

# Optimal Control Problem, Quasi-Assignment Problem and Genetic Algorithm

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**Abstract**— In this paper we apply one of approaches in category of heuristic methods as Genetic Algorithms for obtaining approximate solution of optimal control problems. The first we convert optimal control problem to a quasi Assignment Problem by defining some usual characters as defined in Genetic algorithm applications. Then we obtain approximate optimal control function as an piecewise constant function. Finally the numerical examples are given.

**Keywords**— Optimal control, Integer programming, Genetic algorithm, Discrete approximation, Linear programming.

## I. INTRODUCTION

In early decade Optimal Control Problem (OCP) as one of the most application issues has been taken into consideration. The analytical solution for OCP's are not always available. Thus, to find an approximate solution is the most logical way to solve the problems. Various approaches as, discretization, shooting method [3], [4], using concepts in measure theory [5], using Chebyshev polynomials [2], etc., have been proposed to obtain approximate solutions of OCP's. Some heuristic algorithm and their extensions are applied in topic of control problems. our aim is to apply Genetic Algorithm (GA) to construct approximate optimal control function for OCP's. GA's are stochastic search technique inspired by the principles of natural selection and natural genetics which have revealed a number of characteristics particularly useful for application in optimization, engineering, computer science, among other fields. In this work, in order to apply GA to get an approximate solution for an OCP, we first convert The OCP to a quasi Assignment Problem (QASP) by a suitable linearizing. Then, we apply GA for the quasi assignment problem and we get an approximate solution as an piecewise constant function. Hence, the approximate solution obtained is a precise solution for the OCP.

## II. STATEMENT OF THE OPTIMAL CONTROL PROBLEM

Consider a classical OCP as follows:

$$\text{minimize } I(\mathbf{x}(t), u(t)) = \int_0^{t_f} f_o(t, \mathbf{x}(t), u(t)) \quad (1)$$

subject to

$$\dot{\mathbf{x}}(t) = g(t, \mathbf{x}(t), u(t)) \quad (2)$$

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$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f. \quad (3)$$

where  $t_f$  is known and  $|u| \leq K$ . Our main aim is trajectory planning i.e. it is to find a control function  $u(\cdot)$  such that the corresponding state  $\mathbf{x}(\cdot)$  satisfying (2)-(3) and minimize (1). In order to this work, first, we discretize the interval  $[0, t_f]$  to  $N$  subinterval  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, N$ . Since we tend to detect an approximate optimal control for problem (1)-(2), we focus on finding an optimal piecewise control function  $u(\cdot)$  for the problem. Corresponding each interval  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, N$ , we partition the interval  $[-k, k]$  to  $m$  equal subinterval  $[u_{j-1}, u_j]$ ,  $j = 1, 2, \dots, m$ , where  $u_0 = -K$  and  $u_m = K$ . Of course we need to find  $m+1$  constants  $u_0, u_1, \dots, u_m$ . Thus we consider the space of time and control piecewise constant segments as it is shown in Figure 1.

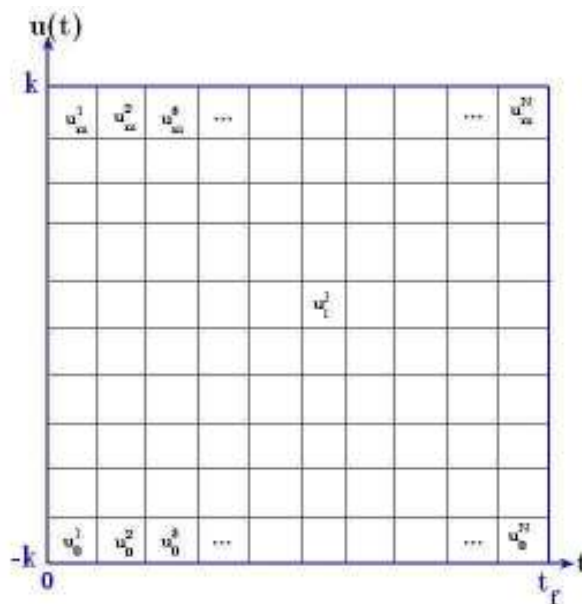


Figure. 1 A discrete form of control space.

One of approach for finding the best selection among of all constants in each interval of  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, N$ , and finally to find the best approximate control is enumerate of all cases. But we will deal with an extreme hard computational complexity. Because we must check  $N^m$  piecewise constant functions.

Our aim is to use of GA approach for detecting the best approximate piecewise constant control function.

If  $u(t) = \sum_{k=1}^N u_k [t_{k-1}, t_k](t)$  be a piecewise constant function then by a numerical method as Euler method or Rung-Kutta, we can find trajectory corresponding  $u(t)$  from (2) with initial condition  $\mathbf{x}(0) = \mathbf{x}_0$ . Thus, if  $(\hat{\mathbf{x}}, \hat{u})$  be a pair

of the trajectory and the control which satisfy in (2) with initial condition  $\mathbf{x}(0) = \mathbf{x}_0$  and for given a small number  $\epsilon > 0$ ,  $\|\hat{\mathbf{x}}(t_f) - \mathbf{x}_f\| \leq \epsilon$ , then we can claim that, we have been found a good approximate pair for minimizing functional  $J$  in (1).

### III. CONVERTING OCP TO QASP

To convert the OCP to QASP, we use a similar framework of solving QAP (Quadratic Assignment Problem) by ACO (Ant colony Optimization). In fact, we decide to assign a constant  $u_k \in \{u_0, u_1, \dots, u_m\}$  for every interval  $[t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, N$ . These constants are selected by GA toolbox in Matlab. The process will be continued until the objective function gets its optimal value. In other word, after discretion of the OCP, the problem is converted to a QASP with an extra objective function, i.e. we add the term,  $\|\hat{\mathbf{x}}(t_f) - \mathbf{x}_f\|$  to the original objective function and then, we apply GA for this new criteria function. Therefore, by applying the method above, the problem (1)-(2) with conditions (3) is converted to:

$$\begin{aligned} \text{minimize } J(\hat{\mathbf{x}}(t), \hat{u}(t)) &= \sum_{i=0}^N f_o(t_i, \hat{\mathbf{x}}(t_i), \hat{u}(t_i)) \\ &+ \mathbf{M} \|\hat{\mathbf{x}}(t_f) - \mathbf{x}_f\| \end{aligned} \quad (4)$$

subject to

$$\hat{\mathbf{x}}(t_{i+1}) = \hat{\mathbf{x}}(t_i) + g(t_i, \hat{\mathbf{x}}(t_i), \hat{u}(t_i)), \quad i = 0, 1, \dots, N, \quad (5)$$

$$\hat{\mathbf{x}}(0) = \mathbf{x}_0. \quad (6)$$

where,  $\mathbf{M}$  is a very large positive number (like as BIG-M Method). To obtain a numerical solution for the problem (4)-(5) with initial condition(6) via GA, we focus just on the space time and control. Because, we use of equation (5) in definition of the fitness function for GA. Now, we are ready to state the numerical algorithm to get the optimal control.

### IV. USING A GENETIC ALGORITHM

In this section, we present an algorithm based on GA, to obtain a solution for problem (4)-(5) with initial condition (6). Because of definition of the fitness function, the solution obtained of GA, is a reasonable approximate solution for original problem.

Corresponding to section 3, after discretizing the problem, including the objective function and the differential equation, to solve the problem via GA, we must add the term  $\|\hat{\mathbf{x}}(t_f) - \mathbf{x}_f\|$  to the new objective function along with a cost coefficient,  $\mathbf{M}$ . This process causes all of the conditions of the original problem be satisfied and also it guaranties the convergence of the approximated solution to the exact solution.

In this work, we use of Genetic Algorithm Toolbox of Matlab (Ver.7.04) to solve examples. The results obtained of using the toolbox are so satisfactory,too.

In this section we propose our method to obtain approximate solutions of some OCP. Before proposing of examples we define an error function as  $e(t_f) = \hat{\mathbf{x}}(t_f) - \mathbf{x}_f$  on  $[0, t_f]$ , where  $\mathbf{x}(t_f)$  and  $\mathbf{x}_f$  are the exact and the approximate solution obtained from each repetition of GA for the example, respectively.

In all examples, we set  $t_f = 1$ ,  $\mathbf{M} = 10^6$ ,  $K = 1$ , and we divide the closed interval  $[0, t_f](= [0, 1])$  into 16 partitions, i.e. the step size  $\Delta t = 0.0625$ . Also, we use Euler method to solve the condition corresponding to the differential equation of the problem.

*Example 1:* Consider the following OCP:

$$\begin{aligned} \text{minimize } J &= \int_0^1 u^2(t) dt \\ \text{subject to} \\ \dot{\mathbf{x}}(t) &= \mathbf{x}^2(t) + u(t) \\ \mathbf{x}(0) &= 0, \quad \mathbf{x}(1) = 0.5. \end{aligned}$$

After solving with the proposed method, we obtain the following results: The final value  $\mathbf{x}(1) = 0.500054$ , the optimal value  $J^* = 0.4447$  and the error function  $e(1) = 5.43 \times 10^{-5}$ . The trajectory and control functions are shown in figure 2 and 3, respectively.

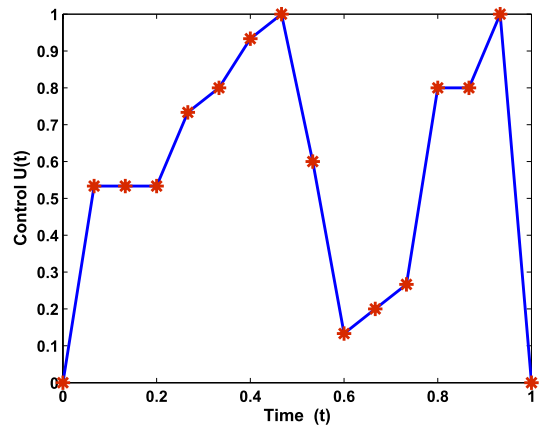


Figure. 2 Control function for Example 1.

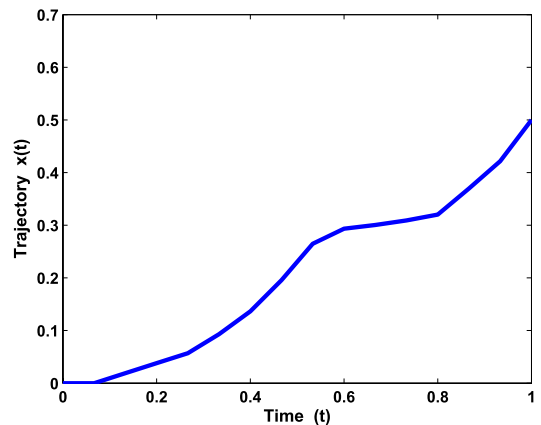


Figure. 2 Trajectory function for Example 1.

Example 2: Consider the following OCP:

$$\text{minimize } I = \int_0^1 u^2(t) dt$$

subject to

$$\dot{x}(t) = \frac{1}{2}x^2(t)\sin(x(t)) + u(t)$$

$$x(0) = 0, \quad x(1) = 0.5.$$

After solving with the proposed method, we obtain the following results:

The final value  $x(1) = 0.500040$ , the optimal value  $I^* = 0.3526$  and the error function  $e(1) = 4.095 \times 10^{-5}$ . The trajectory and control functions are shown in figure 4 and 5, respectively.

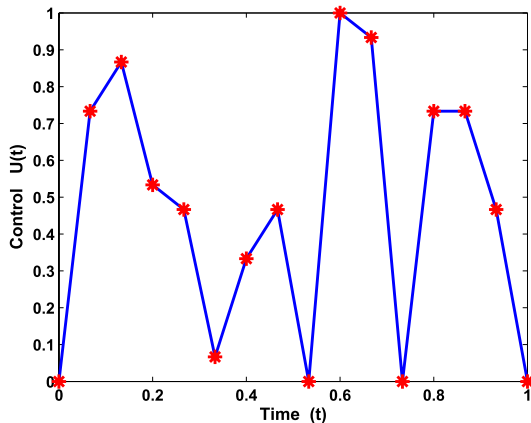


Figure. 4 Control function for Example 2.

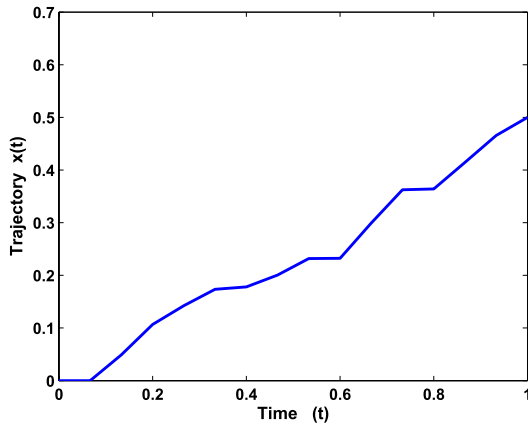


Figure. 5 Trajectory function for Example 2.

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