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Hutchinson-Barnsley Operator in Intuitionistic Fuzzy Metric Spaces

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Abstract—The main purpose of this paper is to prove the intuitionistic fuzzy contraction properties of the Hutchinson-Barnsley operator on the intuitionistic fuzzy hyperspace with respect to the Hausdorff intuitionistic fuzzy metrics. Also we discuss about the relationships between the Hausdorff intuitionistic fuzzy metrics on the intuitionistic fuzzy hyperspaces. Our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces to the intuitionistic fuzzy metric spaces.

Keywords—Contraction, Iterated Function System, Hutchinson-Barnsley Operator, Intuitionistic Fuzzy Metric Space, Hausdorff Intuitionistic Fuzzy Metric.

I. Introduction

Fuzzy set theory was introduced by Zadeh in 1965 [1]. Many authors have introduced and discussed several notions of fuzzy metric space in different ways [2], [3], [4] and also proved fixed point theorems with interesting consequent results in the fuzzy metric spaces [5]. Recently the concept of intuitionistic fuzzy metric space was given by Park [6] and the subsequent fixed point results in the intuitionistic fuzzy metric spaces are investigated by Alaca and et al. [7] and Mohamad [8].

The Fractal Analysis was introduced by Mandelbrot in 1975 [9] and popularized by various mathematicians [10], [11], [12]. Sets with non-integral Hausdorff dimension, which exceeds its topological dimension, are called Fractals by Mandelbrot [9]. Hutchinson [10] and Barnsley [11] initiated and developed the Hutchinson-Barnsley theory (HB theory) in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the Iterated Function System (IFS) of contractions. That is, Hutchinson introduced an operator on hyperspace called as Hutchinson-Barnsley operator (HB operator) to define a fractal set as a unique fixed point by using the Banach Contraction Theorem in the metric spaces. Recently in [13], [14]; HB operator properties were analyzed in fuzzy metric spaces. Here we introduce the concepts and properties of HB operator in the intuitionistic fuzzy metric spaces.

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In this paper, we prove the intuitionistic fuzzy contraction properties of the HB operator on the intuitionistic fuzzy hyperspace with respect to the Hausdorff intuitionistic fuzzy metrics. Also we discuss about the relationships between the Hausdorff intuitionistic fuzzy metrics on the intuitionistic fuzzy hyperspaces. Here our theorems generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces. This paper will help us to develop the HB Theory in order to define a fractal set in the intuitionistic fuzzy metric spaces as a unique fixed point of the Fuzzy HB operator.

II. HB Operator in Metric Space

In this section, we recall the Hutchinson-Barnsley theory (HB theory) to define HB operator in the metric space.

Definition II.1. *([11], [12]) Let (X, d) be a metric space and $\mathscr{K}_o(X)$ be the collection of all non-empty compact subsets of X.*

Define, $d(x, B) := \inf_{y\in B} d(x, y)$ and $d(A, B) := \sup_{x\in A} d(x, B)$ for all $x \in X$ and $A, B \in \mathscr{K}_o(X)$. The Hausdorff metric or Hausdorff distance (H_d) is a function $H_d : \mathscr{K}_o(X) \times \mathscr{K}_o(X) \longrightarrow \mathbb{R}$ defined by

$$H_d(A, B) = \max\Big\{d(A, B), d(B, A)\Big\}.$$

Then H_d is a metric on the hyperspace of compact sets $\mathscr{K}_o(X)$ and hence $(\mathscr{K}_o(X), H_d)$ is called a Hausdorff metric space.

Theorem II.1. *([11], [12]) If (X, d) is a complete metric space, then $(\mathscr{K}_o(X), H_d)$ is also a complete metric space.*

Definition II.2. *([10], [11]) Let (X, d) be a metric space and $f_n : X \longrightarrow X$, $n = 1, 2, 3, ..., N_o$ ($N_o \in \mathbb{N}$) be N_o - contraction mappings with the corresponding contractivity ratios k_n, $n = 1, 2, 3, ..., N_o$. The system $\{X; f_n,\ n = 1, 2, 3, ..., N_o\}$ is called an Iterated Function System (IFS) or Hyperbolic Iterated Function System with the ratio $k = \max_{n=1}^{N_o} k_n$. Then the Hutchinson-Barnsley operator (HB operator) of the IFS is a function $F : \mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ defined by*

$$F(B) = \bigcup_{n=1}^{N_o} f_n(B), \quad \text{for all} \quad B \in \mathscr{K}_o(X).$$

Theorem II.2. *([10], [11]) Let (X, d) be a metric space. Let $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IFS. Then, the HB operator (F) is a contraction mapping on $(\mathscr{K}_o(X), H_d)$.*

Theorem II.3. *(HB Theorem [10], [11]) Let (X, d) be a complete metric space and $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IFS. Then, there exists only one compact invariant set $A_\infty \in \mathscr{K}_o(X)$ of the HB operator (F) or, equivalently, F has a unique fixed point namely $A_\infty \in \mathscr{K}_o(X)$.*

Definition II.3. *([11]) The fixed point $A_\infty \in \mathscr{K}_o(X)$ of the HB operator F described in the Theorem II.3 is called the Attractor (Fractal) of the IFS. Sometimes $A_\infty \in \mathscr{K}_o(X)$ is called as Fractal generated by the IFS and so called as IFS Fractal.*

III. Intuitionistic Fuzzy Metric Space

In [1], Zadeh defined a fuzzy set on X as a function $f : X \longrightarrow [0, 1]$. In order to define the Intuitionistic Fuzzy HB operator, we have to state the required concepts of intuitionistic fuzzy metric spaces as follows:

Definition III.1. *([15]) A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm, if $*$ satisfying the following conditions:*

(a) *$*$ is commutative and associative;*
(b) *$*$ is continuous;*
(c) *$a * 1 = a$ for all $a \in [0, 1]$;*
(d) *$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.*

Definition III.2. *([15]) A binary operation $\diamondsuit : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-conorm, if $\diamondsuit$ satisfying the following conditions:*

(a) *$\diamondsuit$ is commutative and associative;*
(b) *$\diamondsuit$ is continuous;*
(c) *$a\diamondsuit 0 = a$ for all $a \in [0, 1]$;*
(d) *$a\diamondsuit b \leq c\diamondsuit d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.*

Remarks:

(a) If * is a continuous t-norm, it follows from the Definition III.1 that for every $a \in [0, 1], 0 * a \leq 0 * 1 = 0$ and so $0 * a = a * 0 = 0$.
(b) If $\diamondsuit$ is a continuous t-conorm, it follows from the Definition III.2 that for every $a \in [0, 1], 1 = 0\diamondsuit 1 \leq a\diamondsuit 1$ and so $a\diamondsuit 1 = 1\diamondsuit a = 1$.

Definition III.3. *([6], [16]) A 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions:*

(a) $M(x, y, t) + N(x, y, t) \leq 1$;
(b) $M(x, y, t) > 0$;
(c) $M(x, y, t) = 1$ *if and only if* $x = y$;
(d) $M(x, y, t) = M(y, x, t)$;
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(f) $M(x, y, \cdot) : (0, \infty) \longrightarrow (0, 1]$ *is continuous;*
(g) $N(x, y, t) < 1$;
(h) $N(x, y, t) = 0$ *if and only if* $x = y$;
(i) $N(x, y, t) = N(y, x, t)$;
(j) $N(x, y, t)\diamondsuit N(y, z, s) \geq N(x, z, t + s)$;
(k) $N(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1)$ *is continuous;*

for all $x, y, z \in X$ and $t, s > 0$.

Then $(M, N, *, \diamondsuit)$ or simply (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ represents the degree of nearness and the degree of non-nearness between x and y in X with respect to t, respectively.

Definition III.4. *([6]) Let (X, d) be a metric space. Let M_d and N_d be the functions defined on $X^2 \times (0, \infty)$ by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)},$$

for all $x, y \in X$ and $t > 0$. Then $(X, M_d, N_d, *, \diamondsuit)$ is an intuitionistic fuzzy metric space, which is called standard intuitionistic fuzzy metric space, and (M_d, N_d) is called as the standard intuitionistic fuzzy metric induced by the metric d.

Definition III.5. *([6]) Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. The open ball $B(x, r, t)$ with center $x \in X$ and radius $r, 0 < r < 1$, with respect to $t > 0$, is defined as*

$$B(x, r, t) = \Big\{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\Big\}.$$

Define

$$\tau_{(M,N)} \;=\; \{A \subset X : for\ each\ x \in A, \exists\ t > 0\ and\ r \in (0, 1)\ such\ that\ B(x, r, t) \subset A\}.$$

Then $\tau_{(M,N)}$ is a topology on X induced by an intuitionistic fuzzy metric (M, N).

The topologies induced by the metric and the corresponding standard intuitionistic fuzzy metric are the same.

Proposition III.1. *([8]) The metric space (X, d) is complete if and only if the standard intuitionistic fuzzy metric space $(X, M_d, N_d, *, \diamondsuit)$ is complete.*

Definition III.6. *A intuitionistic fuzzy B-contraction (intuitionistic fuzzy Sehgal contraction) on an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ is a self-mapping f on X for which*

$$M(f(x), f(y), kt) \geq M(x, y, t)$$

and

$$N(f(x), f(y), kt) \leq N(x, y, t),$$

for all $x, y \in X$ and $t > 0$, where k is a fixed constant in $(0, 1)$.

A. Hausdorff Intuitionistic Fuzzy Metric Space

In [16], Gregori et al. defined the Hausdorff intuitionistic fuzzy metric on intuitionistic fuzzy hyperspace $\mathscr{K}_o(X)$ and constructed the Hausdorff intuitionistic fuzzy metric space as follows.

Definition III.7. *([16]) Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology induced by the intuitionistic fuzzy metric (M, N).*

We shall denote by $\mathscr{K}_o(X)$, the set of all non-empty compact subsets of $\big(X, \tau_{(M,N)}\big)$.

Define,

$$\begin{aligned} M(x,B,t) &:= \sup_{y\in B} M(x,y,t), \\ M(A,B,t) &:= \inf_{x\in A} M(x,B,t) \\ &\textit{and} \\ N(x,B,t) &:= \inf_{y\in B} N(x,y,t), \\ N(A,B,t) &:= \sup_{x\in A} N(x,B,t) \end{aligned}$$

for all $x \in X$ and $A, B \in \mathscr{K}_o(X)$.

Then we define the Hausdorff intuitionistic fuzzy metric $(H_M, H_N, *, \diamondsuit)$ as

$$H_M(A,B,t) = \min\Big\{M(A,B,t), M(B,A,t)\Big\}$$

and

$$H_N(A,B,t) = \max\Big\{N(A,B,t), N(B,A,t)\Big\}.$$

Here (H_M, H_N) is an intuitionistic fuzzy metric on the hyperspace of compact sets, $\mathscr{K}_o(X)$, and hence $\Big(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit\Big)$ is called a Hausdorff intuitionistic fuzzy metric space.

In [17], Rodriguez-Lopez and Romaguera proved some results and examples for fuzzy metric spaces. Here we generalize their results and examples for intuitionistic fuzzy metric space.

Proposition III.2. *Let (X,d) be a metric space. Then, the Hausdorff intuitionistic fuzzy metric (H_{M_d}, H_{N_d}) of the standard intuitionistic fuzzy metric (M_d, N_d) coincides with the standard intuitionistic fuzzy metric (M_{H_d}, N_{H_d}) of the Hausdorff metric (H_d) on $\mathscr{K}_o(X)$, i.e., $H_{M_d}(A,B,t) = M_{H_d}(A,B,t)$ and $H_{N_d}(A,B,t) = N_{H_d}(A,B,t)$ for all $A, B \in \mathscr{K}_o(X)$ and $t > 0$.*

Proof: Fix $t > 0$ and let $A, B \in \mathscr{K}_o(X)$.

We recall that

$$\sup_{b\in B} M_d(a,b,t) = \frac{t}{t + \inf_{b\in B} d(a,b)}$$

and

$$\inf_{b\in B} N_d(a,b,t) = \frac{1}{1 + \frac{t}{\inf_{b\in B} d(a,b)}}$$

for all $a \in A$.

It follows that

$$M_d(a,B,t) = \frac{t}{t + d(a,B)}$$

and

$$N_d(a,B,t) = \frac{1}{1 + \frac{t}{d(a,B)}}$$

for all $a \in A$. Then

$$\inf_{a\in A} M_d(a,B,t) = \frac{t}{t + \sup_{a\in A} d(a,B)}$$

and

$$\sup_{a\in A} N_d(a,B,t) = \frac{1}{1 + \frac{t}{\sup_{a\in A} d(a,B)}}.$$

It follows that

$$M_d(A,B,t) = \frac{t}{t + d(A,B)}$$

and

$$N_d(A,B,t) = \frac{1}{1 + \frac{t}{d(A,B)}} = \frac{d(A,B)}{t + d(A,B)}.$$

Similarly, we obtain

$$M_d(B,A,t) = \frac{t}{t + d(B,A)}$$

and

$$N_d(B,A,t) = \frac{d(B,A)}{t + d(B,A)}.$$

Therefore

$$H_{M_d}(A,B,t) = M_{H_d}(A,B,t)$$

and

$$H_{N_d}(A,B,t) = N_{H_d}(A,B,t).$$

The proof is complete. ■

Using the Proposition III.2, we can easily compute distances with respect to the Hausdorff intuitionistic fuzzy metric (H_{M_d}, H_{N_d}) of a standard intuitionistic fuzzy metric (M_d, N_d) by computing distances with respect to the Hausdorff metric (H_d) implied by the metric d. Here we illustrate this situation with two examples.

Example III.1. *Let $(\mathbb{R}, d)$ be the Euclidean metric space and let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two compact intervals of $\mathbb{R}$. Then $d(A,B) = |a_1 - b_1|$ and $d(B,A) = |a_2 - b_2|$ and hence $H_d(A,B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$; so, by Proposition III.2, we have*

$$H_{M_d}(A,B,t) = \frac{t}{t + \max\Big\{|a_1 - b_1|, |a_2 - b_2|\Big\}}$$

and

$$H_{N_d}(A,B,t) = \frac{\max\Big\{|a_1 - b_1|, |a_2 - b_2|\Big\}}{t + \max\Big\{|a_1 - b_1|, |a_2 - b_2|\Big\}},$$

for all $t > 0$.

Example III.2. *Let (X,d) be the discrete metric space such that $|X| \geq 2$. Let A and B be two non-empty finite subsets of X, with $A \neq B$. Then $d(A,B) = 1 = d(B,A)$ and hence $H_d(A,B) = 1$; so, by Proposition III.2, we have*

$$H_{M_d}(A,B,t) = \frac{t}{t+1} \quad \text{and} \quad H_{N_d}(A,B,t) = \frac{1}{1+t},$$

for all $t > 0$.

Definition III.8. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology induced by (M,N). We observe that, $\Big(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_M}, \mathscr{H}_{H_N}, *, \diamondsuit\Big)$ is also an intuitionistic fuzzy metric space, where $\mathscr{K}_o(\mathscr{K}_o(X))$ is the hyperspace of all non-empty compact subsets of $(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit)$ and $(\mathscr{H}_{H_M}, \mathscr{H}_{H_N})$ is the Hausdorff intuitionistic fuzzy metric on $\mathscr{K}_o(\mathscr{K}_o(X))$ implied by the*

Hausdorff intuitionistic fuzzy metric (H_M, H_N) *on* $\mathscr{K}_o(X)$. *That is, for all* $A \in \mathscr{K}_o(X)$ *and* $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$,

$$\mathscr{H}_{H_M}(\mathscr{A}, \mathscr{B}) = \min\Big\{H_M(\mathscr{A}, \mathscr{B}), H_M(\mathscr{B}, \mathscr{A})\Big\}$$
and
$$\mathscr{H}_{H_N}(\mathscr{A}, \mathscr{B}) = \max\Big\{H_N(\mathscr{A}, \mathscr{B}), H_N(\mathscr{B}, \mathscr{A})\Big\},$$

where

$$\begin{aligned} H_M(\mathscr{A}, \mathscr{B}) &:= \inf_{A\in\mathscr{A}} H_M(A, \mathscr{B}), \\ H_M(A, \mathscr{B}) &:= \sup_{B\in\mathscr{B}} H_M(A, B) \end{aligned}$$
and
$$\begin{aligned} H_N(\mathscr{A}, \mathscr{B}) &:= \sup_{A\in\mathscr{A}} H_N(A, \mathscr{B}), \\ H_N(A, \mathscr{B}) &:= \inf_{B\in\mathscr{B}} H_N(A, B). \end{aligned}$$

Proposition III.3. *Let* (X, d) *be a metric space and let* $(\mathscr{K}_o(X), H_d)$ *and* $\Big(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_d}\Big)$ *be the corresponding Hausdorff metric spaces. Then, the Hausdorff intuitionistic fuzzy metric* $(\mathscr{H}_{\mathscr{M}_{H_d}}, \mathscr{H}_{\mathscr{N}_{H_d}})$ *of the standard fuzzy metric* $(\mathscr{M}_{H_d}, \mathscr{N}_{H_d})$ *coincides with the standard intuitionistic fuzzy metric* $(\mathscr{M}_{\mathscr{H}_{H_d}}, \mathscr{N}_{\mathscr{H}_{H_d}})$ *of the Hausdorff metric* $(\mathscr{H}_{H_d})$ *on* $\mathscr{K}_o(\mathscr{K}_o(X))$, *i.e.,* $\mathscr{H}_{\mathscr{M}_{H_d}}(\mathscr{A}, \mathscr{B}, t) = \mathscr{M}_{\mathscr{H}_{H_d}}(\mathscr{A}, \mathscr{B}, t)$ *and* $\mathscr{H}_{\mathscr{N}_{H_d}}(\mathscr{A}, \mathscr{B}, t) = \mathscr{N}_{\mathscr{H}_{H_d}}(\mathscr{A}, \mathscr{B}, t)$ *for all* $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$ *and* $t > 0$.

Proof: Proposition III.2 completes the proof. ■

IV. Intuitionistic Fuzzy HB Operator

In this section, we define the Intuitionistic Fuzzy IFS and Intuitionistic Fuzzy HB operator on the intuitionistic fuzzy metric spaces.

Definition IV.1. *Let* $(X, M, N, *, \diamondsuit)$ *be an intuitionistic fuzzy metric space and* $f_n : X \longrightarrow X,\ n = 1, 2, 3, ..., N_o\ (N_o \in \mathbb{N})$ *be* N_o *- intuitionistic fuzzy B-contractions. Then the system* $\{X; f_n,\ n = 1, 2, 3, ..., N_o\}$ *is called an Intuitionistic Fuzzy Iterated Function System (IF-IFS) of intuitionistic fuzzy B-contractions in the intuitionistic fuzzy metric space* $(X, M, N, *, \diamondsuit)$.

Definition IV.2. *Let* $(X, M, N, *, \diamondsuit)$ *be an intuitionistic fuzzy metric space. Let* $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ *be an IF-IFS of intuitionistic fuzzy B-contractions. Then the Intuitionistic Fuzzy Hutchinson-Barnsley operator (IF-HB operator) of the IF-IFS is a function* $F : \mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ *defined by*

$$F(B) = \bigcup_{n=1}^{N_o} f_n(B), \quad \text{for all} \quad B \in \mathscr{K}_o(X).$$

Definition IV.3. *Let* $(X, M, N, *, \diamondsuit)$ *be a complete intuitionistic fuzzy metric space. Let* $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ *be an IF-IFS of intuitionistic fuzzy B-contractions and* F *be the IF-HB operator of the IF-IFS. We say that the set* $A_\infty \in \mathscr{K}_o(X)$ *is Intuitionistic Fuzzy Attractor (Intuitionistic Fuzzy Fractal) of the given IF-IFS, if* A_∞ *is a unique fixed point of the IF-HB operator* F. *Such* $A_\infty \in \mathscr{K}_o(X)$ *is also called as Fractal generated by the IF-IFS and so called as IF-IFS Fractal of intuitionistic fuzzy B-contractions.*

V. Properties of IF-HB Operator

Now we prove the interesting results about the intuitionistic fuzzy B-contraction properties of operators with respect to the Hausdorff intuitionistic fuzzy metric on $\mathscr{K}_o(X)$.

Theorem V.1. *Let* (X, d) *be a metric space. Let* $f : X \longrightarrow X$ *be a contraction function on* (X, d), *with a contractivity ratio* k. *Then,*

$$H_{M_d}\Big(f(A), f(B), t\Big) \geq H_{M_d}\Big(A, B, t\Big)$$
and
$$H_{N_d}\Big(f(A), f(B), t\Big) \leq H_{N_d}\Big(A, B, t\Big)$$

for all $A, B \in \mathscr{K}_o(X)$ *and* $t > 0$.

Proof: Fix $t > 0$ and let $A, B \in \mathscr{K}_o(X)$. Since f is contraction on (X, d) with the contractivity ratio $k \in (0, 1)$ and by Theorem II.2 for the case $N = 1$, we have

$$H_d(f(A), f(B)) \leq kH_d(A, B).$$

Since $t > 0$ and $k \in (0, 1)$,

$$\frac{kt}{kt + H_d(f(A), f(B))} \geq \frac{kt}{kt + kH_d(A, B)} = \frac{t}{t + H_d(A, B)}$$
and
$$\frac{H_d(f(A), f(B))}{kt + H_d(f(A), f(B))} \leq \frac{kH_d(A, B)}{kt + kH_d(A, B)} = \frac{H_d(A, B)}{t + H_d(A, B)}.$$

By using the above inequalities and the Proposition III.2, we have

$$\begin{aligned} H_{M_d}(f(A), f(B), kt) &= M_{H_d}(f(A), f(B), kt) \\ &= \frac{kt}{kt + H_d(f(A), f(B))} \\ &\geq \frac{t}{t + H_d(A, B)} \\ &= M_{H_d}(A, B, t) = H_{M_d}(A, B, t). \end{aligned}$$

Similarly,

$$\begin{aligned} H_{N_d}(f(A), f(B), kt) &= N_{H_d}(f(A), f(B), kt) \\ &= \frac{kH_d(f(A), f(B))}{kt + H_d(f(A), f(B))} \\ &\leq \frac{H_d(A, B)}{t + H_d(A, B)} \\ &= N_{H_d}(A, B, t) = H_{N_d}(A, B, t). \end{aligned}$$

■

The above Theorem V.1 shows that f is a intuitionistic fuzzy B-contraction on $\mathscr{K}_o(X)$ with respect to the Hausdorff intuitionistic fuzzy metric (H_{M_d}, H_{N_d}) implied by the standard fuzzy metric (M_d, N_d), if f is contraction on a metric space (X, d). The following theorem is somewhat generalization of the Theorem V.1.

Theorem V.2. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Let $\Big(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit\Big)$ be the corresponding Hausdorff intuitionistic fuzzy metric space. Suppose $f : X \longrightarrow X$ is an intuitionistic fuzzy B-Contraction on $(X, M, N, *, \diamondsuit)$. Then for $k \in (0, 1)$,*

$$H_M(f(A), f(B), kt) \geq H_M(A, B, t)$$

and

$$H_N(f(A), f(B), kt) \leq H_N(A, B, t)$$

for all $A, B \in \mathscr{K}_o(X)$ and $t > 0$.

Proof: Fix $t > 0$. Let $A, B \in \mathscr{K}_o(X)$.
For given $k \in (0, 1)$, we get

$$\begin{aligned}
M(f(x), f(y), kt) &\geq M(x, y, t), \quad \forall x, y \in X\\
M(f(x), f(y), kt) &\geq M(x, y, t), \quad \forall x \in A \ \&\ y \in B\\
\sup_{y\in B} M(f(x), f(y), kt) &\geq \sup_{y\in B} M(x, y, t), \quad \forall x \in A\\
M(f(x), f(B), kt) &\geq M(x, B, t), \quad \forall x \in A\\
\inf_{x\in A} M(f(x), f(B), kt) &\geq \inf_{x\in A} M(x, B, t)\\
M(f(A), f(B), kt) &\geq M(A, B, t).
\end{aligned}$$

Similarly, $M(f(B), f(A), kt) \geq M(B, A, t)$.
Hence $H_M\Big(f(A), f(B), kt\Big) \geq H_M\Big(A, B, t\Big)$.
Now,

$$\begin{aligned}
N(f(x), f(y), kt) &\leq N(x, y, t), \quad \forall x, y \in X\\
N(f(x), f(y), kt) &\leq N(x, y, t), \quad \forall x \in A \ \&\ y \in B\\
\inf_{y\in B} N(f(x), f(y), kt) &\leq \inf_{y\in B} N(x, y, t), \quad \forall x \in A\\
N(f(x), f(B), kt) &\leq N(x, B, t), \quad \forall x \in A\\
\sup_{x\in A} N(f(x), f(B), kt) &\leq \sup_{x\in A} N(x, B, t)\\
N(f(A), f(B), kt) &\leq N(A, B, t).
\end{aligned}$$

Similarly, $N(f(B), f(A), kt) \leq N(B, A, t)$.
Hence $H_N\Big(f(A), f(B), kt\Big) \leq H_N\Big(A, B, t\Big)$.
This completes the proof. ∎

The above Theorem V.2 shows that f is a intuitionistic fuzzy B-contraction function on $\mathscr{K}_o(X)$ with respect to the Hausdorff intuitionistic fuzzy metric (H_M, H_N), if f is intuitionistic fuzzy B-contraction on an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$.

Lemma V.1. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. If $B, C \subset X$ such that $B \subset C$, then $M(x, B, t) \leq M(x, C, t)$ and $N(x, B, t) \geq N(x, C, t)$ for all $x \in X$ and $t > 0$.*

Proof: Fix $t > 0$. Let $x \in X$ and $B, C \subset X$ such that $B \subset C$.
Then,

$$M(x, B, t) = \sup_{b\in B} M(x, b, t) \leq \sup_{b\in C} M(x, b, t) = M(x, C, t)$$

and

$$N(x, B, t) = \inf_{b\in B} N(x, b, t) \geq \inf_{b\in C} N(x, b, t) = N(x, C, t).$$

∎

Lemma V.2. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. If $B, C \subset X$ such that $B \subset C$, then $M(A, B, t) \leq M(A, C, t)$ and $N(A, B, t) \geq N(A, C, t)$ for all $A \subset X$ and $t > 0$.*

Proof: Fix $t > 0$. Let $A, B, C \subset X$ such that $B \subset C$.
By the Lemma V.1, we have

$$\begin{aligned}
M(A, B, t) &= \inf_{a\in A} M(a, B, t)\\
M(A, B, t) &\leq M(a, B, t), \forall a \in A\\
M(A, B, t) &\leq M(a, C, t), \forall a \in A\\
M(A, B, t) &\leq \inf_{a\in A} M(a, C, t)\\
M(A, B, t) &\leq M(A, C, t).
\end{aligned}$$

Similarly by the Lemma V.1,

$$\begin{aligned}
N(A, B, t) &= \sup_{a\in A} N(a, B, t)\\
N(A, B, t) &\geq N(a, B, t), \forall a \in A\\
N(A, B, t) &\geq N(a, C, t), \forall a \in A\\
N(A, B, t) &\geq \sup_{a\in A} N(a, C, t)\\
N(A, B, t) &\geq N(A, C, t).
\end{aligned}$$

∎

Lemma V.3. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. If $A, B, C \subset X$, then*

$$M\Big(A \cup B, C, t\Big) = \min\Big\{M(A, C, t), M(B, C, t)\Big\}$$

and

$$N\Big(A \cup B, C, t\Big) = \max\Big\{N(A, C, t), N(B, C, t)\Big\}$$

for all $t > 0$.

Proof: Fix $t > 0$. Let $A, B, C \subset X$.
Then,

$$\begin{aligned}
&M\Big(A \cup B, C, t\Big)\\
&\quad = \inf_{x\in A\cup B} M(x, C, t)\\
&\quad = \min\Big\{\inf_{a\in A} M(a, C, t), \inf_{b\in B} M(b, C, t)\Big\}\\
&\quad = \min\Big\{M(A, C, t), M(B, C, t)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&N\Big(A \cup B, C, t\Big)\\
&\quad = \sup_{x\in A\cup B} N(x, C, t)\\
&\quad = \max\Big\{\sup_{a\in A} N(a, C, t), \sup_{b\in B} N(b, C, t)\Big\}\\
&\quad = \max\Big\{N(A, C, t), N(B, C, t)\Big\}.
\end{aligned}$$

∎

Lemma V.4. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Let $\Big(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit\Big)$ be the corresponding Hausdorff intuitionistic fuzzy metric space. If $A, B, C, D \in \mathscr{K}_o(X)$, then*

$$H_M\Big(A \cup B, C \cup D, t\Big) \geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\}$$

and

$$H_N\Big(A \cup B, C \cup D, t\Big) \leq \max\Big\{H_N(A, C, t), H_N(B, D, t)\Big\}$$

for all $t > 0$.

Proof: Fix $t > 0$. Let $A, B, C, D \in \mathscr{K}_o(X)$. By using the Lemmas V.2 and V.3, we get

$$\begin{aligned} M\Big(A \cup B, C \cup D, t\Big) &= \min\Big\{M(A, C \cup D, t), M(B, C \cup D, t)\Big\} \\ &\geq \min\Big\{M(A, C, t), M(B, D, t)\Big\} \\ &\geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\}. \end{aligned}$$

Similarly,

$$M\Big(C \cup D, A \cup B, t\Big) \geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\}.$$

Hence,
$$H_M\Big(A \cup B, C \cup D, t\Big) \geq \min\Big\{H_M(A, C, t), H_M(B, D, t)\Big\}.$$

Again by using the Lemmas V.2 and V.3, we get

$$\begin{aligned} N\Big(A \cup B, C \cup D, t\Big) &= \max\Big\{N(A, C \cup D, t), N(B, C \cup D, t)\Big\} \\ &\leq \max\Big\{N(A, C, t), N(B, D, t)\Big\} \\ &\leq \max\Big\{H_N(A, C, t), H_N(B, D, t)\Big\}. \end{aligned}$$

Similarly,
$$N\Big(C \cup D, A \cup B, t\Big) \leq \max\Big\{H_N(A, C, t), H_N(B, D, t)\Big\}.$$

Hence,
$$H_N\Big(A \cup B, C \cup D, t\Big) \leq \max\Big\{H_N(A, C, t), H_N(B, D, t)\Big\}.$$

This completes the proof. ∎

The following theorem is a generalized version of the Theorem V.2.

Theorem V.3. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Let $\Big(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit\Big)$ be the corresponding Hausdorff intuitionistic fuzzy metric space. Suppose $f_n : X \longrightarrow X, n = 1, 2, ..., N_o; N_o \in \mathbb{N}$, is an intuitionistic fuzzy B-Contraction on $(X, M, N, *, \diamondsuit)$. Then the intuitionistic fuzzy HB operator is also an intuitionistic fuzzy B-Contraction on $\Big(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit\Big)$.*

Proof: Fix $t > 0$. Let $A, B \in \mathscr{K}_o(X)$. By using the Lemma V.4 and the Theorem V.2 for a given $k \in (0, 1)$, we get

$$\begin{aligned} H_M\Big(F(A), F(B), kt\Big) &= H_M\left(\bigcup_{n=1}^{N_o} f_n(A), \bigcup_{n=1}^{N_o} f_n(B), kt\right) \\ &\geq \min_{n=1}^{N_o} H_M\Big(f_n(A), f_n(B), kt\Big) \\ &\geq H_M\Big(A, B, t\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} H_N\Big(F(A), F(B), kt\Big) &= H_N\left(\bigcup_{n=1}^{N_o} f_n(A), \bigcup_{n=1}^{N_o} f_n(B), kt\right) \\ &\leq \max_{n=1}^{N_o} H_N\Big(f_n(A), f_n(B), kt\Big) \\ &\leq H_N\Big(A, B, t\Big). \end{aligned}$$

This completes the proof. ∎

From the above Theorem V.3, we conclude that the operator F is a intuitionistic fuzzy B-contraction on $\mathscr{K}_o(X)$ with respect to the Hausdorff intuitionistic fuzzy metric (H_M, H_N), if f_n is intuitionistic fuzzy B-contraction on an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ for each $n \in \{1, 2, 3, ..., N_o\}$.

VI. Hausdorff Intuitionistic Fuzzy Metrics on $\mathscr{K}_o(X)$ and $\mathscr{K}_o(\mathscr{K}_o(X))$

Now, we investigate the relationships between the hyperspaces $\mathscr{K}_o(X)$ and $\mathscr{K}_o(\mathscr{K}_o(X))$ and the Hausdorff intuitionistic fuzzy metrics H_M and $\mathscr{H}_{H_M}$.

Theorem VI.1. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Let $\Big(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit\Big)$ and $\Big(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_M}, \mathscr{H}_{H_N}, *, \diamondsuit\Big)$ be the corresponding Hausdorff Intuitionistic fuzzy hyperspaces. Let $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$ be such that*

$$\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\} \in \mathscr{K}_o(X).$$

Then

$$H_M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big) \geq \mathscr{H}_{H_M}(\mathscr{A}, \mathscr{B}, t)$$

and

$$H_N\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big) \leq \mathscr{H}_{H_N}(\mathscr{A}, \mathscr{B}, t)$$

for all $t > 0$.

Proof:

Fix $t > 0$.

Firstly, we note that

$$\begin{aligned}
&M\Big(B, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \inf_{b\in B} M\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \inf_{b\in B} \sup_{\{a\in A: A\in\mathscr{A}\}} M(b,a,t)\\
&= \inf_{b\in B} \sup_{A\in\mathscr{A}} \sup_{a\in A} M(b,a,t)\\
&\geq \sup_{A\in\mathscr{A}} \inf_{b\in B} \sup_{a\in A} M(b,a,t)\\
&= \sup_{A\in\mathscr{A}} M(B,A,t).
\end{aligned}$$

It follows that

$$\begin{aligned}
&M\Big(\{b \in B : B \in \mathscr{B}\}, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \inf_{\{b\in B: B\in\mathscr{B}\}} M\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \inf_{B\in\mathscr{B}} \inf_{b\in B} M\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \inf_{B\in\mathscr{B}} M\Big(B, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&\geq \inf_{B\in\mathscr{B}} \sup_{A\in\mathscr{A}} M(B,A,t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big)\\
&\quad \geq \inf_{A\in\mathscr{A}} \sup_{B\in\mathscr{B}} M(A,B,t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&H_M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big)\\
&= \min\Big\{M\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big),\\
&\qquad M\Big(\{b \in B : B \in \mathscr{B}\}, \{a \in A : A \in \mathscr{A}\}, t\Big)\Big\}\\
&\geq \min\Big\{\inf_{A\in\mathscr{A}} \sup_{B\in\mathscr{B}} M(A,B,t), \inf_{B\in\mathscr{B}} \sup_{A\in\mathscr{A}} M(B,A,t)\Big\}\\
&\geq \min\Big\{\inf_{A\in\mathscr{A}} \sup_{B\in\mathscr{B}} H_M(A,B,t), \inf_{B\in\mathscr{B}} \sup_{A\in\mathscr{A}} H_M(B,A,t)\Big\}\\
&= \min\Big\{H_M(\mathscr{A},\mathscr{B},t), H_M(\mathscr{B},\mathscr{A},t)\Big\}\\
&= \mathscr{H}_{H_M}(\mathscr{A},\mathscr{B},t).
\end{aligned}$$

Secondly, we note that

$$\begin{aligned}
&N\Big(B, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \sup_{b\in B} N\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \sup_{b\in B} \inf_{\{a\in A: A\in\mathscr{A}\}} N(b,a,t)\\
&= \sup_{b\in B} \inf_{A\in\mathscr{A}} \inf_{a\in A} N(b,a,t)\\
&\leq \inf_{A\in\mathscr{A}} \sup_{b\in B} \inf_{a\in A} N(b,a,t)\\
&= \inf_{A\in\mathscr{A}} N(B,A,t).
\end{aligned}$$

It follows that

$$\begin{aligned}
&N\Big(\{b \in B : B \in \mathscr{B}\}, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \sup_{\{b\in B: B\in\mathscr{B}\}} N\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \sup_{B\in\mathscr{B}} \sup_{b\in B} N\Big(b, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&= \sup_{B\in\mathscr{B}} N\Big(B, \{a \in A : A \in \mathscr{A}\}, t\Big)\\
&\leq \sup_{B\in\mathscr{B}} \inf_{A\in\mathscr{A}} N(B,A,t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&N\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big)\\
&\quad \leq \sup_{A\in\mathscr{A}} \inf_{B\in\mathscr{B}} N(A,B,t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&H_N\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big)\\
&= \max\Big\{N\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}, t\Big),\\
&\qquad N\Big(\{b \in B : B \in \mathscr{B}\}, \{a \in A : A \in \mathscr{A}\}, t\Big)\Big\}\\
&\leq \max\Big\{\sup_{A\in\mathscr{A}} \inf_{B\in\mathscr{B}} N(A,B,t), \sup_{B\in\mathscr{B}} \inf_{A\in\mathscr{A}} N(B,A,t)\Big\}\\
&\leq \max\Big\{\sup_{A\in\mathscr{A}} \inf_{B\in\mathscr{B}} H_N(A,B,t), \sup_{B\in\mathscr{B}} \inf_{A\in\mathscr{A}} H_N(B,A,t)\Big\}\\
&= \max\Big\{H_N(\mathscr{A},\mathscr{B},t), H_N(\mathscr{B},\mathscr{A},t)\Big\}\\
&= \mathscr{H}_{H_N}(\mathscr{A},\mathscr{B},t).
\end{aligned}$$

The proof is complete. ■

The above Theorem VI.1 declares that H_M is a 'stronger' degree of nearness Hausdorff intuitionistic fuzzy metric than $\mathscr{H}_{H_M}$ and $\mathscr{H}_{H_N}$ is a 'stronger' degree of non-nearness Hausdorff intuitionistic fuzzy metric than H_N.

VII. Conclusion

In this paper, we proved the intuitionistic fuzzy contraction properties of the Hutchinson-Barnsley operator on the intuitionistic fuzzy hyperspace with respect to the Hausdorff intuitionistic fuzzy metrics. Also we discussed about the relationships between the Hausdorff intuitionistic fuzzy metrics on the intuitionistic fuzzy hyperspaces. This paper will lead our direction to develop the Hutchinson-Barnsley Theory in the sense of intuitionistic fuzzy B-contractions in order to define a fractal set in the intuitionistic fuzzy metric spaces as a unique fixed point of the Intuitionistic Fuzzy HB operator.

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