

# New Design Constraints of FIR Filter on Magnitude and Phase of Error Function

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**Abstract**—Exchange algorithm with constraints on magnitude and phase error separately in new way is presented in this paper. An important feature of the algorithms presented in this paper is that they allow for design constraints which often arise in practical filter design problems. Meeting required minimum stopband attenuation or a maximum deviation from the desired magnitude and phase responses in the passbands are common design constraints that can be handled by the methods proposed here. This new algorithm may have important advantages over existing technique, with respect to the speed and stability of convergence, memory requirement and low ripples.

**Keywords**—Least square estimation, Constraints, Exchange algorithm.

## I. INTRODUCTION

DIGITAL filters are integral parts of many digital signal processing systems, including control systems, systems for audio and video processing, communication systems and systems for medical applications. Due to the increasing number of applications involving digital signal processing and digital filtering, the variety of requirements that have to be met by digital filters has increased as well. Consequently, there is a need for flexible techniques that can design digital filters satisfying sophisticated specifications. The design specifications are formulated in the frequency domain by choosing a complex desired frequency response  $D(e^{j\omega})$  which prescribes the desired magnitude and phase response. The complex function  $D(e^{j\omega})$  is defined on  $\Omega$  the domain of approximation. Which is a subset of the interval  $[0, 2\pi]$ . In most cases the domain  $\Omega$  is the union of several disjoint frequency bands which are separated by transition bands where no desired response is specified.

Cortezzo and Lightner [3] apply a multiple criterion optimization technique to a specification of both gain and group delay of FIR filter. But they state that their design method requires considerable computing time and is reliable only for orders not higher than five for FIR filter.

Chen and Parks [12] investigate an approach in which the complex valued response is converted into a real-valued function which is nearly equivalent to the complex function.

They also state that their methods has a large computer memory requirement and CPU time increases exponentially with increasing grid density using a linear programming technique the grid density governs the accuracy with which the solution approaches the optimum.

Xiapoping Lai [13] applied PLS algorithm to the constrained least square design of FIR filter directly. But there was no method for non convex problem.

The method presented here intends to solve the problem of computation time and memory requirement.

The paper is organized as, after the brief introduction in section I, section II gives insight into Least square Approximation. Section III deals with the Constrained Designing of filter. Design Examples are taken in section IV followed by conclusion in section V. Last section shows the references.

## II. CONSTRAINED LEAST SQUARE APPROXIMATION

The union of all passbands is denoted by  $\Omega^p$  :

$$\Omega^p = \{\omega \in \Omega \mid |D(e^{j\omega})| > 0\}.$$

The union of all stopbands is denoted by  $\Omega^s$  :

$$\Omega^s = \{\omega \in \Omega \mid |D(e^{j\omega})| = 0\}$$

If the designed filter is to have real valued coefficients, only the domain  $\Omega \cap [0, \pi]$  is considered. In this case the symmetry  $D(e^{j\omega}) = D^*(e^{-j\omega})$  is assumed implicitly. This paper focus will be on the design of filters with real-valued coefficients. It is, however straightforward to extend the proposed methods to the design of filters with complex coefficients.

For formulating the design problems it is useful to define a complex error function by:

$$E_c(\omega) = H(e^{j\omega}) - D(e^{j\omega}) \quad (1)$$

Where  $H(e^{j\omega})$  is the actual frequency response of the filter. Often a, real valued positive weighting function  $W(\omega)$  is used, e.g. by considering a weighted error function  $W(\omega)E_c(\omega)$ , in order to give different weights on the approximation error in different frequency regions.

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Considering the design of FIR filters; in this case the frequency response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-jn\omega} \quad (2)$$

Where N denotes the length of the impulse response h (n). The degree of the FIR filter is N-1. For further discussions it will be advantageous to use vector/matrix notation. Define a column vector of FIR filter coefficients by:

$$h = [h(0), h(1), \dots, h(N-1)]^T$$

Let define a column vector of complex exponentials:

$$e(\omega) = [1, e^{j\omega}, e^{2j\omega}, \dots, e^{(N-1)j\omega}]^T \quad (3)$$

With these vectors the frequency response  $H(e^{j\omega})$  of an FIR filter given by (2) can be written as:

$$H(e^{j\omega}) = e^H(\omega)h, \quad (4)$$

Where the superscript  $H$  denotes conjugate transposition, the notation  $H(e^{j\omega}, h)$  will be used to explicitly express the dependence of  $H(e^{j\omega})$  on the coefficients h whenever necessary. Likewise, the complex error function will be written as  $E_c(\omega, h)$  whenever its dependence on the filter coefficients is to be emphasized. Note the linear dependence of  $H(e^{j\omega}, h)$  and of  $E_c(\omega, h)$  on the coefficients h in the FIR case. For this reason, FIR filter design problems are much simpler to solve. The ability to design long filters is among the more significant improvements over previous work.

The constrained least squares filter design problem can be posed as an optimization problem where approximation error energy is minimized subject to constraints on error functions of interest.[8] Phase errors are used to define error energy the resulting objective function is non-convex which may cause severe difficulties when solving the design problem and prevents the use of simple and efficient algorithms. Moreover the minimization of error energy is most important in the stopbands of the filter. The use of the complex error for defining the error energy leads to a simple and meaningful objective function. These considerations lead to two constrained least squares design problems those are of practical interest and that result in tractable optimization problems[10]. This can be formulated by

$$\underset{h}{\text{minimize}} \int_{\Omega_{\min}} W(\omega) |E_c(\omega, h)|^2 d\omega \quad (5)$$

Subject to  $|E_c(\omega, h)| \leq \delta_c(\omega), \omega \in \Omega_B$

Where  $\Omega_{\min}$  contains the frequency bands where the mean squared error is to be minimized.  $\Omega_B$  contains the frequency bands where constraints are to be imposed and  $\Omega^p$  is the passband region. The positive function  $\delta_c(\omega)$ , defining the constraints are part of the design specifications. Algorithms for solving problems eq. (5) will be presented in section 3.

This problem formulation seems more suitable for practical applications. Hence after frequency discretization the design problem can be formulated as a standard convex quadratic programming problem with linear constraints. The motivation of the approach in [5] was the minimization of stopband noise power subject to constraints on the maximum passband deviations. This proved the additional advantage that arbitrarily tight error constraints may be formulated and the existence of a solution is always guaranteed. The more restrictive the constraints the wider become the transition widths.

The magnitude of the complex error function  $E_c(\omega, h)$  as defined by (1) is a convex function with respect to h for all frequencies. This is easily shown as follows. Define

$$h(\alpha) = \alpha h_1 + (1 - \alpha)h_2 \quad (6)$$

From linearity of  $E_c(\omega, h)$  with respect to h and from the triangular inequality it follows that

$$\begin{aligned} |E_c(\omega, h(\alpha))| &= |\alpha E_c(\omega, h_1) + (1 - \alpha)E_c(\omega, h_2)| \\ &\leq \alpha E_c(\omega, h_1) + (1 - \alpha)E_c(\omega, h_2) \\ &\quad \forall \alpha \in [0, 1], \end{aligned} \quad (7)$$

must hold. Inequality equation (7) is the definition of a convex function.

### III. CONSTRAINED DESIGN OF LEAST SQUARE FIR FILTER

Emphasis has been put on the constrained least squares problem because it leads to new interesting solutions whereas the constrained Chebyshev problem formulation merely represents a more practical formulation of the design problem than the standard Chebyshev problem formulation.[11] Now to solve equation (5) and minimized the error there is a need of putting constraints on magnitude and phase error separately. It can be done by using two methods for solving equation (5). (1) Linear and Quadratic Programming or (2) Exchange algorithm. The constraints are put on complex error function on both solving methods.

#### A. Constraints on Magnitude and Phase Errors using Linear and Quadratic Programming Approach

As shown constraints on the phase error are exactly represented by linear inequality constraints if the function  $\delta_\phi(\omega)$  constraining the phase response satisfies  $\delta_\phi(\omega) \leq \pi/2, \forall \omega \in \Omega^p$ . Hence only the nonlinear magnitude constraints must be replaced by a finite set of linear constraints. In the stopbands the error region is a circle around the origin of the complex plane. Consequently all results from the previous section still apply to the stopbands. The modifications of the passband error region due to magnitude constraint linearization are shown in Fig. 1. The dashed arcs show the original magnitude constraints. The solid vertical lines correspond to linear constraints replacing the original magnitude constraints. The constraints denoted by '1' in Fig. 1 correspond to a feasible region which is the smallest set

described by two linear inequality constraints per frequency point containing the original feasible set. The constraints denoted by '2' correspond to a feasible region being the largest set described by two linear inequality constraints per frequency point that is contained in the original set. A better approximation to the upper magnitude constraint could be achieved by using more than one linear upper magnitude constraint per frequency point. However this increases the size of the resulting optimization problem. Note that improving the approximation by using more than one linear constraint per frequency point is not possible for the non-convex lower magnitude.

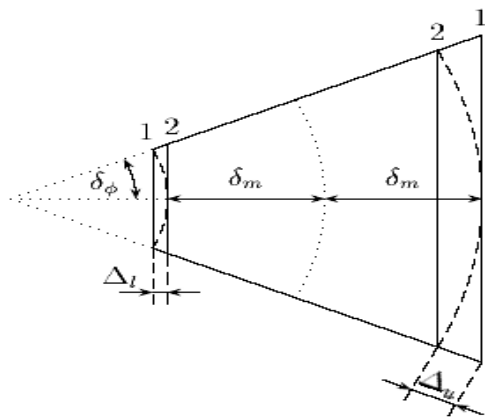


Fig. 1 Replacing nonlinear magnitude constraints (dashed arcs) by linear constraints (solid vertical lines). 1: new feasible region is larger than the original feasible region. 2: new feasible region is smaller than the original feasible region

Constraint. Choosing the linear constraints denoted by '1' in Fig. 1 results in the following maximum violations of the original passband magnitude constraints if  $\delta_\phi(\omega) < \pi/2$  holds

$$\Delta_u(\omega) = \left[ |D(e^{j\omega})| + \delta_m(\omega) \left[ \frac{1}{\cos \delta_\phi(\omega)} - 1 \right] \right]$$

$$= \left[ |D(e^{j\omega})| + \delta_m(\omega) \left[ \frac{1}{2} \delta_\phi^2(\omega) + \frac{5}{24} \delta_\phi^4(\omega) + \dots \right] \right], \quad \omega \in \Omega^p \quad (8)$$

$$\Delta_l(\omega) = \left[ |D(e^{j\omega})| - \delta_m(\omega) [1 - \cos \delta_\phi(\omega)] \right]$$

$$= \left[ |D(e^{j\omega})| - \delta_m(\omega) \left[ \frac{1}{2} \delta_\phi^2(\omega) - \frac{1}{24} \delta_\phi^4(\omega) + \dots \right] \right], \quad \omega \in \Omega^p$$

Where  $\Delta_u(\omega)$  and  $\Delta_l(\omega)$  are the maximum violations of the upper and lower magnitude constraints respectively. It is straightforward to show that  $\Delta_l(\omega) \leq \Delta_u(\omega)$ ,  $\forall \omega \in \Omega^p$ , if  $0 < \delta_\phi(\omega) < \pi/2$ ,  $\forall \omega \in \Omega^p$ . holds. From (9) it is clear that the violations  $\Delta_u(\omega)$  and  $\Delta_l(\omega)$  consist only of mixed and higher order terms of the specified functions  $\delta_m(\omega)$  and  $\delta_\phi(\omega)$ . Hence

for small  $\delta_m(\omega)$  and  $\delta_\phi(\omega)$  the errors introduced by using linearized magnitude constraints are small as well. If the linear constraints denoted by '2' in Fig.1 are used no violations with respect to the original feasible region occur. Using the linearized passband magnitude constraints shown in Fig. 1 linear and quadratic programming problem formulation can be done with approximating the constrained least squares problems (5). If the new feasible region is to completely contain the original feasible region (constraints denoted by '1' in Fig. 1. define positive functions  $U(\omega)$  and  $L(\omega)$  according to:

$$U(\omega) = |D(e^{j\omega})| + \delta_m(\omega), \quad \omega \in \Omega^p \cap \Omega_B$$

$$\delta_m(\omega) \quad \omega \in \Omega^s \cap \Omega_B$$

$$L(\omega) = [|D(e^{j\omega})| - \delta_m(\omega)] \cos \delta_\phi(\omega), \quad \omega \in \Omega^p \cap \Omega_B \quad (9)$$

If no constraint violations with respect to the original feasible region are tolerated (constraints denoted by '2' in Fig. 1 then  $U(\omega)$  and  $L(\omega)$  must be chosen according to;

$$U(\omega) = [|D(e^{j\omega})| + \delta_m(\omega)] \cos \delta_\phi(\omega) \quad \omega \in \Omega^p \cap \Omega_B$$

$$\delta_m(\omega) \cos(\pi/2p) \quad \omega \in \Omega^s \cap \Omega_B$$

$$L(\omega) = |D(e^{j\omega})| - \delta_m(\omega), \quad \omega \in \Omega^p \cap \Omega_B \quad (10)$$

The quadratic programming problem approximating the constrained least squares problem reads

$$\text{minimize}_h \int_{\Omega_{\min}} W(\omega) |E_c(\omega, h)|^2 d\omega \quad \text{subject to}$$

$$\text{Re}[H(e^{j\omega}, h)e^{-j\phi_d(\omega)}] \leq U(\omega), \quad \omega \in \Omega^p \cap \Omega_B$$

$$\text{Re}[H(e^{j\omega}, h)e^{-j\phi_d(\omega)}] \geq L(\omega), \quad \omega \in \Omega^p \cap \Omega_B \quad (11)$$

$$\text{Im}[H(e^{j\omega}, h)e^{-j\phi_d(\omega)}] \leq \tan \delta_\phi(\omega) \text{Re}[H(e^{j\omega}, h)e^{-j\phi_d(\omega)}], \quad \omega \in \Omega^p \cap \Omega_B$$

$$\text{Im}[H(e^{j\omega}, h)e^{-j\phi_d(\omega)}] \geq -\tan \delta_\phi(\omega) \text{Re}[H(e^{j\omega}, h)e^{-j\phi_d(\omega)}], \quad \omega \in \Omega^p \cap \Omega_B$$

$$\text{Re}[H(e^{j\omega}, h)e^{j\alpha_k}] \leq U(\omega), \quad \omega \in \Omega^s \cap \Omega_B$$

Where  $\Omega^p$  and  $\Omega^s$  are passbands and stopbands respectively,  $\Omega_B$  is the union of all bands where constraints are to be imposed and  $\Omega_{\min}$  is the union of all bands where the objective function (maximum error or error energy) is to be minimized. Note that  $\Omega_B$  is assumed to be represented by a discrete set of grid points. The remaining constraints in (11) are arranged as follows: upper and lower passband magnitude constraints upper and lower phase constraints and stopband constraints. Note that the phase constraints in equation (11) are only valid if  $\delta_\phi(\omega) < \pi/2$  holds. The results obtained by using problem formulation (11) will only be satisfactory if the phase

constraint function  $\delta\phi(\omega)$  (in radians) is not considerably larger than the magnitude constraint function  $\delta_m(\omega)$ . Otherwise the deviations of the linearized magnitude constraints from the original magnitude constraints will be large.

### B. Constraints on Magnitude and Phase using Exchange Approach

In the passbands the linearization errors depend on second and higher order terms involving the constraint functions  $\delta_m(\omega)$  and  $\delta\phi(\omega)$ . Hence these errors are small if  $\delta_m(\omega)$  and  $\delta\phi(\omega)$  are small. However especially in situations where  $\delta\phi(\omega)/\text{rad} \gg \delta_m(\omega)$  holds in parts of the passbands the linearization errors may become quite large. Exchange algorithms can solve optimization problems with an arbitrarily large or even infinite number of constraints. During the iteration process a set of constraints containing those constraints that will be active at the optimum solution is identified. The solution of the last sub problem subject to these constraints equals the solution of the original problem. The efficiency of these methods strongly depends on the efficiency of the algorithms used for solving the sub problems. Hence it is desirable to solve sub problems with linear constraints since there exist fast algorithms for linearly constrained problems. The amount of memory required by exchange algorithms is independent of the total number of constraints. Hence the first drawback of the standard quadratic programming formulation is eliminated. Also the linearization errors can be eliminated. We have to distinguish between problems with convex and non-convex feasible regions. Since any convex feasible region can be represented by an infinite number of linear constraints such problems can directly be handled by exchange algorithms that solve a sequence of linearly constrained sub problems. These algorithms can be viewed as generalizations of cutting plane methods for convex programming problems. Linearization of nonlinear constraints is no longer necessary. The only requirement for algorithms based on cutting plane methods to be applicable is the convexity of the feasible region is satisfied for upper bounds on the magnitude response and for phase constraints if  $\delta\phi(\omega) \leq \pi/2$  holds. Only lower bounds on the magnitude response as used in the passbands result in a non-convex feasible region. Hence for exchange algorithms based on cutting plane methods to be applicable the lower bounds on the magnitude error have to be replaced by constraints such that the feasible region is convex. Non-convex feasible regions cannot be represented by an infinite number of linear constraints and exchange algorithms based on cutting plane methods cannot be applied.

An exchange algorithm that exactly solves the non-convex constrained least squares problem with magnitude and phase constraints is presented. The algorithms used in literature so far solve a constrained optimization problem in every iteration step where the set of constraints is composed of a part of the constraints used in the previous iteration step and new constraints determined by evaluating the original semi-infinite constraints at the current solution. Reusing a part of the old constraints is crucial for convergence. A nonlinear semi-infinite constraint can be written in the form

$$c(\omega, h) \leq 0, \quad \forall \omega \in \Omega_B \quad (12)$$

Each constraint used in a certain iteration step is derived from a first order Taylor expansion of  $c(\omega, h)$  about some coefficient vector  $h_i$  evaluated at some frequency point  $\omega$ :

$$c(\omega, h_i) + (h - h_i)^T \nabla c(\omega, h_i) \leq 0, \quad (13)$$

Where  $\nabla c(\omega, h)$  is the gradient vector of  $c(\omega, h)$  with respect to  $h$ . If  $c(\omega, h)$  is convex, and hence also the set defined by (12) is convex, the inequality

$$c(\omega, h_i) + (h - h_i)^T \nabla c(\omega, h_i) \leq c(\omega, h) \quad \forall \omega \in \Omega_B \quad (14)$$

is satisfied for any  $h_i$ . Constraints from previous iteration steps may become unnecessary and could be thrown away but they will never cut away parts of the original feasible region. However if  $c(\omega, h)$  is not convex the feasible set defined by (12) is non-convex in general and linear constraints as formulated might cut away parts of the original feasible region because (14) is not satisfied in general. Hence old constraints must be removed because they might cut away the part of the feasible region that contains the optimum solution. Instead of reusing old constraints some other constraints related to these old constraints must be used. The exchange algorithm presented in Section (III b) reuses the active constraints of the previous iteration step. A logical extension is to formulate new constraints at the frequency points corresponding to active constraints of the previous iteration step. These constraints are used instead of the old active constraints. Hence in every iteration step all constraints are formulated anew and the propagation of old constraints cutting away parts of the feasible region are prevented.

Note that all these considerations only apply to non-convex constraint functions. In the problem under consideration only the exchange rule for the lower bounds on the magnitude response must be adapted. The exchange rule for upper magnitude bounds and for phase constraints remains unchanged. The modified exchange algorithm for exactly solving the constrained least squares problem with magnitude and phase constraints works as follows:

### Algorithm:

- 1)  $k = 0$  Solve the unconstrained quadratic minimization problem for  $h^{(0)}$ .
- 2) Determine the local maxima of  $E_m(\omega, h^{(k)}) - \delta_m(\omega)$ ,  $\omega \in \Omega_B$  and of  $E_\phi(\omega, h^{(k)}) - \delta_\phi(\omega)$ ,  $\omega \in \Omega^p \cap \Omega_B$  and the local minima of  $E_m(\omega, h^{(k)}) + \delta_m(\omega)$ ,  $\omega \in \Omega^p \cap \Omega_B$ , and of  $E_\phi(\omega, h^{(k)}) + \delta_\phi(\omega)$ ,  $\omega \in \Omega^p \cap \Omega_B$ . If  $|E_m(\omega, h^{(k)})| \leq \delta_m(\omega)$ ,  $\omega \in \Omega_B$  and  $E_\phi(\omega, h^{(k)}) \leq \delta_\phi(\omega)$ ,  $\omega \in \Omega^p \cap \Omega_B$ , is satisfied up to some specified tolerance, stop Otherwise go to 3.

Determine the sets of frequencies at which local maxima or minima of the functions considered in step 1 violate the respective constraints

- 3) Formulate a new set of constraints for the next iteration step:
  - a) Reuse the current active upper magnitude bounds and active phase constraints.
  - b) Impose new magnitude constraints at passband frequencies corresponding to active lower magnitude bounds in the current iteration step.
  - c) Impose new magnitude and phase constraints at the respective sets of frequencies determined in step 2. Formulate new magnitude constraints using a first order Taylor series of  $E_m(\omega, h)$  about the current solution  $h^{(k)}$ :
 
$$E_m(\omega, h) \approx \text{Re}[H(e^{j\omega}, h)e^{-j\phi_d(\omega)}] - |D(e^{j\omega})|,$$
- 4)  $k: k+1$ , Compute  $h^{(k)}$  by solving the quadratic programming problem subject to the constraints determined in step 4. Go to 2.

The algorithm presented in this section solves an unconstrained least squares problem as an initial problem and then adds all constraints that are necessary to compute the optimum solution to the semi-infinite programming problem. This greatly reduces memory requirements and computational effort.

#### IV. DESIGN EXAMPLE

We design a chirp-lowpass filter according to the following specification:

$$D(e^{j\omega}) = e^{j\phi_d(\omega)}, \quad 0 \leq \omega \leq \omega_p \quad (15)$$

$$= 0 \quad \omega_s \leq \omega \leq \pi$$

With the desired phase response  $\phi_d(\omega)$  given by

$$\phi_d(\omega) = -\frac{N-1}{2}\omega - 8\pi\left(\frac{1}{\omega_p} - \frac{1}{2}\right)^2 \quad 0 \leq \omega \leq \omega_p \quad (16)$$

Where  $\omega_p$  and  $\omega_s$  are the passband and stopband edges respectively and  $N$  is the filter length. The desired phase response  $\phi_d(\omega)$  corresponds to a linearly ascending desired group delay response. Its minimum and maximum values are given by:

$$\tau_d(0) = \frac{N-1}{2} - \frac{8\pi}{\omega_p},$$

$$\tau_d(\omega_p) = \frac{N-1}{2} + \frac{8\pi}{\omega_p},$$

Choosing  $\omega_p = 0.2\pi$ ,  $\omega_s = 0.225\pi$  and  $N=201$ . With these choices the desired group delay linearly ascends from 60 samples to 140 samples in the passband. The constraints are

chosen in such a way that the maximum passband magnitude error is less than 0.007 and the minimum stopband attenuation is 45 dB. The passband phase error constraint is chosen as 0.007 radians. The filter is designed by independently constraining magnitude and phase errors. The constraint functions are chosen as

$$\delta_\phi(\omega) = \delta_m(\omega) = 0.007, \quad 0 \leq \omega \leq \omega_p, \text{ and}$$

$$\delta_m(\omega) = 10^{-45/20}, \quad \omega_s \leq \omega \leq \pi$$

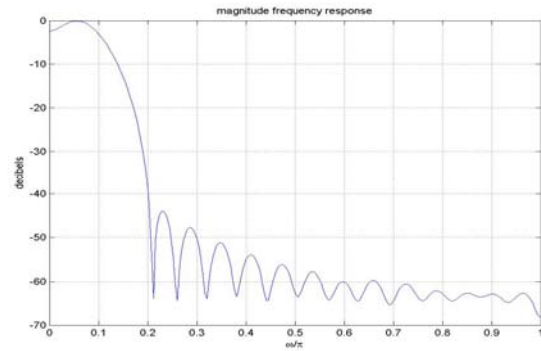


Fig. 2 Magnitude Response of filter without exchange algorithm with constraints on magnitude and phase of errors

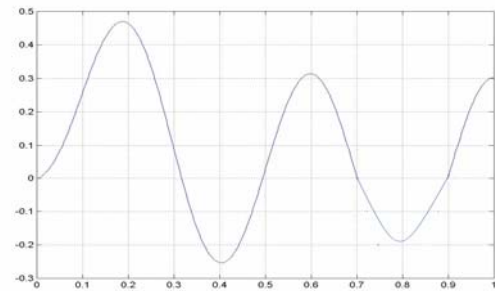


Fig. 3 Phase Error of Filter without exchange algorithm with constraints on magnitude and phase of errors

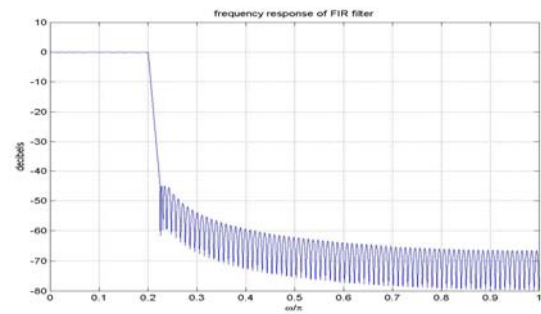


Fig. 4 Frequency Response using exchange algorithm

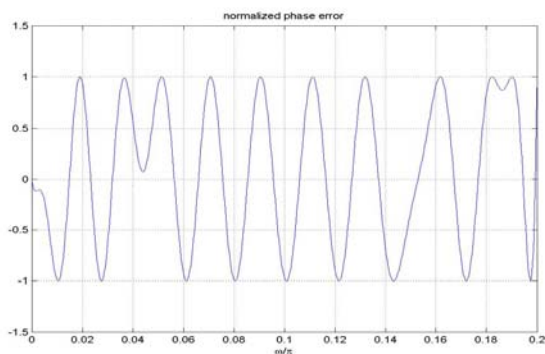


Fig. 5 Frequency response of filter with using

## V. CONCLUSION

The two drawbacks of constraints on magnitude and phase error without exchange algorithm first is the large number of constraints resulting in a high computational effort and in high memory requirements. The second drawback is the fact that replacing the nonlinear magnitude constraints by linear constraints introduces errors.

The stopband error energy is smaller for the filter designed with independent magnitude and phase constraints in exchange algorithm than simple constraints on magnitude and phase error.

Introduced exchange algorithms with constraints on magnitude and phase of error solve a sequence of small sub-problems in order to compute the optimum solution to the constrained filter design problem. This greatly reduces the computational effort and the memory requirements compared to the linear and quadratic programming approach with constraints on magnitude and phase of error. The errors resulting from this approximation are small as long as the phase constraint function is small. It can eliminate linearize error. Along with the above qualities it also makes ripples to die out.

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