well-known expansion [2, ch. 5]:

$$(4/\pi)\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{2\nu+1}=1,\quad (0<x<\pi) \quad (10)$$

For $x=\pi/2$ one have $K_0=1$. By integration of both sides of (10) in $[0,x]$ it follows

$$-(4/\pi)\sum_{\nu=0}^{\infty}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^2}+\frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^2}=x=D_1(x) \quad (11)$$

For $x=\pi$ from (11) one obtain $2K_1=\pi=D_1(\pi)$, and consequently $K_1=D_1(\pi)/2=\pi/2$. The same results for K_0 and K_1 can be achieved starting from the equality [4]

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\pi/2-x/2=\pi/2-D_1(x)/2,\quad (0<x<2\pi) \quad (12)$$

For $x=\pi/2$ we find again $K_0=1$. After integration of the both sides of (12)

$$-\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}+T_2=-x^2/4+\pi x/2=-D_2(x)/2+\pi D_1(x)/2,$$
$$(0\le x\le 2\pi). \quad (13)$$

For $x=\pi$ it is easy to find $K_1=\pi/2$. Next after integration of the both sides of (13)

$$-\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}+T_2x=-D_3(x)/2+\pi D_2(x)/2,\quad (0\le x\le 2\pi) \quad (14)$$

which for $x=\pi$ gives

$$T_2=-(1/2\pi)D_3(\pi)+(1/2)D_2(\pi)=\pi^2/6 \text{ b} \quad (15)$$

Now putting $x=\pi/2$ in the same equality (14) and making a little processing we obtain $K_2=\pi^2/8$.

On the other hand, after integration of both sides of (11) it follows

$$-(4/\pi)\sum_{\nu=0}^{\infty}\frac{\sin[(2\nu+1)x]}{(2\nu+1)^3}+K_1x=D_2(x),\quad (0\le x\le\pi) \quad (16)$$

In (16) for $x=\pi/2$ one get $-K_2+(\pi/2)K_1=D_2(\pi/2)$ and consequently

$$K_2=\pi K_1/2-D_2(\pi/2)=\pi^2/8 \quad (17)$$

For the constant K_3 the integration of both sides of (14) leads to

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^4}-T_4+T_2D_2(x)=-D_4(x)/2+\pi D_3(x)/2,$$
$$(0\le x\le 2\pi) \quad (18)$$

Putting in (18) $x=\pi$, or integrating both sides of (16) it can be found

$$(4/\pi)\sum_{\nu=1}^{\infty}\frac{\cos[(2\nu+1)x]}{(2\nu+1)^4}-\frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{1}{(2\nu+1)^4}+K_1D_2(x)=D_3(x),$$
$$(0\le x\le\pi) \quad (19)$$

For $x=\pi$ in (19) this gives

$$K_3=D_4(\pi)/\pi-3D_3(\pi)/2+\pi D_2(\pi)/2$$
$$=(\pi^2/4)K_1-D_3(\pi)/2=\pi^3/24 \quad (20)$$

Going on the indicated procedure on the base of induction one arrive at the recursive representations (8), (9) which completes the proof.

Remark. The scheme of this proof is valid for the most of the other statements in this study.

In connection with theorem 1 we would like to note another representation of K_r (see, for instance, [18]). It can be written in terms of Lerch transcendent, or as it is shown in [2, ch. 5.1.4] by means of the special functions in (3), (4):

$$K_{2s-1}=2/(\pi(2s)!)(2^{2s}-1)\pi^{2s}|B_{2s}|$$
$$K_{2s}=2/(\pi(2s)!)(\pi/2)^{2s+1}|E_{2s}|$$

For more details see also [5], [2, ch 5.1.4].

Calculated values of magnitudes in (8), (9) are shown in Table I. Symbolic and numerical calculations are made by using *Mathematica* [19].

It is easy to see that the constants K_r satisfy the following inequalities (see also [1]):

$$1=K_0<K_2<K_4<...<4/\pi<...<K_5<K_3<K_1=\pi/2 \quad (21)$$

and $\lim_{r\to\infty} K_r = 4/\pi$

III. RECURSIVE REPRESENTATIONS FOR NUMERICAL SERIES T_r, Q_r

The equalities (12)–(15) outline a procedure for summing up the numerical series T_r and Q_r defined in (7). It leads to the assertion

Corollary 1. The following recursive finite representation holds:

$$T_{2s} = (-1)^s \left[D_{2s+1}(\pi)/(2\pi) - D_{2s}(\pi)/2 \right] + \sum_{p=1}^{s-1} \frac{(-1)^{p+1}\pi^{2p}}{(2p+1)!} T_{2s-2p} \tag{22}$$

where $s = 1,2,3,...$ and for $s = 1$ by definition $T_0 = 0$.

It can be reminded for comparison the well-known formula (see [2], 5.1.2) for T_{2s}

$$T_{2s} = \left(2^{2s-1}\pi^{2s}/(2s)! \right) \left| B_{2s} \right|$$

The same procedure applied on the base of the equality

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n} = -D_1(x)/2, \quad (-\pi < x < \pi) \tag{23}$$

leads to

Corollary 2. The following recursive finite representation holds:

$$Q_{2s} = (-1)^s D_{2s+1}(\pi)/(2\pi) + \sum_{p=1}^{s-1} \frac{(-1)^{p+1}\pi^{2p}}{(2p+1)!} Q_{2s-2p} \tag{24}$$

where $s = 1,2,3,...$ and for $s = 1$ by definition $Q_0 = 0$.

In this connection it can be reminded the explicit formula ([2], section 5.2.1)

$$Q_{2s} = (1 - 2^{2s-1}) \left(\pi^{2s}/(2s)! \right) \left| B_{2s} \right|, \quad s = 1,2,3,...$$

Numerical results obtained by finite representations (22) and (24) using symbolic and numerical computations carried out by means of *Mathematica* software are shown in Table II.

IV. RECURSIVE REPRESENTATIONS FOR SOME FOURIER SERIES

The procedure of getting the representations (13), (14) and (18) with the help of (12) gives an opportunity to lay down

Theorem 2. The following recursive representations hold

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{2s-1}} = (-1)^s \left[D_{2s-1}(x) - \pi D_{2s-2}(x) \right]/2 + \sum_{p=1}^{s-1} (-1)^{p+1} T_{2s-2p} D_{2p-1}(x) \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^{2s}} = (-1)^{s-1} \left[D_{2s}(x) - \pi D_{2s-1}(x) \right]/2 + \sum_{p=0}^{s-1} (-1)^p T_{2s-2p} D_{2p}(x) \tag{26}$$

where $s = 1,2,3,...$ $(0 \le x \le 2\pi)$, and for $s = 1$ by definition $D_0(x) = 1$, $T_0 = 0$, $0 < x < 2\pi$.

TABLE I
EXACT AND APPROXIMATE VALUES OF THE FAVARD CONSTANTS K_r, CALCULATED BY THE RECURSIVE FORMULAS (8), (9) USING MATHEMATICA SOFTWARE PACKAGE

r	Exact values of K_r	Approximate values of K_r
1	$\pi/2$	1.5707963267948961923132169
2	$\pi^2/8$	1.2337005501361698273543113 7
3	$\pi^3/24$	1.2919281950124925073115131 4
4	$5\pi^4/384$	1.2683475395052400681828168 3
5	$\pi^5/240$	1.2750820199386727219280887 9
6	$61\pi^6/46080$	1.2726723265645306132561498 7
7	$17\pi^7/40320$	1.2734371248066831633864461 9
8	$277\pi^8/2064384$	1.2731754806526058136347769 7
9	$31\pi^9/725760$	1.2732612424724875463814366 7
10	$50521\pi^{10}/3715891200$	1.2732323827293948495082797 1
11	$691\pi^{11}/159667200$	1.2732419458721540967715077 9
12	$540553\pi^{12}/392398110720$	1.2732387471572495304117396 9

TABLE II
APPROXIMATE VALUES OF THE NUMERIC SERIES T_{2s}, Q_{2s}, CALCULATED BY THE RECURSIVE FORMULAS (22), (24) USING MATHEMATICA SOFTWARE PACKAGE

s	T_{2s}	Q_{2s}
1	1.6449340668482262	-0.8224670334241131
2	1.082323233711138	-0.9470328294972458
3	1.017343061984449	-0.9855510912974348
4	1.0040773561979441	-0.9962330018526475
5	1.0009945751278175	-0.9990395075982711
6	1.0002460865533072	-0.9997576851438577
7	1.0000612481350577	-0.9999391703459792
8	1.0000152822594075	-0.9999847642149055
9	1.0000038172932637	-0.9999961878696093
10	1.0000009539620325	-0.9999990466115808
11	1.0000002384505013	-0.9999997616132299
12	1.0000000596081875	-0.9999999403988914

It can be mentioned that the both series in (25), (26)) have the well-known representations ([2], 5.4.2)

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{2s+1}} = \frac{(-1)^{s-1}}{2(2s+1)!} (2\pi)^{2s+1} B_{2s+1}\left(\frac{x}{2\pi}\right)$$

where $s = 1,2,3,...$ $(0 \le x \le 2\pi)$, and $0 < x < 2\pi$ for $s = 0$.

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s}}=\frac{(-1)^{s-1}}{2(2s)!}(2\pi)^{2s}B_{2s}(\frac{x}{2\pi}),\ (0\le x\le 2\pi)\ ,s=1,2,3,...$$

The application of the above stated procedure for obtaining (25), (26) on the strength of (23) leads to the assertion

Theorem 3. The following recursive representations hold:

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^{2s-1}}=\frac{(-1)^s}{2}D_{2s-1}(x)+\sum_{p=1}^{s-1}(-1)^{p+1}Q_{2s-2p}D_{2p-1}(x) \quad (27)$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^{2s}}=\frac{(-1)^{s-1}}{2}D_{2s}(x)+\sum_{p=0}^{s-1}(-1)^{p}Q_{2s-2p}D_{2p}(x) \quad (28)$$

where $s=1,2,3,...\ (-\pi\le x\le\pi)$, and for $s=1$ by definition $D_0(x)=1$, $Q_0=0, -\pi<x<\pi$.

In the same time the both series (27), (28) have the following well-known representations (see [2], 5.4.2)

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^{2s+1}}=\frac{(-1)^{s-1}}{2(2s+1)!}(2\pi)^{2s+1}B_{2s+1}(\frac{x+\pi}{2\pi}),$$
$$(-\pi<x\le\pi),\quad s=0,1,2,...$$
$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^{2s}}=\frac{(-1)^{s-1}}{2(2s)!}(2\pi)^{2s}B_{2s}(\frac{x+\pi}{2\pi}),$$
$$(-\pi\le x\le\pi),\quad s=1,2,3,...$$

By analogy with the previous the procedure for obtaining (11), (16) and (19) with the help of (10) leads to

Theorem 4. The following recursive representations hold:

$$\sum_{v=0}^{\infty}\frac{\sin\left[(2v+1)x\right]}{(2v+1)^{2s-1}}=\left(\pi(-1)^{s-1}/4\right)\left[D_{2s-2}(x)+\sum_{p=1}^{s-1}(-1)^pK_{2p-1}D_{2s-2p-1}(x)\right] \quad (29)$$

$$\sum_{v=0}^{\infty}\frac{\cos\left[(2v+1)x\right]}{(2v+1)^{2s}}=\left(\pi(-1)^{s}/4\right)\left[D_{2s-1}(x)+\sum_{p=1}^{s}(-1)^pK_{2p-1}D_{2s-2p}(x)\right] \quad (30)$$

where $s=1,2,3,...\ (0\le x\le\pi)$, and for $s=1$ by definition $D_0(x)=1$, $(0<x<\pi)$.

Note, that the series (29), (30) have the well-known formulas ([2], 5.4.6)

$$\sum_{v=0}^{\infty}\frac{\sin\left[(2v+1)x\right]}{(2v+1)^{2s+1}}=\frac{(-1)^s\pi^{2s+1}}{4(2s)!}E_{2s}(\frac{x}{\pi}),$$
$$(0<x<\pi;\ s=0,1,2,...)$$

$$\sum_{v=0}^{\infty}\frac{\cos\left[(2v+1)x\right]}{(2v+1)^{2s}}=\frac{(-1)^s\pi^{2s}}{4(2s-1)!}E_{2s-1}(\frac{x}{\pi}),$$
$$(0\le x\le\pi;\ s=1,2,...)$$

The same procedure applied on the base of the equality

$$\left(4/\pi\right)\sum_{v=0}^{\infty}(-1)^v\frac{\cos\left[(2v+1)x\right]}{2v+1}=1,\ (-\pi/2<x<-\pi/2) \quad (31)$$

leads to

Theorem 5. The following recursive representations hold:

$$\sum_{v=0}^{\infty}(-1)^v\frac{\cos\left[(2v+1)x\right]}{(2v+1)^{2s-1}}=\pi(-1)^{s-1}\left[D_{2s-2}(x)+\sum_{p=1}^{s-1}(-1)^pK_{2p}D_{2s-2p-2}(x)\right]/4 \quad (32)$$

$$\sum_{v=0}^{\infty}(-1)^v\frac{\sin\left[(2v+1)x\right]}{(2v+1)^{2s}}=\pi(-1)^{s-1}\left[D_{2s-1}(x)+\sum_{p=1}^{s-1}(-1)^pK_{2p}D_{2s-2p-1}(x)\right]/4 \quad (33)$$

where $s=1,2,3,...\ (-\pi/2\le x\le\pi/2)$, and for $s=1:K_0$ declines; $D_0(x)=1$, $(-\pi/2\le x\le\pi/2)$.

In the same time the both series in (32), (33) have the well-known representations ([2], 5.4.6)

$$\sum_{v=0}^{\infty}(-1)^v\frac{\cos\left[(2v+1)x\right]}{(2v+1)^{2s+1}}=\frac{(-1)^s\pi^{2s+1}}{4(2s)!}E_{2s}(\frac{x}{\pi}+\frac{1}{2}),$$
$$(-\pi/2<x<\pi/2);\quad s=0,1,...$$
$$\sum_{v=0}^{\infty}(-1)^v\frac{\sin\left[(2v+1)x\right]}{(2v+1)^{2s}}=\frac{(-1)^{s-1}\pi^{2s}}{4(2s-1)!}E_{2s-1}(\frac{x}{\pi}+\frac{1}{2}),$$
$$(-\pi/2\le x\le\pi/2);\quad s=1,2,...$$

Meanwhile it is important to note that the number of addends in recursive representations (8), (9), (25), (26), (29), (30), (32), and (33) is two times less than the number of the addends in the corresponding cited formulas from [2].

This way the derived in this study method appears to be more economic and effective than the existing ones in the literature.

V. Additional Recursive Representations

Using the theorems 2 – 5 it is possible to obtain many other representations of the Favard constants K_r and numerical series T_{2s}, Q_{2s} putting, in particular, $x=\pi/2$ or $x=\pi$. For completeness some of the main results are given in this section.

From theorem 2 for $x=\pi/2$ and $x=\pi$ immediately follows

Corollary 3. For the Favard constants K_r the following recursive representations hold:

$$K_{2s-2} = \frac{4}{\pi}(-1)^s \left\{ \left[(1/2) D_{2s-1}(\pi/2) - (\pi/2) D_{2s-2}(\pi/2) \right] + \sum_{p=1}^{s-1} (-1)^{p+1} T_{2s-2p} D_{2p-1}(\pi/2) \right\} \quad (34)$$

$$K_{2s-1} = \frac{2}{\pi}\left\{ (-1)^s \left[(1/2) D_{2s}(\pi) - (\pi/2) D_{2s-1}(\pi) \right] + \sum_{p=1}^{s-1} (-1)^{p+1} T_{2s-2p} D_{2p}(\pi) \right\} \quad (35)$$

where $s = 1,2,3,...$, and for $s = 1$: $D_0(x) = 1$, $T_0 = 0$.

For $x = \pi$ one can get (22) too by replacing previously s by $s+1$.

For $x = \pi/2$ it is easy to obtain

Corollary 4. For numbers Q_{2s} the following recursive representations hold:

$$Q_{2s} = 4^s \left\{ (-1)^{s-1} \left[(1/2) D_{2s}(\pi/2) - (\pi/2) D_{2s-1}(\pi/2) \right] + \sum_{p=0}^{s-1} (-1)^p T_{2s-2p} D_{2p}(\pi/2) \right\} \quad (36)$$

where $s = 1,2,3,...$, and for $s = 1$ $D_0(x) = 1$.

Similarly, from theorem 3 for $x = \pi/2$ and $x = \pi$ respectively follows

Corollary 5. For the Favard constants K_r the following recursive representations hold:

$$K_{2s-2} = (4/\pi)(-1)^s \left\{ (-1)^{s-1} D_{2s-1}(\pi/2)/2 + \sum_{p=1}^{s-1} (-1)^p Q_{2s-2p} D_{2p-1}(\pi/2) \right\} \quad (37)$$

$$K_{2s-1} = (2/\pi)\left\{ (-1)^{s-1} D_{2s}(\pi)/2 + \sum_{p=1}^{s-1} (-1)^p Q_{2s-2p} D_{2p}(\pi) \right\} \quad (38)$$

where $s = 1,2,3,...$, and for $s = 1$: $D_0(x) = 1$, $Q_0 = 0$.

For $x = \pi$ one can get (24) again by replacing previously s by $s+1$.

For $x = \pi/2$ by applying theorem 3 and formula (28) it can be analogously obtained

Corollary 6. For numbers Q_{2s} the following recursive representations hold:

$$Q_{2s} = 4^s / \left(4^s - 1\right) \left\{ (-1)^s D_{2s}(\pi/2)/2 + \sum_{p=1}^{s-1} (-1)^{p+1} Q_{2s-2p} D_{2p}(\pi/2) \right\} \quad (39)$$

here $s = 1,2,3,...$, and for $s = 1$: $D_0(x) = 1$, $Q_0(x) = 0$.

By the same manner from theorem 4 for $x = \pi/2$ and $x = \pi$ can be obtained respectively the formulas for different K_r from these in theorem 1.

Corollary 7. For the Favard constants K_{2s-3} and K_{2s-1} the following recursive representations take place:

$$K_{2s-3} = (-1)^s \left\{ D_{2s-2}(\pi) + \sum_{p=1}^{s-2} (-1)^p K_{2p-1} D_{2s-2p-1}(\pi) \right\} / \pi \quad (40)$$

for $s = 2,3,...$, and for $s = 2$: $K_1 D_1(\pi)$ must be canceled, and

$$K_{2s-1} = (-1)^{s-1} \left\{ D_{2s-1}(\pi/2) + \sum_{p=1}^{s-1} (-1)^p K_{2p-1} D_{2s-2p}(\pi/2) \right\} \quad (41)$$

where $s = 1,2,3,...$, and for $s = 1$: $K_1 D_0(\pi/2)$ must be canceled.

The rest cases for $x = \pi/2$ and $x = \pi$ immediately lead to theorem 1 after replacing s by $s+1$.

From theorem 5 for $x = \pi/2$ it can get respectively other representations for K_r $(r = 1,2,3,...)$, different from these in theorem 1.

Corollary 8. For the Favard constants K_r the following recursive representations hold:

$$K_{2s-2} = (-1)^s \left\{ D_{2s-2}(\pi/2) + \sum_{p=1}^{s-2} (-1)^p K_{2p} D_{2s-2p-2}(\pi/2) \right\} \quad (42)$$

$$K_{2s-1} = (-1)^{s-1} \left\{ D_{2s-1}(\pi/2) + \sum_{p=1}^{s-1} (-1)^p K_{2p} D_{2s-2p-1}(\pi/2) \right\} \quad (43)$$

where $s = 1,2,3,...$, for $s = 1$: $D_0(\pi/2) = -1$, and for $s = 2$: $K_2 D_0(\pi/2)$ must be canceled.

In addition it is valid

Corollary 9. From the difference $T_{2s} - Q_{2s} = (\pi/2) K_{2s-1}$ $(s = 1,2,...)$ and after replacing s by $s+1$ in the obtained expression the following formulae holds

$$K_{2s+1} = \left((-1)^s/\pi\right) D_{2s+2}(\pi) + \sum_{p=1}^{s} \frac{(-1)^{p+1}\pi^{2p}}{(2p+1)!} K_{2s-2p+1}, \quad (s = 0,1,...). \quad (44)$$

This is somewhat better than the corresponding formula in Theorem 1, because $(2s+1)! > 2(2s)!$ for $s = 1,2,...$

VI. Numerical and Computer Implementation of the Derived Theoretical Representations

We will consider some aspects of numerical and symbolic calculations of the constants K_r and summation of series.

In Fig.1, we provide a *Mathematica* code for symbolic and numerical calculation of the Favard constants K_r for $r = 1, 2, 3, ..., m$ for a given arbitrary integer m>0, based on formulas (8) and (9). The obtained results are shown in Table I. We have to note that this code is not the most economic. It can be seen that the thrifty code will take about $2m^2 + 8m$ or $O(m^2)$ arithmetic operations in (8), (9).

$$\mathtt{m = 12;\quad Array[K, m];\ } K_1 = \frac{\pi}{2};\ \mathtt{d[n\_, x\_] :=} \frac{x^n}{n!}$$

$$\mathtt{For}\Big[\mathtt{s = 1,\ s \le m/2,\ s = s + 1,}$$

$$K_{2s+1} = \frac{1}{2}\left((-1)^s\, \mathtt{d}[2s+1, \pi] + \sum_{p=1}^{s}\left(K_{2s+1-2p}\,\frac{(-1)^{p-1}\pi^{2p}}{(2p)!}\right)\right);$$

$$K_{2s} = \left((-1)^s\, \mathtt{d}\left[2s, \frac{\pi}{2}\right] + \sum_{p=1}^{s}\left(K_{2s+1-2p}\,\frac{(-1)^{p-1}\left(\frac{\pi}{2}\right)^{2p-1}}{(2p-1)!}\right)\right)\Big]$$

$$\mathtt{For[s = 1,\ s \le m,\ s{+}{+},\ Print[}K_s\mathtt{,\ "\quad",\ N[}K_s\mathtt{, 30]]]}$$

Fig. 1 *Mathematica* code for exact symbolic and approximate computation with optional 30 digits accuracy of the Favard constants by (8), (9)

The basic advantage of using formulas (8) and (9) or other derived in this study representations is that they contain a finite number of terms (i.e., finite number of arithmetic operations) in comparison with the initial formula (5), (7) which needs the calculation of the slowly convergent infinite sums. It must be also mentioned, that in *Mathematica*, Maple and other powerful mathematical software packages, the Favard constants are represented by sums of Zeta (see (2)) and related functions, which are calculated by the use of Euler-Maclaurin summation and functional equations. Near the critical strip they also use the Riemann-Siegel formula (see, for instance, [19], A.9.4).

It can be concluded, that the derived recursive representations of the considered constants and series have both theoretical and practical importance. Future work can be addressed to obtain more properties and application of the Favard constants to numerical integration of singular integrals and summation of series.

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