World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:6, 2011

# Regular Generalized Star Star closed sets in Bitopological Spaces

K. Kannan, D. Narasimhan, K. Chandrasekhara Rao and R. Ravikumar

*Abstract*—The aim of this paper is to introduce the concepts of $\boldsymbol{\tau_1\tau_2}$-regular generalized star star closed sets , $\boldsymbol{\tau_1\tau_2}$-regular generalized star star open sets and study their basic properties in bitopological spaces.

*Keywords*—$\tau_1\tau_2$-regular closed sets; $\tau_1\tau_2$-regular open sets; $\tau_1\tau_2$-regular generalized closed sets; $\tau_1\tau_2$-regular generalized star closed sets; $\tau_1\tau_2$-regular generalized star star closed sets.



## I. Introduction

IN 1963, J.C. Kelly [11] initiated the study of bitopoloical spaces as a natural structure by studying quasi metrics and its conjugate. This structure ia a richer structure than that of a topological space. Considerable effort had been expended in obtaining appropriate generalizations of standard topological properties to bitopological category by various authors. Most of them deal with the theory itself but very few with applications.

K. Chandrasekhara Rao and N. Palaniappan [6] introduced the concepts of regular generalized star closed sets and regular generalized star open sets in a topological space and they are extended to bitopological settings by K. Chandrasekhara Rao and K. Kannan [1]. On the other hand Chandrasekhara Rao and N. Palaniappan [7] introduced the concept of regular generalized star star closed sets and regular generalized star star open sets in topological spaces and study their properties. In this sequel, we introduce the concepts of $\tau_1\tau_2$-regular generalized star star closed sets ($\tau_1\tau_2$-$rg^*$ closed sets) and $\tau_1\tau_2$-regular generalized star star open sets ($\tau_1\tau_2$-$rg^*$ open sets) and study their basic properties in bitopological spaces.

Throughout this paper, $(X, \tau_1, \tau_2)$ or simply $X$ denote a bitopological space. The intersection (resp. union) of all $\tau_i$-closed sets containing $A$ (resp. $\tau_i$-open sets contained in $A$) is called the $\tau_i$-closure (resp. $\tau_i$-interior) of $A$, denoted by $\tau_i$-cl($A$) {resp. $\tau_i$-int($A$)}. The closure and interior of $B$ relative to $A$ with respect to the topology $\tau_i$ are written as $\tau_i$-cl$_A(B)$ and $\tau_i$-int$_A(B)$ respectively.

For any subset $A \subseteq X$, $\tau_i$-rint($A$) and $\tau_i$-rcl($A$) denote the regular interior and regular closure of a set $A$ with respect to the topology $\tau_i$ respectively. The regular closure and regular interior of $B$ relative to $A$ with respect to the topology $\tau_i$ are written as $\tau_i$-rcl$_A(B)$ and $\tau_i$-rint$_A(B)$ respectively. The set of all $\tau_2$-regular closed sets in X is denoted by $\tau_2$-R.C$(X, \tau_1, \tau_2)$. The set of all $\tau_1\tau_2$ -regular open sets in X is denoted by $\tau_1\tau_2$-R.O$(X, \tau_1, \tau_2)$. $A^C$ denotes the complement of $A$ in $X$ unless explicitly stated.

K. Kannan, D. Narasimhan, K. Chandrasekhara Rao are with Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA University, Kumbakonam, India, E.Mail:anbukkannan@rediffmail.com, dnsastra@rediffmail.com and k.chandrasekhara@rediffmail.com

R. Ravikumar is with Department of Mathematics, A.M.A. College of Engineering, Vadamavandal, India, E.Mail: dr.ravikumar@rediffmail.com

We shall require the following known definitions and results:

*Definition 1.1:* [1] A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called

(a) $\tau_1\tau_2$-*regular closed* if $\tau_1$-cl[$\tau_2$-int$(A)$] $=A$.
(b) $\tau_1\tau_2$-*regular open* if $\tau_1$-int[$\tau_2$-cl$(A)$] $=A$.
(c) $\tau_1\tau_2$-*regular generalized closed* ($\tau_1\tau_2$-$rg$ closed) in $X$ if $\tau_2$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$.
(d) $\tau_1\tau_2$-*regular generalized open* ($\tau_1\tau_2$-$rg$ open) in $X$ if $F \subseteq \tau_2$-int$(A)$ whenever $F \subseteq A$ and $F$ is $\tau_1\tau_2$-regular closed in $X$.
(e) $\tau_1\tau_2$-*regular generalized star closed* ($\tau_1\tau_2$-$rg^*$ closed) in $X$ if and only if $\tau_2$-rcl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$.
(f) $\tau_1\tau_2$-*regular generalized star open* ($\tau_1\tau_2$-$rg^*$ open) in $X$ if and only if its complement is $\tau_1\tau_2$-*regular generalized star closed* ($\tau_1\tau_2$-$rg^*$ closed) in $X$.

*Lemma 1.2:* [1] Let $A$ be a $\tau_1$-open set in $(X, \tau_1, \tau_2)$ and let $U$ be $\tau_1\tau_2$-regular open in $A$. Then $U = A \cap W$ for some $\tau_1\tau_2$-regular open set $W$ in $X$.

*Lemma 1.3:* [1] If $A$ is $\tau_1\tau_2$-open and $U$ is $\tau_1\tau_2$-regular open in $X$ then $U \cap A$ is $\tau_1\tau_2$-regular open in $A$.

## II. $\tau_1\tau_2$-Regular generalized star star closed sets

*Definition 2.1:* A subset A of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-regular generalized star star closed ($\tau_1\tau_2$-$rg^{**}$ closed) in $X$ if and only if $\tau_2$-$cl[\tau_1$-$int(A)] \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$.

*Example 2.2:* Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then the set of all subsets $P(X)$ of $X$ are $\tau_1\tau_2$-$rg^{**}$ closed sets in $(X, \tau_1, \tau_2)$.

*Theorem 2.3:* Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. If $A$ is $\tau_1\tau_2$-$rg^{**}$ closed then $\tau_2$-$cl[\tau_1$-$int(A)] - A$ does not contain non empty $\tau_1\tau_2$-regular closed sets.

*Proof:* Suppose that $A$ is $\tau_1\tau_2$-$rg^{**}$ closed . Let $F$ be a $\tau_1\tau_2$-regular closed set such that $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)] - A$. Then $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)] \cap A^C$. Since $F \subseteq A^C$, we have $A \subseteq F^C$ . Since $F$ is $\tau_1\tau_2$-regular closed set, we have $F^C$ is $\tau_1\tau_2$-regular open . Since $A$ is $\tau_1\tau_2$-$rg^{**}$ closed, we have $\tau_2$-$cl[\tau_1$-$int(A)] \subseteq F^C$ . Therefore, $F \subseteq [\tau_2$-$cl[\tau_1$-$int(A)]]^C$. Also $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)]$ . Hence $F \subseteq \phi$ . Therefore, $F = \phi$. ■

*Theorem 2.4:* If $A$ is $\tau_1\tau_2$-$rg^{**}$ closed and $B$ is $\tau_1\tau_2$-$g$ closed, then $A \cup B$ is $\tau_1\tau_2$-$rg^{**}$ closed.

*Proof:* Let $A \cup B \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$. Since $A \subseteq U$ and $A$ is $\tau_1\tau_2$-$rg^{**}$ closed, we have

$\tau_2$-$cl[\tau_1$-$int(A)] \subseteq U$. Since $B \subseteq U$ and $B$ is $\tau_1\tau_2$-$g$ closed, we have $\tau_2$-$cl(B) \subseteq U$. Now, $\tau_2$-$cl[\tau_1$-$int(A \cup B)] \subseteq \tau_2$-$cl[\tau_1$-$int[A \cup \tau_2$-$cl(B)]] \subseteq U$. Therefore, $A \cup B$ is $\tau_1\tau_2$-$rg^{**}$ closed. ■

*Theorem 2.5:* If a subset $A$ is $\tau_1\tau_2$-$rg$ closed then $A$ is $\tau_1\tau_2$-$rg^{**}$ closed.

*Proof:* Let $A \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open. Since $A$ is $\tau_1\tau_2$-$rg$ closed, we have $\tau_2$-$cl(A) \subseteq U$. Hence $\tau_2$-$cl[\tau_1$-$int(A)] \subseteq U$ . Therefore, $A$ is $\tau_1\tau_2$-$rg^{**}$ closed. ■

*Definition 2.6:* Let $B \subseteq Y \subseteq X$. A subset $B$ of $Y$ is said to be $\tau_1\tau_2$-$rg^{**}$ closed relative to $Y$ if $B$ is $\tau_1\tau_2$-$rg^{**}$ closed in the subspace $Y$.

*Theorem 2.7:* Let $(Y, \tau_{1/Y}, \tau_{2/Y})$ be a subspace of $(X, \tau_1, \tau_2)$. Suppose that a subset $B$ of $Y$ is $\tau_1\tau_2$-$rg^{**}$ closed relative to $(Y, \tau_{1/Y}, \tau_{2/Y})$ and $Y$ is $\tau_1\tau_2$-open and $\tau_1\tau_2$-$g$ closed in $(X, \tau_1, \tau_2)$ then $B$ is $\tau_1\tau_2$-$rg^{**}$ closed in $(X, \tau_1, \tau_2)$.

*Proof:* Let $B \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$. Since $Y$ is $\tau_1\tau_2$-open in $X$, we have $U \cap Y$ is $\tau_1\tau_2$-regular open in $Y$ {by Lemma 1.3} and $B \subseteq U \cap Y$. Since $B$ is $\tau_1\tau_2$-$rg^{**}$ closed relative to $(Y, \tau_{1/Y}, \tau_{2/Y})$, we have $\tau_2$-$cl_Y[\tau_1$-$int_Y(B)] \subseteq U \cap Y$. Hence $\tau_2$-$cl[\tau_1$-$int(B)] \cap Y \subseteq U \cap Y$. Let $G = U \cup \{X - \tau_2$-$cl[\tau_1$-$int(B)]\ \}$. Then $G$ is $\tau_1$-open and $Y \subseteq G$. Since $Y$ is $\tau_1\tau_2$-$g$ closed, we have $\tau_2$-$cl(Y) \subseteq G$. Now, $\tau_2$-$cl[\tau_1$-$int(B]= \tau_2$-$cl(Y) \cap \tau_2$-$cl[\tau_1$-$int(B)] \subseteq G \cap \tau_2$-$cl[\tau_1$-$int(B)]= U \cap \tau_2$-$cl[\tau_1$-$int(B)] \cup \phi \subseteq U$. Therefore, $B$ is $\tau_1\tau_2$-$rg^{**}$ closed in $(X, \tau_1, \tau_2)$. ■

*Theorem 2.8:* Suppose that a subset $B$ of $Y$ is $\tau_1\tau_2$-$rg^{**}$ closed in $(X, \tau_1, \tau_2)$ and $Y$ is $\tau_1\tau_2$-open in $(X, \tau_1, \tau_2)$ then $B$ is $\tau_1\tau_2$-$rg^{**}$ closed relative to $(Y, \tau_{1/Y}, \tau_{2/Y})$.

*Proof:* Let $B \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $Y$. Since $Y$ is $\tau_1$-open, we have $U = Y \cap W$ for some $\tau_1\tau_2$-regular open set $W$ in $(X, \tau_1, \tau_2)$ {by Lemma 1.2}. Since $B \subseteq Y \cap W \subseteq W$ and $B$ is $\tau_1\tau_2$-$rg^{**}$ closed in $(X, \tau_1, \tau_2)$, we have $\tau_2$-$cl[\tau_1$-$int(B)] \subseteq W$. Therefore, $\tau_2$-$cl_Y[\tau_1$-$int_Y(B)] = \tau_2$-$cl[\tau_1$-$int(B)] \cap Y \subseteq W \cap Y = U$. Hence $B$ is $\tau_1\tau_2$-$rg^{**}$ closed relative to $Y$. ■

*Theorem 2.9:* Let $A$ and $B$ are subsets such that $A \subseteq B \subseteq \tau_2$-$cl[\tau_1$-$int(A)]$. If $A$ is $\tau_1\tau_2$-$rg^{**}$ closed, then $B$ is $\tau_1\tau_2$-$rg^{**}$ closed

*Proof:* Let $B \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$ . Since $A \subseteq B$, we have $A \subseteq U$. Since $A$ is $\tau_1\tau_2$-$rg^{**}$ closed, we have $\tau_2$-$cl[\tau_1$-$int(A)] \subseteq U$ . Since $B \subseteq \tau_2$-$cl[\tau_1$-$int(A)]$, we have $\tau_2$-$cl[\tau_1$-$int(B)] \subseteq \tau_2$-$cl(B) \subseteq \tau_2$-$cl[\tau_1$-$int(A)] \subseteq U$. Therefore, $B$ is $\tau_1\tau_2$-$rg^{**}$ closed. ■

*Theorem 2.10:* Suppose that $\tau_1\tau_2$-$R.O(X, \tau_1, \tau_2) \subseteq \tau_2$-$C(X, \tau_1, \tau_2)$. Then every subset of $X$ is $\tau_1\tau_2$-$rg^{**}$ closed.

*Proof:* Let $A$ be a subset of $X$. Let $A \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$. Since $\tau_1\tau_2$-$R.O(X, \tau_1, \tau_2) \subseteq \tau_2$-$C(X, \tau_1, \tau_2)$, we have $U$ is $\tau_2$-closed in $X$. Since $A \subseteq U$, we have $\tau_2$-$cl(A) \subseteq \tau_2$-$cl(U) = U$. Therefore, $\tau_2$-$cl[\tau_1$-$int(A)] \subseteq \tau_2$-$cl[A] \subseteq U$. Hence $A$ is $\tau_1\tau_2$-$rg^{**}$ closed . ■

## III. $\tau_1\tau_2$-REGULAR GENERALIZED STAR STAR OPEN SETS

*Definition 3.1:* A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-regular generalized star star open ($\tau_1\tau_2$-$rg^{**}$ open) in $X$ if and only if its complement is $\tau_1\tau_2$-regular generalized star star closed ($\tau_1\tau_2$-$rg^{**}$ closed) in $X$ .

*Example 3.2:* Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then the set of all subsets $P(X)$ are $\tau_1\tau_2$-$rg^{**}$ open sets in $(X, \tau_1, \tau_2)$.

A necessary and sufficient condition for a set $A$ to be a $\tau_1\tau_2$-$rg^{**}$ open set is obtained in the next theorem.

*Theorem 3.3:* A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_1\tau_2$-$rg^{**}$ open if and only if $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$ whenever $F \subseteq A$ and $F$ is $\tau_1\tau_2$-regular closed in $X$.

*Proof:* **Necessity:** Let $F \subseteq A$ and $F$ is $\tau_1\tau_2$-regular closed in $X$ . Then $A^C \subseteq F^C$ and $F^C$ is $\tau_1\tau_2$-regular open in $X$ . Since $A$ is $\tau_1\tau_2$-$rg^{**}$ open, we have $A^C$ is $\tau_1\tau_2$-$rg^{**}$ closed. Hence, $\tau_2$-$cl[\tau_1$-$int(A^C)] \subseteq F^C$ . Consequently, $[\tau_2$-$int[\tau_1$-$cl(A)]]^C \subseteq F^C$. Therefore, $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$.

**Sufficiency:** Let $A^C \subseteq U$ and $U$ is $\tau_1\tau_2$-regular open in $X$ . Then $U^C \subseteq A$ and $U^C$ is $\tau_1\tau_2$-regular closed in $X$. By our assumption, we have $U^C \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$. Hence $[\tau_2$-$int[\tau_1$-$cl(A)]]^C \subseteq U$. Therefore, $\tau_2$-$cl[\tau_1$-$int(A^C)] \subseteq U$. Consequently $A^C$ is $\tau_1\tau_2$-$rg^{**}$ closed. Hence $A$ is $\tau_1\tau_2$-$rg^{**}$ open. ■

*Theorem 3.4:* Let $A$ and $B$ be subsets such that $\tau_2$-$int[\tau_1$-$cl(A)] \subseteq B \subseteq A$. If $A$ is $\tau_1\tau_2$-$rg^{**}$ open, then $B$ is $\tau_1\tau_2$-$rg^{**}$ open.

*Proof:* Let $F \subseteq B$ and $F$ is $\tau_1\tau_2$-regular closed in $X$. Since $B \subseteq A$, we have $F \subseteq A$. Since $A$ is $\tau_1\tau_2$-$rg^{**}$ open, we have, $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)]$ {by Theorem 3.3}. Since $\tau_2$-$int[\tau_1$-$cl(A)] \subseteq B$, we have $\tau_2$-$int[\tau_2$-$int[\tau_1$-$cl(A)]] \subseteq \tau_2$-$int(B) \subseteq \tau_2$-$int[\tau_1$-$cl(B)]$. Hence $F \subseteq \tau_2$-$int[\tau_1$-$cl(A)] \subseteq \tau_2$-$int[\tau_1$-$cl(B)]$. Therefore, $B$ is $\tau_1\tau_2$-$rg^{**}$ open. ■

*Theorem 3.5:* If a subset $A$ is $\tau_1\tau_2$-$rg^{**}$ closed, then $\tau_2$-$cl[\tau_1$-$int(A)] - A$ is $\tau_1\tau_2$-$rg^{**}$ open .

*Proof:* Let $F \subseteq \tau_2$-$cl[\tau_1$-$int(A)] - A$ and $F$ is $\tau_1\tau_2$-regular closed. Since $A$ is $\tau_1\tau_2$-$rg^{**}$ closed, we have $\tau_2$-$cl[\tau_1$-$int(A)] - A$ does not contain nonempty $\tau_1\tau_2$-regular closed {by Theorem 2.3}. Therefore, $F = \phi$. Hence $\tau_2$-$cl[\tau_1$-$int(A)] - A$ is $\tau_1\tau_2$-$rg^{**}$ open. ■

## REFERENCES


[1] K. Chandrasekhara Rao and K. Kannan, *Regular generalized star closed sets in bitopological spaces*, Thai Journal of Mathematics, Vol 4, (2) (2006), 341–349.

[2] K. Chandrasekhara Rao and K. Kannan, *Generalized star regular closed sets in bitopological spaces*, Antarctica Math. J., 4, (2)(2007), 139–146.

[3] K. Chandrasekhara Rao, K. Kannan and D. Narasimhan, *Generalized star star regular closed sets,* Natl. Acad. Sci. Lett., Vol.31, No. 1 & 2 , (2008), 31–33.

[4] K. Chandrasekhara Rao and M. Mariasingam, *On Bitopological spaces*, Acta Ciencia Indica, Vol. XXVI M (4) (2000), 283–288.

[5] K. Chandrasekhara Rao and N. Palaniappan, *Regular generalized closed sets*, Kyungpook Mathematical Journal, 33(2)(1993) 211–219.

[6] K. Chandrasekhara Rao and N. Palaniappan, *Regular generalized star closed sets*, Bulletin of Pure and Applied Sciences, 19(2)(2000), 291–306.

[7] K. Chandrasekhara Rao and N. Planiappan, *Ph. D. Thesis,* Alagappa University, (2000).

[8] T. Fukutake, *On generalized closed sets in bitopological spaces,* Bull. Fukuoka Univ. Ed. Part III, 35 (1986), 19–28.

[9] T. Fukutake, *Semi open sets on bitopological spaces*, Bull. Fukuoka Uni. Education, 38(3)(1989), 1–7.

[10] T. Fukutake, P. Sundaram and M. Sheik John, *$w$-closed sets, $w$-open sets and $w$-continuity in bitopological spaces*, Bull. Fukuoka Univ. Ed. Part III, 51 (2002), 1–9.
[11] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Society, 13 (1963), 71–89.
[12] M. Rajamani and K. Vishwanathan, *On $\alpha$-generalized semi closed sets in bitopological spaces*, Bulletin of Pure and Applied Sciences, Vol. 24E (No.1) (2005), 39–53.
[13] M. Sheik John and P. Sundaram, *$g^*$-closed sets in Bitopological Spaces*, Indian J. Pure Appl. Math., 35(1) (2004), 71–80.