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The Bipartite Ramsey Numbers $b(C_{2m}; C_{2n})$

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Abstract—Given bipartite graphs H_1 and H_2, the bipartite Ramsey number $b(H_1; H_2)$ is the smallest integer b such that any subgraph G of the complete bipartite graph $K_{b,b}$, either G contains a copy of H_1 or its complement relative to $K_{b,b}$ contains a copy of H_2. It is known that $b(K_{2,2}; K_{2,2}) = 5, b(K_{2,3}; K_{2,3}) = 9, b(K_{2,4}; K_{2,4}) = 14$ and $b(K_{3,3}; K_{3,3}) = 17$. In this paper we study the case that both H_1 and H_2 are even cycles, prove that $b(C_{2m}; C_{2n}) \geq m + n - 1$ for $m \neq n$, and $b(C_{2m}; C_6) = m + 2$ for $m \geq 4$.

Keywords—bipartite graph; Ramsey number; even cycle

I. Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex-set $V(G)$ and edge-set $E(G)$, we denote the order and the size of G by $p(G) = |V(G)|$ and $q(G) = |E(G)|$. $\delta(G)$ and $\Delta(G)$ are the minimum degree and the maximum degree of G respectively.

Let $K_{m,n}$ be a complete m by n bipartite graph, that is, $K_{m,n}$ consists of $m + n$ vertices, partitioned into sets of size m and n, and the mn edges between them. P_k is a path on k vertices, and C_k is a cycle of length k. Let H_1 and H_2 be bipartite graphs, the bipartite Ramsey number $b(H_1; H_2)$ is the smallest integer b such that given any subgraph G of the complete bipartite graph $K_{b,b}$, either G contains a copy of H_1 or there exists a copy of H_2 in the complement of G relative to $K_{b,b}$. Obviously, we have $b(H_1; H_2) = b(H_2; H_1)$.

Beineke and Schwenk[1] showed that $b(K_{2,2}; K_{2,2}) = 5, b(K_{2,4}; K_{2,4}) = 13, b(K_{3,3}; K_{3,3}) = 17$. In particular, they proved that $b(K_{2,n}; K_{2,n}) = 4n - 3$ for n odd and less than 100 except $n = 59$ or $n = 95$. Carnielli and Carmelo[2] proved that $b(K_{2,n}; K_{2,n}) = 4n - 3$ if $4n - 3$ is a prime power. They also showed that $b(K_{2,2}; K_{1,n}) = n + q$ for $q^2 - q + 1 \leq n \leq q^2$, where q is a prime power. Irving[6] showed that $b(K_{4,4}; K_{4,4}) \leq 48$. Hattingh and Henning[4] proved that $b(K_{2,2}; K_{3,3}) = 9, b(K_{2,2}; K_{4,4}) = 14$. They also determined the values of $b(P_m; K_{1,n})$[5]. Faudree and Schelp proved the values of $b(H_1; H_2)$ when both H_1 and H_2 are two paths[3]. It was shown that $b(C_6; K_{2,2}) = 5$ and $b(C_{2m}; K_{2,2}) = m + 1$ for $m \geq 4$ in [7].

Let G_i be the subgraph of G whose edges are in the i-th color in an r-coloring of the edges of G. If there exists an r-coloring of the edges of G such that $H_i \nsubseteq G_i$ for all $1 \leq i \leq r$, then G is said to be r-colorable to $(H_1, H_2, \ldots, H_r)$. The neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) | uv \in E(G)\}$, and let $d(v) = |N(v)|$. G^c denotes the complement of G relative to $K_{b,b}$. $G\langle W\rangle$ denotes the subgraph of G induced by $W \subseteq V(G)$. Let $G \cup H$ denote a disjoint sum of G and H, and nG is a disjoint sum of n copies of G.

Obviously, if H_1 and H_2 are cycles, then they are both even cycles. In this paper we study the case that both H_1 and H_2 are even cycles. Firstly, we prove that $b(C_{2m}; C_{2n}) \geq m + n - 1$ for $m \neq n$ and $b(C_{2m}; C_{2m}) \geq 2m$. Then setting $n = 3$, we prove that $b(C_6; C_6) = 6$ and $b(C_{2m}; C_6) = m + 2$ for $m \geq 4$. For the sake of convenience, let $V(K_{m,n}) = X \cup Y$, where $X = \{x_i | 1 \leq i \leq m\}$, $Y = \{y_j | 1 \leq j \leq n\}$, and $E(K_{m,n}) = \{x_i y_j | 1 \leq i \leq m, 1 \leq j \leq n\}$.

II. The lower bounds of $b(C_{2m}; C_6)$

Theorem 1: $b(C_{2m}; C_{2n}) \geq \begin{cases} m + n - 1, & m \neq n, \\ 2m, & m = n. \end{cases}$

Proof: If $m \neq n$, let G_1 and G_2 be subgraphs of $K_{m+n-2,m+n-2}$, where G_1 is a complete $m-1$ by $m+n-2$ bipartite graph, and G_2 is a complete $n - 1$ by $m + n - 2$ bipartite graph. And let $V(G_1) = X_1 \cup Y$, where$X_1 = \{x_i | 1 \leq i \leq m - 1\}$ and $Y = \{y_i | 1 \leq i \leq m + n - 2\}; V(G_2) = X_2 \cup Y$, where$X_2 = \{x_i | m \leq i \leq m + n - 2\}$, $Y = \{y_i | 1 \leq i \leq m + n - 2\}$. Then we have $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(K_{m+n-2,m+n-2})$. Note that $C_{2m} \nsubseteq G_1$ and $C_{2n} \nsubseteq G_2$. So $K_{m+n-2,m+n-2}$ is 2-colorable to (C_{2m}, C_{2n}), that is, $b(C_{2m}; C_{2n}) \geq m + n - 1$.

If $m = n$, let G_1 and G_2 be the spanning subgraphs of $K_{2m-1,2m-1}$. And let $E(G_1) = \{x_i y_j | 1 \leq i, j \leq m - 1\} \cup \{x_i y_j | m \leq i, j \leq 2m - 2\} \cup \{x_{2m-1} y_j | 1 \leq j \leq 2m - 1\}; E(G_2) = \{x_i y_j | 1 \leq i \leq m - 1, m \leq j \leq 2m - 2\} \cup \{x_i y_j | m \leq i \leq 2m - 2, 1 \leq j \leq m - 1\} \cup \{x_i y_{2m-1} | 1 \leq i \leq 2m - 2\}$. Then we have $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(K_{2m-1,2m-1})$. Note that $C_{2m} \nsubseteq G_1$ and $C_{2m} \nsubseteq G_2$. So $K_{2m-1,2m-1}$ is 2-colorable to (C_{2m}, C_{2m}), that is, $b(C_{2m}; C_{2m}) \geq 2m$. ∎

Setting $n = 3$ in Theorem 1, we have

Corollary 1: $b(C_{2m}; C_6) \geq \begin{cases} m + 2, & m \neq 3, \\ 6, & m = 3. \end{cases}$

III. The upper bounds of $b(C_{2m}; C_6) (m \geq 3)$

Lemma 1: Let G be a spanning subgraph of $K_{3,3}$, if $C_6 \nsubseteq G^c$, then $P_3 \subseteq G$.

Proof: If $P_3 \nsubseteq G$, then G is isomorphic to one graph of $\{6P_1, 4P_1 \cup P_2, 2P_1 \cup 2P_2, 3P_2\}$. In any case, we have $C_6 \subseteq G^c$. ∎

Lemma 2: $b(C_6; C_6) \leq 6$.

Proof: By contradiction, we assume that $b(C_6; C_6) > 6$, that is, $K_{6,6}$ is 2-colorable to C_6. Let $V(K_{5,5})$=$V(K_{6,6}) - \{x_6, y_6\}$. By Theorem 1, $K_{5,5}$ is 2-colorable to C_6, and $E(G_1\langle V(K_{5,5})\rangle) = \{x_i y_j | 1 \leq i, j \leq 2\} \cup \{x_i y_j | 3 \leq i, j \leq$

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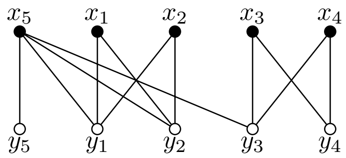
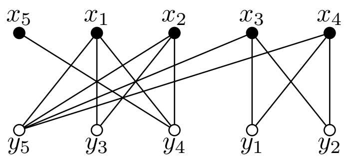


(a) $G'_1\langle V(K_{5,5})\rangle$

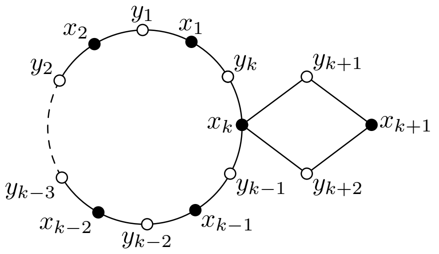


(b) $G'_2\langle V(K_{5,5})\rangle$

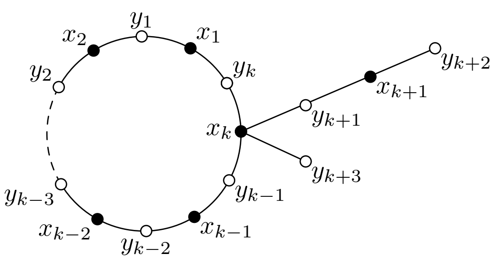
Fig. 1. The graphs $G'_1\langle V(K_{5,5})\rangle$ and $G'_2\langle V(K_{5,5})\rangle$

$4\} \cup \{x_5y_j|1 \leq j \leq 5\}; E(G_2\langle V(K_{5,5})\rangle) = \{x_iy_j|1 \leq i \leq 2, 3 \leq j \leq 4\} \cup \{x_iy_j|3 \leq i \leq 4, 1 \leq j \leq 2\} \cup \{x_iy_5|1 \leq i \leq 4\}$. Besides this, there is one coloring way without resulting monosubgraph C_6 in the 2-coloring edges of $K_{5,5}$, namely $G'_1\langle V(K_{5,5})\rangle \cong G_1\langle V(K_{5,5})\rangle - x_5y_4$ and $G'_2\langle V(K_{5,5})\rangle \cong G_2\langle V(K_{5,5})\rangle + x_5y_4$(see Figure 1). Now we consider the vertices x_6 and y_6. Since $C_6 \nsubseteq G_2$(or G'_2), x_6 is adjacent to at most one vertex of $\{y_1, y_2, y_3, y_4\}$. Hence x_6 has to be adjacent to at least three vertices of $\{y_1, y_2, y_3, y_4\}$ in G_1(or G'_1), we have $C_6 \subseteq G_1$(or G'_1), a contradiction. So, $K_{6,6}$ is not 2-colorable to C_6, that is, $b(C_6; C_6) \leq 6$. ■

In order to prove Lemma 3, we need the following claims. Let H_{2k+3} and H_{2k+4} denote the two graphs as shown in Figure 2, and G be a spanning subgraph of $K_{k+3,k+3}$ for $k \geq 3$ such that $C_{2(k+1)} \nsubseteq G$ and $C_6 \nsubseteq G^c$, then we have



(a) H_{2k+3}



(b) H_{2k+4}

Fig. 2. The graphs H_{2k+3} and H_{2k+4}

Claim 1: $H_{2k+3} \nsubseteq G$.

Proof: By contradiction, we assume that $H_{2k+3} \subseteq G$, and label the vertices of H_{2k+3} as shown in Fig. 2(a). Let x_{k+2}, x_{k+3} and y_{k+3} be the remaining vertices of $V(G)$. Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+3}\}\rangle$. Since $C_{2(k+1)} \nsubseteq G$, x_{k+1} is nonadjacent to y_{k-1} or y_k. By symmetry, it is sufficient to consider the five cases. We may assume $x_{k+2}y_{k-1}, x_{k+2}y_{k+3} \in E(G)$, $y_{k-1}x_{k+2}, y_{k-1}x_{k+3} \in E(G)$, $x_{k+2}y_{k-1}, x_{k+2}y_k \in E(G)$, $y_{k+3}x_{k+1}, y_{k+3}x_{k+2} \in E(G)$ or $y_{k+3}x_{k+2}, y_{k+3}x_{k+3} \in E(G)$.

Case 1. Suppose $x_{k+2}y_{k-1}, x_{k+2}y_{k+3} \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{y_{k+1}, y_{k+2}\}$ is nonadjacent to any vertex of $\{x_1, x_{k-1}, x_{k+2}\}$, and y_{k+3} is nonadjacent to x_{k-1}. And since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. Hence y_{k+3} has to be adjacent to x_1. Since $C_{2(k+1)} \nsubseteq G$, y_k is nonadjacent to any vertex of $\{x_{k-1}, x_{k+2}\}$. Hence we have we have $P_3 \nsubseteq G\langle\{x_1, x_{k-1}, x_{k+2}, y_k, y_{k+1}, y_{k+2}\}\rangle$. By Lemma 1, we have $C_6 \subseteq G^c$, a contradiction.

Case 2. Suppose $y_{k-1}x_{k+2}, y_{k-1}x_{k+3} \in E(G)$. Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k-1}, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{y_{k+1}, y_{k+2}\}$ is nonadjacent to any vertex of $\{x_{k-1}, x_{k+2}, x_{k+3}\}$. Hence y_{k+3} has to be adjacent to at least one vertex of $\{x_{k+2}, x_{k+3}\}$. The proof is same as Case 1.

Case 3. Suppose $x_{k+2}y_{k-1}, x_{k+2}y_k \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{y_{k+1}, y_{k+2}\}$ is nonadjacent to any vertex of $\{x_1, x_{k-1}, x_{k+2}\}$. And since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. Hence y_{k+3} is adjacent to at least two vertices of $\{x_1, x_{k-1}, x_{k+2}\}$. Therefore since $C_{2(k+1)} \nsubseteq G$, we have y_{k+3} has to be adjacent to x_1 and x_{k-1}. Similarly, since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_k, x_{k+1}, x_{k+3}, y_1, y_k, y_{k+3}\}\rangle$. Since $C_{2(k+1)} \nsubseteq G$, x_k is nonadjacent to y_1 or y_{k+3}, and x_{k+1} is nonadjacent to any vertex of $\{y_1, y_k, y_{k+3}\}$. If x_{k+3} is adjacent to y_k, the proof is same as Case 2. If x_{k+3} is adjacent to both y_1 and y_{k+3}, we have $C_{2(k+1)} \subseteq G$, a contradiction.

Case 4. Suppose $y_{k+3}x_{k+1}, y_{k+3}x_{k+2} \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{x_1, x_{k-1}\}$ is nonadjacent to any vertex of $\{y_{k+1}, y_{k+2}, y_{k+3}\}$. And since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. Hence x_{k+2} is adjacent to at least one vertex of $\{y_{k+1}, y_{k+2}\}$, say $x_{k+2}y_{k+1} \in E(G)$ as shown in Fig. 3. And since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_1, x_{k-1}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. Hence we have x_{k+3} is adjacent to at least two vertices of $\{y_{k+1}, y_{k+2}, y_{k+3}\}$. In any case, since $C_{2(k+1)} \nsubseteq G$, x_{k+3} is nonadjacent to any vertex of $\{y_{k-2}, y_{k-1}, y_k\}$. And each vertex of $\{x_{k+1}, x_{k+2}\}$ is nonadjacent to any vertex of $\{y_{k-2}, y_{k-1}, y_k\}$. Hence we have $P_3 \nsubseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-2}, y_{k-1}, y_k\}\rangle$. By Lemma 1, we have $C_6 \subseteq G^c$, a contradiction.

Case 5. Suppose $y_{k+3}x_{k+2}, y_{k+3}x_{k+3} \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, x_{k+1} is nonadjacent to y_{k-1} or y_k. Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+1}\}\rangle$. If there is one edge between $\{x_{k+2}, x_{k+3}\}$ and $\{y_{k-1}, y_k\}$, the proof is same as Case 2. Hence y_{k+1} has to be adjacent to at least one vertex of $\{x_{k+2}, x_{k+3}\}$, say $y_{k+1}x_{k+2} \in E(G)$. And since $C_{2(k+1)} \nsubseteq G$, y_1 is nonadjacent to x_{k+2} or x_{k+3}. Therefore we have $P_3 \nsubseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_1, y_{k-1}, y_k\}\rangle$. By Lemma 1, we have

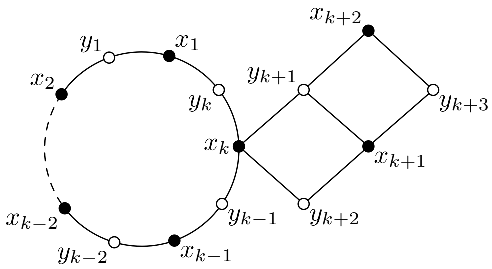


Fig. 3. x_{k+2} being adjacent to y_{k+1}

$C_6 \subseteq G^c$, a contradiction.

By Cases 1-5, we have $H_{2k+3} \nsubseteq G$. ■

Claim 2: $H_{2k+4} \nsubseteq G$.

Proof: By contradiction, we assume that $H_{2k+4} \subseteq G$, and label the vertices of H_{2k+4} as shown in Fig. 2(b). Let x_{k+2} and x_{k+3} be the remaining vertices of $V(G)$. Since $C_{2(k+1)} \nsubseteq G$, x_{k+1} is nonadjacent to any vertex of $\{y_{k-1}, y_k\}$. If x_{k+1} is adjacent to y_{k+3}, then we have $H_{2k+3} \subseteq G$, a contradiction to Claim 1. And since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+3}\}\rangle$. If x_{k+2}(or x_{k+3}) is adjacent to both y_{k-1} and y_{k+3}, we have $C_{2(k+1)} \subseteq G$, a contradiction. By symmetry, we may assume $x_{k+2}y_{k-1}, x_{k+2}y_k \in E(G)$, $y_{k-1}x_{k+2}, y_{k-1}x_{k+3} \in E(G)$ or $y_{k+3}x_{k+2}, y_{k+3}x_{k+3} \in E(G)$.

Case 1. Suppose $x_{k+2}y_{k-1}, x_{k+2}y_k \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{y_{k+1}, y_{k+3}\}$ is nonadjacent to any vertex of $\{x_1, x_{k-1}, x_{k+2}\}$, and y_{k+2} is nonadjacent to any vertex of $\{x_1, x_{k-1}\}$. Then $P_3 \nsubseteq G\langle\{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. By Lemma 1, we have $C_6 \subseteq G^c$, a contradiction.

Case 2. Suppose $y_{k-1}x_{k+2}, y_{k-1}x_{k+3} \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{y_{k+1}, y_{k+3}\}$ is nonadjacent to any vertex of $\{x_{k+2}, x_{k+3}\}$, and y_k is nonadjacent to x_{k+1}. If y_k is adjacent to one vertex of $\{x_{k+2}, x_{k+3}\}$, the proof is same as Case 1. If $x_{k+1}y_{k+3} \in E(G)$, then $H_{2k+3} \subseteq G$, a contradiction to Claim 1. Hence $P_3 \nsubseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+3}\}\rangle$. By Lemma 1, we have $C_6 \subseteq G^c$, a contradiction.

Case 3. Suppose $y_{k+3}x_{k+2}, y_{k+3}x_{k+3} \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{y_{k-1}, y_k\}$ is nonadjacent to any vertex of $\{x_{k+1}, x_{k+2}, x_{k+3}\}$. And since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+1}\}\rangle$. Hence y_{k+1} is adjacent to at least one vertex of $\{x_{k+2}, x_{k+3}\}$. In any case, we have $H_{2k+3} \subseteq G$, a contradiction to Claim 1.

By Cases 1-3, we have $H_{2k+4} \nsubseteq G$. ■

By an argument similar to the above proofs, we can prove Claim 3 and 4. However, their proofs are more complicated than Claim 2.

Claim 3: $(C_{2k} \cup C_4) \nsubseteq G$.

Claim 4: $(C_{2k} \cup P_5) \nsubseteq G$.

Lemma 3: Let G be a spanning subgraph of $K_{k+3,k+3}$ for $k \geq 3$. If $C_{2k} \subseteq G$ and $C_6 \nsubseteq G^c$, then $C_{2(k+1)} \subseteq G$.

Proof: We may assume that $C_{2(k+1)} \nsubseteq G$. Without loss of generality, let $E(C_{2k}) = \{x_1y_1, y_1x_2, x_2y_2, \ldots, x_ky_k, y_kx_1\}$. Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$, say $x_{k+1}y_{k+1}, x_{k+1}y_{k+2} \in E(G)$. Similarly, since $C_6 \nsubseteq G^c$, we have $P_3 \subseteq G\langle\{x_k, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. If x_k is adjacent to both y_{k+1} and y_{k+2}, then $H_{2k+3} \subseteq G$, a contradiction to Claim 1. If x_k is adjacent to both y_{k+1} and y_{k+3}(or both y_{k+2} and y_{k+3}), then $H_{2k+4} \subseteq G$, a contradiction to Claim 2. If there exists one vertex of $\{x_{k+2}, x_{k+3}\}$ being adjacent to both y_{k+1} and y_{k+2}, then $(C_{2k} \cup C_4) \subseteq G$, a contradiction to Claim 3. If there exists one vertex of $\{x_{k+2}, x_{k+3}\}$ being adjacent to both y_{k+1} and y_{k+3}(or both y_{k+2} and y_{k+3}), then $(C_{2k} \cup P_5) \subseteq G$, a contradiction to Claim 4. So, by symmetry, it is sufficient to consider the four cases as follows.

Case 1. Suppose $y_{k+1}x_k, y_{k+1}x_{k+2} \in E(G)$. Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+2}\}\rangle$. Since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{x_{k+1}, x_{k+2}\}$ is nonadjacent to any vertex of $\{y_{k-1}, y_k\}$. If x_{k+2} is adjacent to y_{k+2}, then we have $(C_{2k} \cup C_4) \subseteq G$, a contradiction to Claim 3. If x_{k+3} is adjacent to y_{k+2}, then we have $(C_{2k} \cup P_5) \subseteq G$, a contradiction to Claim 4. Hence x_{k+3} has to be adjacent to both y_{k-1} and y_k. Similarly since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_1, x_{k-1}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. Since $C_{2(k+1)} \nsubseteq G$, y_{k+1} is nonadjacent to any vertex of $\{x_1, x_{k-1}, x_{k+3}\}$, y_{k+2} is nonadjacent to any vertex of $\{x_1, x_{k-1}\}$. If $y_{k+2}x_{k+3} \in E(G)$, we have $(C_{2k} \cup P_5) \subseteq G$, a contradiction to Claim 4. If y_{k+3} is adjacent to both x_1 and x_{k+3}(or both x_{k-1} and x_{k+3}), we have $C_{2(k+1)} \subseteq G$, a contradiction too. Hence we have $y_{k+3}x_1, y_{k+3}x_{k-1} \in E(G)$ as shown in Fig. 4. However, since $C_{2(k+1)} \nsubseteq G$, each vertex of $\{x_{k+1}, x_{k+2}\}$ is nonadjacent to any vertex of $\{y_1, y_{k-1}, y_{k+3}\}$ and x_{k+3} is nonadjacent to any vertex of $\{y_1, y_{k+3}\}$. So, we have $P_3 \nsubseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_1, y_{k-1}, y_{k+3}\}\rangle$. By Lemma 1, we have $C_6 \subseteq G^c$, a contradiction.

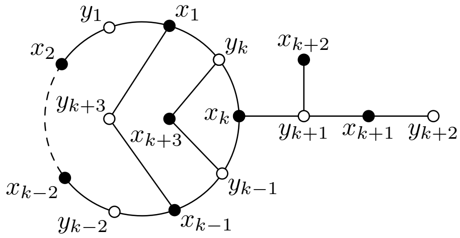


Fig. 4. y_{k+3} being adjacent to both x_1 and x_{k-1}

Case 2. Suppose $y_{k+1}x_{k+2}, y_{k+1}x_{k+3} \in E(G)$. Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_{k+1}, x_{k+2}, x_{k+3}, y_k, y_{k+2}, y_{k+3}\}\rangle$. If x_{k+1} is adjacent to y_k, the proof is same as Case 1. If there exists one vertex of $\{x_{k+2}, x_{k+3}\}$ being adjacent to y_{k+2}, then we have $(C_{2k} \cup C_4) \subseteq G$, a contradiction to Claim 3. If there exists one vertex of $\{x_{k+2}, x_{k+3}\}$ being adjacent to y_{k+3}, then we have $(C_{2k} \cup P_5) \subseteq G$, a contradiction to Claim 4. If y_k is adjacent to both x_{k+2} and x_{k+3}, we have $H_{2k+3} \subseteq G$, a contradiction to Claim 1. Hence y_{k+3} has to be adjacent to x_{k+1}. Similarly, since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_1, x_k, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. If there exists one vertex of $\{x_1, x_k\}$ being adjacent to y_{k+1}, the proof is same as Case 1. If there exists one vertex of $\{x_1, x_k\}$ being adjacent to both y_{k+2} and y_{k+3}, then we have $H_{2k+3} \subseteq G$, a contradiction to Claim 1. If x_{k+2} is adjacent to y_{k+2} or y_{k+3}, then we have $(C_{2k} \cup C_4) \subseteq G$, a contradiction to Claim 3. If there exists one vertex of $\{y_{k+2}, y_{k+3}\}$ being adjacent to both x_1 and x_k, the proof is same as Case 1.

Case 3. Suppose $y_{k+3}x_k, y_{k+3}x_{k+2} \in E(G)$. And since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_k, x_{k+2}, x_{k+3}, y_{k-1}, y_{k+1}, y_{k+2}\}\rangle$. If x_k is adjacent to y_{k+1} or y_{k+2}, then we have $H_{2k+4} \subseteq G$, a contradiction to Claim 2. If x_{k+2} is adjacent to y_{k+1} or y_{k+2}, then we have $(C_{2k} \cup P_5) \subseteq G$, a contradiction to Claim 4. If x_{k+3} is adjacent to both y_{k+1} and y_{k+2}, then we have $(C_{2k} \cup C_4) \subseteq G$, a contradiction to Claim 3. Since $C_{2(k+1)} \nsubseteq G$, y_{k-1} is nonadjacent to x_{k+2}. Hence x_{k+3} has to be adjacent to y_{k-1}. Similarly, we have $y_kx_{k+3} \in E(G)$, since otherwise $P_3 \nsubseteq G\langle\{x_k, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+2}\}\rangle$.

Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_1, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\}\rangle$. If there exists one vertex of $\{x_1, x_{k+3}\}$ being adjacent to both y_{k+1} and y_{k+2}, then we have $H_{2k+3} \subseteq G$, a contradiction to Claim 1. If x_{k+2} is adjacent to y_{k+1} or y_{k+2}, then we have $(C_{2k} \cup P_5) \subseteq G$, a contradiction to Claim 4. Since $C_{2(k+1)} \nsubseteq G$, y_{k+3} is nonadjacent to x_1 or x_{k+3}. If there exists one vertex of $\{y_{k+1}, y_{k+2}\}$ being adjacent to both x_1 and x_{k+3}, we have $C_{2(k+1)} \subseteq G$, a contradiction.

Case 4. Suppose $y_{k+3}x_{k+2}, y_{k+3}x_{k+3} \in E(G)$. Since $C_6 \nsubseteq G^c$, by Lemma 1, we have $P_3 \subseteq G\langle\{x_k, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+2}\}\rangle$. If there exists one edge between $\{x_{k+2}, x_{k+3}\}$ and $\{y_{k+1}, y_{k+2}\}$, we have $(C_{2k} \cup P_5) \subset G$, a contradiction to Claim 4. If x_k is adjacent to y_{k+1} or y_{k+2}, the proof is same as Case 3. If y_k is adjacent to x_{k+2} or x_{k+3}, the proof is also same as Case 3.

By Cases 1-4, we have $C_{2k+1} \subseteq G$. ■

Let G be a spanning subgraph of $K_{6,6}$. If $C_6 \nsubseteq G^c$, by Lemma 2, we have $C_6 \subseteq G$. Hence we have the following corollary by Lemma 3.

Corollary 2: $b(C_8; C_6) \leq 6$.

Lemma 4: If $m \geq 4$, we have $b(C_{2m}; C_6) \leq m + 2$.

Proof: We will prove it by induction.

(1) For $m = 4$, the lemma holds by Corollary 2.

(2) Suppose that $b(C_{2k}; C_6) \leq k + 2$ for $k \geq 5$. We assume that $b(C_{2(k+1)}; C_6) > k + 3$ for $k \geq 5$. Since $C_6 \nsubseteq G^c$, we have $C_{2k} \subseteq G$. By Lemma 3, we have $C_{2(k+1)} \subseteq G$, a contradiction. So the assumption does not hold, that is, $b(C_{2(k+1)}; C_6) \leq k + 3$. This completes the induction step, and the proof is finished. ■

IV. Conclusion

Setting $m = 3$ in Corollary 1, we have $b(C_6; C_6) \geq 6$. By Theorem 1, Lemma 2 and Lemma 4, we obtain the values of $b(C_{2m}; C_6)$ as follows.

Theorem 2: $b(C_{2m}; C_6) = \begin{cases} 6, & m = 3, \\ m + 2, & m \geq 4. \end{cases}$

Furthermore, we have the following conjecture,

Conjecture 1: $b(C_{2m}; C_{2n}) = m + n - 1$ for $m > n$.

By the results in [7] and Theorem 2, it is true for $n = 2$ and 3.

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