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On the Central Limit Theorems for Forward and Backward Martingales

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Abstract—Let $\{X_i\}_{i\geq 1}$ be a martingale difference sequence with $X_i = S_i - S_{i-1}$. Under some regularity conditions, we show that $(X_1^2+\cdots+X_N^2\ )^{-1/2}S_N$ is asymptotically normal, where $\{N_i\}_{i\geq 1}$ is a sequence of positive integer-valued random variables tending to infinity. In a similar manner, a backward (or reverse) martingale central limit theorem with random indices is provided.

Keywords—central limit theorem, martingale difference sequence, backward martingale.

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I. Introduction

FORWARD and backward martingale limit theory has very useful unifying properties in the sense that many specific limit theorems follow as special cases of martingale's. The classical form of the forward martingale central limit theorem, as presented by Brown [7] and amplified by Dvoretzky [8], Scott [21] and McLeish [17], closely resembles the theorem of Lindeberg and Feller [3], [4]. Analogous backward martingale (also known as reverse martingale) central limit theorems are presented by Loynes [14].

Let $n \geq 1$ be a fixed integer. Consider a finite sequence $\{X_1, \cdots, X_n\}$ of martingale difference random variables (i.e., X_i is $\mathcal{F}_i$-measurable and $E(X_i|\mathcal{F}_{i-1}) = 0$ a.s., where $\{\mathcal{F}_i\}_{0\leq i\leq n}$ is an increasing filtration and $\mathcal{F}_0$ is the trivial σ-algebra). Let $S_n = X_1 + X_2 + \cdots + X_n$ and $v_n^2 = \sum_{i=1}^n E(X_i^2)$. The central limit theorem established by Brown [7] and Dvoretzky [8] states that under some Lindeberg-type condition,

$$\triangle_n = \sup_{x\in\mathbb{R}} \left| P\left(\frac{S_n}{v_n} < x\right) - \Phi(x)\right| \to 0 \qquad (1)$$

as $n \to \infty$, where $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-u^2/2}\mathrm{d}u$ is the standard normal distribution function. More recent studies on martingale cental limit theorems and their convergence rates can be find in e.g. [13], [15], [16], [18]. We refer the reader to books [10] and [6] for more about martingale central limit theorems.

A classical (forward) martingale central limit theorem with random norming is the following. See [10] for a proof.

Theorem 1. *Let $\{S_i\}_{i\geq 0}$ be a zero-mean martingale sequence relative to $\{\mathcal{F}_i\}_{i\geq 0}$. Let $c_i = \sqrt{Var(S_i)} < \infty$, $X_i = S_i - S_{i-1}$ and $S_0 = 0$. Suppose*

(i) $\frac{\max_{1\leq\ \leq\ X}}{c} \xrightarrow{P} 0$ *(in probability),*

(ii) There exists a real-valued random variable η such that $\frac{\sum_{=1} X^2}{c^2} \xrightarrow{P} \eta$.

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If $P(\eta = 0) = 0$, then

$$\frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} \xrightarrow{L} N(0,1) \qquad (2)$$

as $n \to \infty$, where "$\xrightarrow{L} N(0,1)$" denotes convergence in distribution to standard normal distribution.

The following is an analogous backward martingale central limit theorem. Also see [10] for a proof.

Theorem 2. *Let $\{S_i\}_{i\geq 0}$ be a zero-mean backward martingale sequence relative to a sequence of decreasing σ-fields $\{\mathcal{G}_i\}_{i\geq 0}$, and let $s_i^2 = E(X_i^2|\mathcal{G}_{i+1})$, where $X_i = S_i - S_{i-1}$. Suppose*

(i) $\frac{\sum_{=}^{\infty} s^2}{E(\sum_{=}^{\infty} s^2)} \xrightarrow{a.s.} 1$ *(almost surely),*

(ii) $\frac{\sum_{=}^{\infty} X^2}{\sum_{=}^{\infty} s^2} \xrightarrow{a.s.} 1$.

(iii) There exists a real-valued random variable η such that $\frac{\sum_{=}^{\infty} X^2}{n} \xrightarrow{P} \eta$.

If $P(\eta = 0) = 0$, then

$$\frac{S_n}{\sqrt{E(\sum_{i=n}^{\infty} s_i^2)}} \xrightarrow{L} N(0,1) \qquad (3)$$

as $n \to \infty$.

In this paper, we want to generalize Theorem 1 and 2 to cover the random indices. In other words, we investigate the sum of a random number of forward (and backward) martingale difference sequence $\{X_i\}$. This question is important not only in probability theory itself but in sequential analysis, random walk problems, Monte Carlo methods, etc. Central limit problems for the sum of a random number of independent random variables have been addressed in the pioneer work of Anscombe [2], Rényi [20] and Blum et. al. [5]. More recent study can be found in e.g. [9], [11], [12], [19], most of which, nevertheless, deals with simple independent cases.

The rest of the paper is organized as follows. In Section 2, we present our forward and backward martingale cental limit theorems and in Section 3, we give the proofs.

II. Main results

The following is our martingale central limit theorem with random indices.

Theorem 3. *Let $\{S_i\}_{i\geq 0}$ be a zero-mean martingale sequence relative to $\{\mathcal{F}_i\}_{i\geq 0}$. Let $c_i = \sqrt{Var(S_i)} < \infty$, $X_i = S_i - S_{i-1}$ and $S_0 = 0$. Denote by $\{N_n\}_{n\geq 1}$ a sequence of positive integer-valued random variables such that*

$$\frac{N_n}{\omega_n} \xrightarrow{P} \omega \qquad (4)$$

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as $n \to \infty$, where $\{\omega_n\}_{n\geq 1}$ is an arbitrary positive sequence tending to $+\infty$ and ω is a positive constant. Suppose

(i) $\frac{\max_{1\leq \ \leq \ n} X}{c} \xrightarrow{P} 0$,

(ii) There exists a real-valued random variable η such that $\frac{\sum_{=1} X^2}{c^2} \xrightarrow{P} \eta$,

(iii) There exists some $k_0 \geq 0$ and $c > 0$ such that, for any $\lambda > 0$ and $n > k_0$, we have

$$P\big(\max_{k_0<k_1\leq k_2\leq n} |S_{k_2} - S_{k_1}| \geq \lambda\big) \leq \frac{c \cdot E(S_n - S_{k_0})^2}{\lambda^2}, \tag{5}$$

(iv) $Cov(X_i, X_j) \geq 0$ for all i and j,

(v) There exists some $k_1 > 0$ and $\alpha > 0$ such that, for any $n > k_1$, we have

$$\Big(\sum_{i\geq n} c_i\Big)^2 \leq \alpha c_n^2. \tag{6}$$

If $P(\eta = 0) = 0$, then

$$\frac{S_N}{\sqrt{\sum_{i=1}^N X_i^2}} \xrightarrow{L} N(0,1) \tag{7}$$

as $n \to \infty$.

We give some remarks here. Firstly, note that the assumption (iii) is for sufficiently large index of martingale difference sequence X_i, i.e., $\{X_i\}_{i>k_0}$. Secondly, if $\{X_i\}_{i\geq 1}$ is independent, then (iv) automatically holds for $k_0 = 0$ and $c = 1$ by using the Kolmogorov inequality or Doob martingale inequality (see e.g. [4]). Therefore, the assumption (iii) may be regarded as a "relaxed" Kolmogrov inequality. Thirdly, the assumption (iv) says that each pair X_s, X_t of $\{X_i\}_{i\geq 1}$ are positively correlated. In view of the independent case [5], it seems likely that the assertion of Theorem 3 still holds when ω is a positive random variable.

Our backward martingale central limit theorem reads as follows. Similar remarks as above may apply.

Theorem 4. *Let $\{S_i\}_{i\geq 0}$ be a zero-mean backward martingale sequence relative to a sequence of decreasing σ-fields $\{\mathcal{G}_i\}_{i\geq 0}$, and let $s_i^2 = E(X_i^2|\mathcal{G}_{i+1})$, where $X_i = S_i - S_{i-1}$. Denote by $\{N_n\}_{n\geq 1}$ a sequence of positive integer-valued random variables such that*

$$\frac{N_n}{\omega_n} \xrightarrow{P} \omega \tag{8}$$

as $n \to \infty$, where $\{\omega_n\}_{n\geq 1}$ is an arbitrary positive sequence tending to $+\infty$ and ω is a positive constant. Suppose

(i) $\frac{\sum_{=}^{\infty} s^2}{E(\sum_{=}^{\infty} s^2)} \xrightarrow{a.s.} 1$,

(ii) $\frac{\sum_{=}^{\infty} X^2}{\sum_{=}^{\infty} s^2} \xrightarrow{a.s.} 1$,

(iii) There exists a real-valued random variable η such that $\frac{\sum_{=}^{\infty} X^2}{n} \xrightarrow{P} \eta$,

(iv) There exists some $k_0 \geq 0$ and $c > 0$ such that, for any $\lambda > 0$ and $n > k_0$, we have

$$P\big(\max_{k_0<k_1\leq k_2\leq n} |S_{k_2} - S_{k_1}| \geq \lambda\big) \leq \frac{c \cdot E(\sum_{i=k_0+1}^n X_i^2)}{\lambda^2}, \tag{9}$$

(v) $Cov(X_i, X_j) \geq 0$ for all i and j.

Then

$$\frac{S_N}{\sqrt{E(\sum_{i=N}^{\infty} s_i^2)}} \xrightarrow{L} N(0,1) \tag{10}$$

as $n \to \infty$.

III. Proofs

In this section, we provide the proofs of Theorem 3 and Theorem 4, which are similar in spirit.

Proof of Theorem 3. Let $0 < \varepsilon < 1/2$. From (4) it follows that there exists some n_0, for any $n \geq n_0$,

$$P(|N_n - \omega\omega_n| > \varepsilon\omega\omega_n) \leq \varepsilon. \tag{11}$$

For any $x \in \mathbb{R}$, we obtain

$$P\left(\frac{S_N}{\sqrt{\sum_{i=1}^N X_i^2}} < x\right) = \sum_{n=1}^{\infty} P\left(\frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} < x, N_n = n\right). \tag{12}$$

By (11) and (12), we obtain for $n \geq n_0$,

$$\left| P\left(\frac{S_N}{\sqrt{\sum_{i=1}^N X_i^2}} < x\right) - \sum_{|n-\omega\omega \ |\leq \varepsilon\omega\omega} P\left(\frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} < x, N_n = n\right) \right| \leq \varepsilon. \tag{13}$$

Let $n_1 = [\omega(1-\varepsilon)\omega_n]$ and $n_2 = [\omega(1+\varepsilon)\omega_n]$. Since ω_n tends to infinity, we have $n_1 \geq k_0$ for large enough n. Note that $S_{n_1} + \sum_{n_1<k\leq n} X_k = S_n$. Then we have for $|n - \omega\omega_n| \leq \varepsilon\omega\omega_n$,

$$P\left(\frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} < x, N_n = n\right) \leq P\left(S_{n_1} < x\sqrt{\sum_{i=1}^{n_2} X_i^2} + Y, N_n = n\right), \tag{14}$$

where

$$Y = \max_{n_1<n\leq n_2} \left| \sum_{n_1<k\leq n} X_k \right|. \tag{15}$$

Likewise, we have

$$P\left(\frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} < x, N_n = n\right) \geq P\left(S_{n_1} < x\sqrt{\sum_{i=1}^{n_1} X_i^2} - Y, N_n = n\right), \tag{16}$$

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Involving the assumptions (i)-(iii), (iv) and (15), we obtain for large enough n,

$$P\left(Y \geq \varepsilon^{\frac{1}{3}}\sqrt{\sum_{i=1}^{n_1} X_i^2}\right) \leq \frac{cE(\sum_{i=n_1+1}^{n_2} X_i)^2}{\varepsilon^{\frac{2}{3}}\sum_{i=1}^{n_1} X_i^2} \leq \frac{5c\alpha\varepsilon^{\frac{1}{3}}}{\eta}, \quad (17)$$

the right-hand side of which is less than 1 when ε is small enough. Note that $\eta \neq 0$ by assumption, which justifies the expression (17).

Denote by E the event that $Y < \varepsilon^{1/3}\sqrt{\sum_{i=1}^{n_1} X_i^2}$. By virtue of (13), (14), (17) and assumption (ii), we get

$$\begin{aligned} & P\left(\frac{S_N}{\sqrt{\sum_{i=1}^N X_i^2}} < x\right) \\ \leq\ & P\left(\frac{S_{n_1}}{\sqrt{\sum_{i=1}^{n_1} X_i^2}} < x\sqrt{\frac{\sum_{i=1}^{n_2} X_i^2}{\sum_{i=1}^{n_1} X_i^2}} + \varepsilon^{\frac{1}{3}}, E\right) \\ & + \frac{5c\alpha\varepsilon^{\frac{1}{3}}}{\eta} + \varepsilon \\ \leq\ & P\left(\frac{S_{n_1}}{\sqrt{\sum_{i=1}^{n_1} X_i^2}} < x(1+\varepsilon)\frac{c_{n_2}}{c_{n_1}} + \varepsilon^{\frac{1}{3}}\right) \\ & + \left(\frac{5c\alpha}{\eta} + 1\right)\varepsilon^{\frac{1}{3}}. \end{aligned} \quad (18)$$

Similarly, from (13), (16) and (17) it follows that

$$\begin{aligned} & P\left(\frac{S_N}{\sqrt{\sum_{i=1}^N X_i^2}} < x\right) \\ & \geq P\left(\frac{S_{n_1}}{\sqrt{\sum_{i=1}^{n_1} X_i^2}} < x - \varepsilon^{\frac{1}{3}}, E\right) - \varepsilon. \end{aligned} \quad (19)$$

Using (19), (17) and the assumption (iv), we may derive

$$\begin{aligned} P\left(\frac{S_N}{\sqrt{\sum_{i=1}^N X_i^2}} < x\right) \geq\ & P\left(\frac{S_{n_1}}{\sqrt{\sum_{i=1}^{n_1} X_i^2}} < x - \varepsilon^{\frac{1}{3}}\right) \\ & \cdot P(E) - \varepsilon \\ \geq\ & \left(1 - \frac{5c\alpha\varepsilon^{\frac{1}{3}}}{\eta}\right) \\ & \cdot P\left(\frac{S_{n_1}}{\sqrt{\sum_{i=1}^{n_1} X_i^2}} < x - \varepsilon^{\frac{1}{3}}\right) \\ & -\varepsilon, \end{aligned} \quad (20)$$

where the first inequality is due to an application of the FKG inequality (see e.g. [1]).

Now by Theorem 1 we obtain

$$\lim_{n_1\to\infty} P\left(\frac{S_{n_1}}{\sqrt{\sum_{i=1}^{n_1} X_i^2}} < x\right) = \Phi(x), \quad (21)$$

where $\Phi(x)$ is the standard normal distribution function as defined above. Combining (18), (20) and (21), we then ends the proof of Theorem 3. □

Proof of Theorem 4. Let $0 < \varepsilon < 1/2$. From (8) it follows that there exists some n_0, for any $n \geq n_0$,

$$P(|N_n - \omega\omega_n| > \varepsilon\omega\omega_n) \leq \varepsilon. \quad (22)$$

For any $x \in \mathbb{R}$, we obtain

$$\begin{aligned} & P\left(\frac{S_N}{\sqrt{E(\sum_{i=N}^{\infty} s_i^2)}} < x\right) \\ & = \sum_{n=1}^{\infty} P\left(\frac{S_n}{\sqrt{E(\sum_{i=n}^{\infty} s_i^2)}} < x, N_n = n\right). \end{aligned} \quad (23)$$

By (22) and (23), we obtain for $n \geq n_0$,

$$\begin{aligned} & \left| P\left(\frac{S_N}{\sqrt{E(\sum_{i=N}^{\infty} s_i^2)}} < x\right) \right. \\ & \left. - \sum_{|n-\omega\omega|\leq\varepsilon\omega\omega} P\left(\frac{S_n}{\sqrt{E(\sum_{i=n}^{\infty} s_i^2)}} < x, N_n = n\right) \right| \\ & \leq \varepsilon. \end{aligned} \quad (24)$$

Let $n_1 = [\omega(1-\varepsilon)\omega_n]$ and $n_2 = [\omega(1+\varepsilon)\omega_n]$. Since ω_n tends to infinity, we have $n_1 \geq k_0$ for large enough n. Note that $S_{n_1} + \sum_{n_1<k\leq n} X_k = S_n$. Then we have for $|n-\omega\omega_n| \leq \varepsilon\omega\omega_n$,

$$\begin{aligned} & P\left(\frac{S_n}{\sqrt{E(\sum_{i=n}^{\infty} s_i^2)}} < x, N_n = n\right) \\ & \leq P\left(S_{n_1} < x\sqrt{E\Big(\sum_{i=n_1}^{\infty} s_i^2\Big)} + Y, N_n = n\right), \end{aligned} \quad (25)$$

where

$$Y = \max_{n_1<n\leq n_2}\left|\sum_{n_1<k\leq n} X_k\right|. \quad (26)$$

Likewise, we have

$$\begin{aligned} & P\left(\frac{S_n}{\sqrt{E(\sum_{i=n}^{\infty} s_i^2)}} < x, N_n = n\right) \\ & \geq P\left(S_{n_1} < x\sqrt{E\Big(\sum_{i=n_2}^{\infty} s_i^2\Big)} - Y, N_n = n\right), \end{aligned} \quad (27)$$

Involving the assumptions (i)-(iv) and (26), we obtain for large enough n,

$$P\left(Y \geq \varepsilon^{\frac{1}{3}}\sqrt{E\Big(\sum_{i=n_2}^{\infty} s_i^2\Big)}\right) \leq \frac{cE(\sum_{i=n_1+1}^{n_2} X_i^2)}{\varepsilon^{\frac{2}{3}}E(\sum_{i=n_2}^{\infty} s_i^2)} \leq 5c\varepsilon^{\frac{1}{3}}, \quad (28)$$

the right-hand side of which is less than 1 when ε is small enough.

Denote by E the event that $Y < \varepsilon^{1/3}\sqrt{E(\sum_{i=n_2}^{\infty} s_i^2)}$. By virtue of (24), (25), (28), and assumption (i)-(iii), we get

$$\begin{aligned} P & \left(\frac{S_N}{\sqrt{E(\sum_{i=N}^{\infty} s_i^2)}} < x \right) \\ \leq & P\left(\frac{S_{n_1}}{\sqrt{E(\sum_{i=n_2}^{\infty} s_i^2)}} < x\sqrt{\frac{E(\sum_{i=n_1}^{\infty} s_i^2)}{E(\sum_{i=n_2}^{\infty} s_i^2)}} + \varepsilon^{\frac{1}{3}}, E \right) \\ & +5c\varepsilon^{\frac{1}{3}} + \varepsilon \\ \leq & P\left(\frac{S_{n_1}}{\sqrt{E(\sum_{i=n_2}^{\infty} s_i^2)}} < x\sqrt{\frac{1+2\varepsilon}{1-2\varepsilon}} + \varepsilon^{\frac{1}{3}} \right) \\ & +(5c+1)\varepsilon^{\frac{1}{3}}. \end{aligned} \tag{29}$$

Similarly, from (24), (27) and (28) it follows that

$$\begin{aligned} & P\left(\frac{S_N}{\sqrt{E(\sum_{i=N}^{\infty} s_i^2)}} < x \right) \\ & \geq P\left(\frac{S_{n_1}}{\sqrt{E(\sum_{i=n_2}^{\infty} s_i^2)}} < x - \varepsilon^{\frac{1}{3}}, E \right) - \varepsilon. \end{aligned} \tag{30}$$

Using (30), (28) and the assumption (v), we may derive

$$\begin{aligned} P & \left(\frac{S_N}{\sqrt{E(\sum_{i=N}^{\infty} s_i^2)}} < x \right) \\ \geq & P\left(\frac{S_{n_1}}{\sqrt{E(\sum_{i=n_2}^{\infty} s_i^2)}} < x - \varepsilon^{\frac{1}{3}} \right) \cdot P(E) - \varepsilon \\ \geq & (1 - 5c\varepsilon^{\frac{1}{3}}) \cdot P\left(\frac{S_{n_1}}{\sqrt{E(\sum_{i=n_2}^{\infty} s_i^2)}} < x - \varepsilon^{\frac{1}{3}} \right) \\ & -\varepsilon, \end{aligned} \tag{31}$$

where the first inequality is due to an application of the FKG inequality.

Now by Theorem 2 we obtain

$$\lim_{n_1 \to \infty} P\left(\frac{S_{n_1}}{\sqrt{E(\sum_{i=n_1}^{\infty} s_i^2)}} < x \right) = \Phi(x), \tag{32}$$

where $\Phi(x)$ is the standard normal distribution function as defined above. Combining (29), (31) and (32), we then concludes the proof of Theorem 4. □

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