The stability of Almost n-multiplicative maps in fuzzy normed spaces

E. Ansari-Piri *and* N. Eghbali

Abstract—Let A and B be two linear algebras. A linear map $\varphi : A \rightarrow B$ is called an n-homomorphism if $\varphi(a_1...a_n) = \varphi(a_1)...\varphi(a_n)$ for all $a_1, ..., a_n \in A$. In this note we have a verification on the behavior of almost n-multiplicative linear maps with $n > 2$ in the fuzzy normed spaces.

Keywords—Almost multiplicative maps, n-homomorphism maps, Almost n-multiplicative maps, Fuzzy normed space, Stability.

I. INTRODUCTION

LET A and B be two linear algebras. A linear mapping $\varphi : A \rightarrow B$ is called an n-homomorphism if $\varphi(a_1...a_n) = \varphi(a_1)...\varphi(a_n)$ for each $a_1, ..., a_n$ in A, (See [6]).

The stability of functional equations is an interesting area of research for mathematicians, but it can be also of importance to persons who work outside of the realm of pure mathematics. For example, physicists are interested in the stability of the mathematical formulae which they use to model physical processes. More precisely, physicists and other scientists are interested in determining when a small change in an equation used to the model of a phenomenon gives a large changes in the results.

It seems that the stability problem of functional equations had been first raised by Ulam [12]: For what metric groups G is it true that an approximate additive of G is necessarily near to a strict linear map?

An answer to the above problem has been given as follows [11]. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \rightarrow E_2$ is a mapping. If there exist $\delta \geq 0$ and $0 \leq p < 1$ such that $||f(x + y) - f(x) - f(y)|| \leq \delta(||x||^p + ||y||^p)$ for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \rightarrow E_2$ such that $||f(x) - T(x)|| \leq 2\delta||x||^p/|2 - 2^p|$ for every $x \in E_1$.

In 1991, Gajda [3] gave a solution to this question for $p > 1$. In 1992, Gavruta [4] generalized the result of Rassias for the admissible control functions.

Moreover the approximated mappings have been studied extensively in several papers. (See for instance [7], [8]).

Fuzzy notion introduced firstly by Zadeh [13] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to [0, 1]. Goguen in [5] generalized the notion of a fuzzy subset of X to that of an L-fuzzy subset, namely a function from X to a lattice L.

E. Amsari-Piri is with the Department of Mathematics, Faculty of science, Guilan University, P.O.Box 1914, Rasht, Iran.

N. Eghbali is with the Department of Mathematics, University of Mohaghegh Ardabili, P. O. Box 179, Ardabil, Iran. e-mail: (nasrineghbali@gmail.com).

Manuscript received

Later in 1984, Katsaras [9] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Defining the class of approximate solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated by an exact solution of the considered equation in the fuzzy Banach algebra.

To answer this question, we use here the definition of fuzzy normed spaces given in [9] to exhibit some reasonable notions of fuzzy approximately additive and almost n-multiplicative functions in fuzzy normed algebras and we will prove that under some suitable conditions an approximately additive and almost n-multiplicative f from an algebra X into a fuzzy Banach algebra Y can be approximated in a fuzzy sense by an n-homomorphism mapping T from X to Y.

II. PRELIMINARIES

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1: Let A and B be Banach algebras and, $\varphi : A \longrightarrow B$ a linear map. We say φ is an almost multiplicative map if there exists an $\epsilon > 0$ such that for all $x, y \in A$, $||\varphi(xy) - \varphi(x)\varphi(y)|| \leq \epsilon||x||||y||$.

Definition 2.2: Let A and B be two linear algebras and $n > 2$ an integer. A linear map $\varphi : A \rightarrow B$ is an n- homomorphism if for all $a_1, a_2, ..., a_n \in A$,

$\varphi(a_1 a_2...a_n) = \varphi(a_1)\varphi(a_2)...\varphi(a_n)$.

In [1] we define the concept of almost n-multiplicative maps:

Definition 2.3: Let A and B be Banach algebras and $n > 2$ an integer. A linear map $\varphi : A \rightarrow B$ is called an almost n-multiplicative map if there exists $\epsilon > 0$ such that for all $a_1, a_2, ..., a_n \in A$,

$$||\varphi(a_1 a_2...a_n) - \varphi(a_1)\varphi(a_2)...\varphi(a_n)|| \leq \epsilon||a_1||||a_2||...||a_n||.$$

Definition 2.4: Let X be a real linear space. A function $N : X \times R \rightarrow [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in R$,

(N1) $N(x, c) = 0$ for $c \leq 0$;

(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N4) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;

(N5) $N(x, .)$ is a non-decreasing function on R and $lim_{t \rightarrow \infty} N(x, t) = 1$;

(N6) for $x \neq 0$, $N(x, .)$ is (upper semi) continuous on R.

The pair (X, N) is called a fuzzy normed linear space.

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International Journal of Mathematical and Computational Sciences
Vol:5, No:8, 2011

Example 2.5: Let $(X, ||.||)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} 0, & t \leq 0; \\ \frac{t}{||x||}, & 0 < t \leq ||x||; \\ 1, & t > ||x||. \end{cases}$$

is a fuzzy norm on X.

Definition 2.6: Let (X, N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n \to \infty} x_n = x$.

Definition 2.7: A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Let X be an algebra and (X, N) be complete fuzzy normed space. The pair (X, N) is said to be a fuzzy Banach algebra if for every $x, y \in X$ and $s, t \in R$ we have: $N(xy, st) \geq min\{N(x, s), N(y, t)\}$.

Example 2.8: Let $(X, ||.||)$ be a Banach algebra. Define,

$$N(x, a) = \begin{cases} 0, & a \leq ||x||; \\ 1, & a > ||x||. \end{cases}$$

Then (X, N) is a fuzzy Banach algebra.

Theorem 2.9: Let X be a linear space and (Y, N) be a fuzzy Banach space. Let $\varphi : X \times X \to [0, \infty)$ be a control function such that

$$\tilde{\varphi}(x, y) = \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty,$$

for all $x, y \in X$. Let $f : X \to Y$ be a uniformly approximately additive function with respect to φ in the sense that

$$lim_{t \to \infty} N(f(x + y) - f(x) - f(y), t\varphi(x, y)) = 1$$

uniformly on $X \times X$. Then $T(x) = N - lim_{n \to \infty} \frac{f(2^n x)}{2^n}$ for all $x \in X$ exists and defines a unique additive mapping $T : X \to Y$ such that if for some $\delta > 0$, $\alpha > 0$

$$N(f(x + y) - f(x) - f(y), \delta\varphi(x, y)) > \alpha,$$

for all $x, y \in X$; then

$$N(T(x) - f(x), \delta/2\tilde{\varphi}(x, x)) > \alpha,$$

for every $x \in X$.

Proof: [10] ∎

corollary 2.10: Let X be a normed linear space and (Y, N) a fuzzy Banach space. Let $\theta \geq 0$ and $0 \leq q < 1$. Suppose that $f : X \to Y$ is a function such that

$$lim_{t \to \infty} N(f(x + y) - f(x) - f(y), t\theta(||x||^q + ||y||^q)) = 1$$

uniformly on $X \times X$. Then there is a unique additive mapping $T : X \to Y$ such that

$$lim_{t \to \infty} N(T(x) - f(x), \frac{2\theta t||x||^q}{1 - 2^{q-1}}) = 1$$

uniformly on X.

Proof: [10] ∎

Remark 2.11: Using the sequence $\{2^n f(2^{-n} x)\}$, one can get dual version of Theorem 2.9 and Corollary 2.10 when the control function satisfies

$$\sum_{n=0}^{\infty} 2^n \varphi(2^{-n} x, 2^{-n} y) < \infty.$$

In particular, the similar results hold for $\varphi(x, y) = ||x||^q + ||y||^q$, where $q > 1$.

Theorem 2.12: Let X be a linear space and let (Z, N') be a fuzzy normed space. Let $\psi : X \times X \to Z$ be a function such that for some $0 < \alpha < 2$,

$$N'(\psi(2x, 2y), t) \geq N'(\alpha\psi(x, y), t)$$

for all $x, y \in X$ and $t > 0$. Let (Y, N) be a fuzzy Banach space and let $f : X \to Y$ be a mapping in the sense that

$$N(f(x + y) - f(x) - f(y), t) \geq N'(\psi(x, y), t)$$

for each $t > 0$ and $x, y \in X$. Then there exists unique additive mapping $T : X \to Y$ such that

$$N(f(x) - T(x), t) \geq N'(\frac{2\psi(x, x)}{2 - \alpha}, t),$$

where $x \in X$ and $t > 0$.

Proof: [10] ∎

III. Uniform version of fuzzy stability of almost n-multiplicative mappings

Eshaghi Gordji [2] investigate the stability of almost n-multiplicative and almost n-multiplicative derivations. We start our work with definition of fuzzy approximately n-multiplicative maps.

Definition 3.1: Let X be a linear algebra, (Y, N) a fuzzy Banach algebra and $\theta \geq 0$. We say that $f : X \to Y$ is a fuzzy approximately n-multiplicative map if

$$lim_{t \to \infty} N(f(x_1...x_n) - f(x_1)...f(x_n), t\theta||x_1||^q...||x_n||^q) = 1,$$

uniformly on X.

Theorem 3.2: Let X be a normed linear algebra and (Y, N) a fuzzy Banach algebra. Let $\theta \geq 0$ and $q \geq 0, q \neq 1$. Suppose that $f : X \to Y$ is a function such that,

$$lim_{t \to \infty} N(f(x + y) - f(x) - f(y), t\theta(||x||^q + ||y||^q)) = 1$$

uniformly on $X \times X$ and

$$lim_{t \to \infty} N(f(x_1...x_n) - f(x_1)...f(x_n), t\theta||x_1||^q...||x_n||^q) = 1$$

uniformly on $X \times ... \times X$.

Then there is a unique n-multiplicative additive mapping $T : X \to Y$ such that

$$lim_{t \to \infty} N(T(x) - f(x), \frac{2\theta t||x||^q}{|1 - 2^{q-1}|}) = 1$$

uniformly on X.

Proof: Theorem 2.9 and Corollary 2.10 show that there exists a unique additive mapping $T = N - lim_{m \to \infty} \frac{f(2^m x)}{2^m}$ such that

$$lim_{t \to \infty} N(T(x) - f(x), \frac{2\theta t||x||^q}{|1 - 2^{q-1}|}) = 1$$

uniformly on X.

We will show that T is an n-multiplicative map. For every $x_1, ..., x_n \in X$ we have

$$N(n^{-2s} f(n^{2s} a^2) - n^{-2s} T(n^{2s} a^2), \frac{n^{-2s} \delta\theta||n^{2s} a^2||^q}{|1 - 2^{q-1}|}) \geq \alpha.$$

Now, since

$$N(n^{-2s} f(n^{2s} a^2) - n^{-2s} T(n^{2s} a^2), \frac{n^{-2s} \delta\theta||n^{2s} a^2||^q}{|1 - 2^{q-1}|}) =$$

$$N(n^{-2s} f(n^{2s} a^2) - T(a^2), \frac{n^{-2s} \delta\theta||n^{2s} a^2||^q}{|1 - 2^{q-1}|}).$$

So,

$$N(n^{-2s} f(n^{2s} a^2) - T(a^2), \frac{n^{-2s} \delta\theta||n^{2s} a^2||^q}{|1 - 2^{q-1}|}) \geq \alpha,$$

for all $n \in N$.

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As same as the above argument, one can show that

$$N(n^{-s}f(n^s a) - T(a), \tfrac{n^{s(q-1)}\delta\theta||a||^q}{|1-2^{q-1}|}) \geq \alpha$$

for every $n \in N$.

The latter inequalities state that:

$$lim_{t\to\infty} N(n^{-2s}f(n^{2s}a^2) - T(a^2), \frac{2n^{2s(q-1)}t\theta||a^2||^q}{|1-2^{q-1}|}) = 1, \tag{1}$$

and

$$lim_{t\to\infty} N(n^{-s}f(n^s a) - T(a), \frac{2n^{s(q-1)}t\theta||a||^q}{|1-2^{q-1}|}) = 1. \tag{2}$$

By assumption we get

$$lim_{t\to\infty} N(f(a^2) - f(a)^2, t\theta||a||^{2q}) = 1,$$

so

$$lim_{t\to\infty} N(f(n^{2s}a^2) - f(n^s a)^2, n^{2s}t\theta||a||^{2q}) = 1,$$

for all $n \in N$. Hence

$$lim_{t\to\infty} N(n^{-2s}(f(n^{2s}a^2) - f(n^s a)^2), n^{2s(q-1)}t\theta||a||^{2q}) = 1.$$

For all $t > 0$ we have:
$N(T(a^2) - T(a)^2, t) \geq min\{N(T(a^2) - n^{-2s}f(n^{2s}a^2), t/3), N(n^{-2s}f(n^{2s}a^2) - n^{-2s}f(n^s a)^2, t/3), N(n^{-2s}f(n^s a)^2 - T(a)^2, t/3)\}$.

Given $\varepsilon > 0$, we can find some $t_1 > 0$ such that

$$N(T(a^2) - n^{-2s}f(n^{2s}a^2), t/3) \geq 1 - \varepsilon,$$

for all $t/3 \geq t_1$ and all $a \in X$.

Since $lim_{n\to\infty} n^{2s(q-1)}t\theta||a||^{2q} = 0$, there is some n_0 such that $n^{2s(q-1)}t\theta||a||^{2q} < t/3$ for all $n \geq n_0$. On the other hand since N is a nondecreasing function hence for each $n \geq n_0$,
$N(n^{-2s}(f(n^{2s}a^2) - f(n^s a)^2), t/3) > N(n^{-2s}(f(n^{2s}a^2) - f(n^s a)^2), n^{2s(q-1)}t\theta||a||^{2q})$.

By the hypothesis, for given $\varepsilon > 0$ we can find some $t_2 > 0$ such that

$$N(n^{-2s}(f(n^{2s}a^2) - f(n^s a)^2), t/3) \geq 1 - \varepsilon,$$

for all $t/3 \geq t_2$ and all $a \in X$.

Also, by Lemma **??** for given $\varepsilon > 0$, we can find some $t_3 > 0$ such that

$$N(n^{-2s}f(n^s a)^2 - T(a)^2, t/3) \geq 1 - \varepsilon,$$

for all $t/3 \geq t_3$ and all $a \in X$.

Let $t_0 = min\{t_1, t_2, t_3\}$. So $N(T(a^2) - T(a)^2, t) \geq 1 - \varepsilon$ for all $t > 0$, and by the item (N2) in the Definition 2.4 we have $T(a^2) = T(a)^2$.

It remains to show that if $a \neq 0$ and $a^2 = 0$ then $T(a^2) = T(a)^2$.

As follows from Corollary 2.10, there exists an additive mapping $T : X \to Y$ such that

$$lim_{t\to\infty} N(f(x) - T(x), \frac{2\theta t||x||^q}{|1-2^{q-1}|}) = 1,$$

for all $x \in X$.

It suffices to show that $T(a^2) = T(a)^2$ for all $a \in X$. Pick $a \in X - \{0\}$ arbitrarily. In this case, we can not apply the pervious proof. In fact, if $a^2 = 0$ then $||a^2||^q = 0$ and hence the relation 1 is meaningless by the item (N1) in the Definition 2.4. We will show that $T(a)^2 = 0$ whenever $a^2 = 0$. It follows from $lim_{t\to\infty} N(f(a^2) - f(a)^2, t\theta||a||^{2q}) = 1$ and the hypothesis $f(0) = 0$, that

$$lim_{t\to\infty} N(n^{-2}f(na)^2, t\theta n^{-2}||na||^{2q}) = 1.$$

Hence,

$$lim_{t\to\infty} N(n^{-2}f(na)^2, t\theta n^{2(q-1)}||a||^{2q}) = 1. \tag{3}$$

Note also that

$$lim_{t\to\infty} N(f(x) - T(x), \frac{2\theta t||x||^q}{|1-2^{q-1}|}) = 1,$$

and

$$lim_{t\to\infty} N(n^{-1}f(na) - n^{-1}T(na), \frac{2\theta t||na||^q}{|1-2^{q-1}|}) = 1.$$

So we have $lim_{t\to\infty} N(n^{-1}f(na) - T(a), \frac{2\theta t n^{q-1}||a||^q}{|1-2^{q-1}|}) = 1$ for all $n \in N$.

In the remaining of the proof we will take into consideration the following two cases:

Case 1. If $0 \leq q < 1$, temporarily fix $t > 0$. Given $\varepsilon > 0$ we can find some $t_1 > 0$ such that $N(n^{-1}f(na) - T(a), \frac{2\theta t n^{q-1}||a||^q}{|1-2^{q-1}|}) \geq 1 - \varepsilon$ for all $t \geq t_1$. Since $lim_{n\to\infty} \frac{2\theta t n^{q-1}||a||^q}{|1-2^{q-1}|} = 0$, there is some n_0 such that $\frac{2\theta t n^{q-1}||a||^q}{|1-2^{q-1}|} < t$ for all $n \geq n_0$. Hence for all $n \geq n_0$

$$N(n^{-1}f(na) - T(a), t) > \\ N(n^{-1}f(na) - T(a), \frac{2\theta t n^{q-1}||a||^q}{|1-2^{q-1}|}) \geq 1 - \varepsilon.$$

We have $N - lim_{n\to\infty} n^{-1}f(na) = T(a)$. By Lemma **??** one gets

$$N - lim_{n\to\infty} n^{-2}f(na)^2 = T(a)^2.$$

Given $\varepsilon > 0$ we can find some $t_2 > 0$ such that $N(T(a)^2 - n^{-2}f(na)^2, t/2) \geq 1 - \varepsilon$ for all $t/2 \geq t_2$. On the other hand for given $\varepsilon > 0$ we can find some $t_3 > 0$ such that $N(n^{-2}f(na)^2, t/2) \geq 1 - \varepsilon$ for all $t/2 \geq t_3$. Hence taking $t_0 = min\{t_1, t_2, t_3\}$ we have
$N(T(a)^2 - 0, t) \geq min\{N(T(a)^2 - n^{-2}f(na)^2, t/2), N(n^{-2}f(na)^2, t/2)\} \geq 1 - \varepsilon$.

This completes the proof for $0 \leq q < 1$.

Case 2. If $q > 1$, by assumption we have $lim_{t\to\infty} N(f(a^2) - f(a)^2, t\theta||a||^{2q}) = 1$ and since $f(0) = 0$ it holds

$$lim_{t\to\infty} N(n^2 f(n^{-1}a)^2, t\theta n^{2(1-q)}||a||^{2q}) = 1.$$

Note also that $lim_{t\to\infty} N(nf(n^{-1}a) - nT(n^{-1}a), \frac{2\theta t||n^{-1}a||^q}{|1-2^{q-1}|}) = 1$.

So $lim_{t\to\infty} N(nf(n^{-1}a) - T(a), \frac{2\theta t n^{1-q}||a||^q}{|1-2^{q-1}|}) = 1$ for all $n \in N$. Following an argument such as that of the proof of case 1 verifies case 2.

Finally suppose that $T^* : X \to Y$ is another Jordan additive function such that $lim_{t\to\infty} N(f(x) - T^*(x), \frac{2\theta t||x||^q}{|1-2^{q-1}|}) = 1$ for all $x \in X$.

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Then it holds

$$lim_{t\to\infty}N(n^{-s}f(n^sa) - T(a), \frac{2n^{s(q-1)}t\theta||a||^q}{|1-2^{q-1}|}) = 1$$

for $T = T^*$. We thus obtain

$N(T(a) - T^*(a),t) \geq min\{N(T(a) - n^{-s}f(n^{sa}),t/2), N(n^{-s}f(n^sa) - T^*(a),t/2)\}$.

Since $lim_{n\to\infty}\frac{2n^{s(q-1)}t\theta||a||^q}{|1-2^{q-1}|} = 0$, there is some n_0 such that $\frac{2n^{s(q-1)}t\theta||a||^q}{|1-2^{q-1}|} < t/2$ for all $n \geq n_0$. Given $\varepsilon > 0$, we can find some $t_0 > 0$ such that

$N(T(a) - n^{-s}f(n^{sa}),t/2) \geq 1 - \varepsilon$ and also $N(n^{-s}f(n^sa) - T^*(a),t/2) \geq \varepsilon$ for all $t \geq t_0$. Hence $N(T(a) - T^*(a),t) \geq 1 - \varepsilon$ for all $t > 0$ and again by the item (N2) in the Definition 2.4, we have $T(a) = T^*(a)$. ∎

Remark 3.3: Using the similar argument such as that of the proof of Theorem 3.2 one can get the similar results where $q < 1$.

In the following example we will show that Theorem 3.2 does not necessarily hold for $q = 1$.

Example 3.4: Let X be a Banach algebra, $x_0 \in X$ and α, β are real numbers such that $|\alpha| \geq 1 - ||x||^2$ and $\beta \leq ||x||$, for every $x \in X$. Put

$f(x) = \alpha x + \beta x_0||x||,$ $\qquad\qquad (x \in X).$

Moreover, for each fuzzy norm N on X, we have

$N(f(x + y) - f(x) - f(y), t(||x|| + ||y||))$
$= N(\beta x_0(||x + y|| - ||x|| - ||y||), t(||x|| + ||y||)),$
$= N(\beta x_0, \frac{t(||x||+||y||)}{||x+y||-||x||-||y||}) \geq$
$N(\beta x_0, t)$ $\qquad (x, y \in X, \qquad t \in R).$

Therefore by the item (N5) of the Definition 2.4, $lim_{t\to\infty}N(f(x + y) - f(x) - f(y), t(||x|| + ||y||)) = 1$, uniformly on $X \times X$.

Also

$N(f(x^2) - f(x)^2, t||x||^2) =$
$N(\alpha x^2 + \beta x_0||x^2|| - \alpha^2x^2 - \beta^2x_0^2||x||^2 - 2\alpha\beta xx_0||x||, t||x||^2) \geq min\{N((1-\alpha)\alpha x^2, \frac{t||x||^2}{4}), N(||x^2||\beta x_0, \frac{t||x||^2}{4})$
$, N(-\beta^2x_0^2||x||^2, \frac{t||x||^2}{4}), N(-2\alpha\beta xx_0||x||, \frac{t||x||^2}{4})\}$

where $x \in X$ and $t \in R$.

Taking into account the following inequalities

$$N((1-\alpha)\alpha x^2, \frac{t||x||^2}{4}) = N(\alpha x^2, \frac{t||x||^2}{4|1-\alpha|}) \geq N(\alpha x^2, t/4), \tag{4}$$

$$N(||x^2||\beta x_0, \frac{t||x||^2}{4}) = N(\beta x_0, \frac{t||x||^2}{4||x^2||}) \geq N(\beta x_0, t/4), \tag{5}$$

$$N(-\beta^2x_0^2||x||^2, \frac{t||x||^2}{4}) = N(\beta^2x_0^2, t) \geq N(\beta^2x_0^2, t/4), \tag{6}$$

$$N(-2\alpha\beta xx_0||x||, \frac{t||x||^2}{4}) = N(2\alpha xx_0, \frac{t||x||}{4|\beta|}) \geq N(2\alpha xx_0, t/4) \tag{7}$$

it can be easily seen that $lim_{t\to\infty}N(f(x^2) - f(x)^2, t||x||^2) = 1$, uniformly on X and therefore the conditions of Theorem 3.2 are fulfilled.

Now, we suppose that there exists a unique linear Jordan map T satisfying the conditions of Theorem 3.2. By the equation

$$lim_{t\to\infty}N(f(x + y) - f(x) - f(y), t(||x|| + ||y||)) = 1, \tag{8}$$

for given $\varepsilon > 0$, we can find some $t_0 > 0$ such that

$N(f(x + y) - f(x) - f(y), t(||x|| + ||y||)) \geq 1 - \varepsilon,$

for all $x, y \in X$ and all $t \geq t_0$. By using the simple induction on n, we shall show that

$$N(f(2^nx) - 2^nf(x), tn2^n||x||) \geq 1 - \varepsilon. \tag{9}$$

putting $y = x$ in 8, we get 9 for $n = 1$. Let 9 holds for some positive integer n. Then

$N(f(2^{n+1}x) - 2^{n+1}f(x), t(n + 1)2^{n+1}||x||) \geq$
$min\{N(f(2^{n+1}x) - 2f(2^nx), t(||2^nx|| + ||2^nx||)), N(2f(2^nx) - 2^{n+1}f(x), 2tn(||2^{n-1}x|| + ||2^{n-1}x||)) \geq 1 - \varepsilon.$

This completes the induction argument. We observe that

$lim_{n\to\infty}N(T(x) - f(x), nt||x||) \geq 1 - \varepsilon.$

Hence

$$lim_{n\to\infty}N(T(x) - f(x), nt||x||) = 1. \tag{10}$$

One may regard $N(x,t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t'. So 10 is a contradiction with the non-fuzzy sense. This means that there is no such a T.

IV. Non-uniform type of stability of fuzzy approximately Jordan mappings

We are in a position to give non-uniform type of Theorem 3.2.

Theorem 4.1: Let X be a linear algebra, (Z, N') a fuzzy Banach algebra and $\varphi : X \times X \to Z$ a function such that for some $0 < \alpha < 2$,

$N'(\varphi(2x, 2y), t) \geq N'(\varphi(x, y), t)$

for all $x, y \in X$ and $t > 0$. Let (Y, N) be a fuzzy Banach algebra and let $f : X \to Y$ be a function such that

$N(f(x + y) - f(x) - f(y), t) \geq N'(\varphi(x, y), t),$

and

$N(f(x^2) - f(x)^2, s) \geq N'(\varphi(x, x), s),$

for each $t, s > 0$ and $x, y \in X$. Then there exists a unique additive Jordan mapping $T : X \to Y$ such that

$N(f(x) - T(x), t) \geq N'(\frac{2\varphi(x,x)}{2-\alpha}, t),$

where $x \in X$ and $t > 0$.

Proof: Theorem 2.12 shows that there exists an additive function $T : X \to Y$ such that

$N(f(x) - T(x), t) \geq N'(\frac{2\varphi(x,x)}{2-\alpha}, t),$

where $x \in X$ and $t > 0$. Now we only need to show that T is a Jordan map. If $a = 0$, since $T(0) = 0$ it is obvious. In the other case,

$N(n^{-2}f(n^2a^2) - n^{-2}T(n^2a^2), n^{-2}t) \geq N'(\frac{2\varphi(n^2a^2, n^2a^2)}{2-\alpha}, t),$

for all $a \in X$, $t > 0$ and $n \in N$. by the additivity of T it is easy to see that

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
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$$N(n^{-2}f(n^2a^2) - T(a^2), t) \geq N'(\frac{2\varphi(n^2a^2, n^2a^2)}{2-\alpha}, n^2t),$$

$$(11)$$

for all $a \in X$, $t > 0$ and $n \in N$. Letting n tend to infinity in 11 and using the items (N2) and (N5) of the Definition 2.4, we see that

$$T(a^2) = N - lim_{n\to\infty}n^{-2}f(n^2a^2).$$ (12)

Also with a similar argument represented above shows that:

$$N(n^{-1}f(na) - T(a), t) \geq N'(\frac{2\varphi(na,na)}{2-\alpha}, nt),$$

for all $a \in X$, $t > 0$ and $n \in N$. Hence we have

$$T(a) = N - lim_{n\to\infty}n^{-1}f(na).$$ (13)

On the other hand

$$N(f(n^2a^2) - f(na)^2, s) \geq N'(\varphi(na, na), s),$$

for all $a \in X$, $s > 0$ and $n \in N$. We observe that

$$N(n^{-2}f(n^2a^2) - n^{-2}f(na)^2, s) \geq N'(\varphi(na, na), n^2s),$$

for all $a \in X$, $s > 0$ and $n \in N$. So again by taking n tend to infinity we have

$$N - lim_{n\to\infty}n^{-2}f(n^2a^2) = N - lim_{n\to\infty}n^{-2}f(na)^2.$$ (14)

Applying 12, 13 and 14 we have

$$T(a^2) = N - lim_{n\to\infty}n^{-2}f(n^2a^2) =$$
$$N - lim_{n\to\infty}n^{-2}f(na)^2 =$$
$$(N - lim_{n\to\infty}n^{-1}f(na))^2 = T(a)^2.$$

To prove the uniqueness property of T, assume that T^* is another additive Jordan mapping satisfying $N(f(x)-T(x),t) \geq N'(\frac{2\varphi(x,x)}{2-\alpha}, t)$. Since both T and T^* are additive we deduce that

$$N(T(a) - T^*(a), t) \geq min\{N(T(a) -$$
$$n^{-1}f(na), t/2), N(n^{-1}f(na) - T^*(a), t/2)\} \geq$$
$$N'(\frac{2\varphi(na,na)}{2-\alpha}, nt/2)$$

for all $a \in X$ and all $t > 0$. Letting n tend to infinity we find that $T(a) = T^*(a)$ for all $a \in X$. ∎

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