World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:7, 2011

Strongly screenableness and its Tychonoff products

Jianjun Wang, Peiyong Zhu

Abstract—In this paper, we prove that if X is regular strongly screenable $\mathcal{DC}$-like ($\mathcal{C}$-scattered), then $X \times Y$ is strongly screenable for every strongly screenable space Y. We also show that the product $\prod_{i\in\omega} Y_i$ is strongly screenable if every Y_i is a regular strongly screenable $\mathcal{DC}$-like space. Finally, we present that the strongly screenableness are poorly behaved with its Tychonoff products.

Keywords—topological game; strongly screenable; scattered; $\mathcal{C}$-scattered.

I. INTRODUCTION

ALL spaces are assumed to be T_1-spaces without any separation axiom. $\mathcal{DC}$ denotes the class of all spaces which have a discrete cover by compact sets. ω denotes the set of natural numbers.

In 1975, Telgarsky [3] presented and studied the topological game $G(\mathcal{DC}, X)$ and its applications to different problems in general topology. In particular, making use of it to products, he shown that if X is a paracompact $\mathcal{DC}$-like T_2-space, then $X \times Y$ is paracompact for each paracompact space Y. In these connection, Yajima [4, 8] obtained an analogous result by replacing paracompactness with subparacompactness, and submetacompactness. Gruenhage and Yajima [10] have proved that analogues of Telgarsky's theorem assuming the regularity of X for metacompactness, submetacompactness and weakly submetacompactness. In addition, these covering properties of countable products have been investigated by several authors. Tanaka [11, 12] improved and extended the Gruenhage and Yajima's result by proving that if Y_i is a regular submetacompact (metacompact, resp.) $\mathcal{DC}$-like space for each $i \in \omega$, then $\prod_{i\in\omega} Y_i$ is submetacompact (metacompact, resp.). In 1968, Greever [1] introduced the concept of strongly screenableness as a certain generalized Lindelof properties, and he studied several equivalent characterization of it. For strongly screenableness, however, the corresponding products has not yet been proved in anywhere.

The central point of this note is to study the strongly screenableness of Tychonoff products with topological game $G(\mathcal{DC}, X)$. The rest of this paper is organized as follows. In Section 2, we state some of the definitions, terminology and notation used in this note. In Section 3, we present that X is a regular strongly screenable $\mathcal{DC}$-like ($\mathcal{C}$-scattered) space, then $X \times Y$ is strongly screenable for each strongly screenable space Y. With regard to strongly screenableness of countable product, in Section 4, we obtain an analogue of Tanaka's results. In connections to this results, we also prove a similar result by replacing $\mathcal{DC}$-like with a σ-closure-preserving cover by compact sets and σ-scattered. In Section 5, we give two examples to illustrate that the strongly screenableness are poorly behaved with its products. Furthermore, the condition $\mathcal{DC}$-like can not be omitted in the above conclusions.

Jianjun Wang, Peiyong Zhu is with School of Mathematical sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, CHNIA e-mail: (wangjianjun02@163.com).

Supported by the Nature Scientific Fund Committee of China, grant No.10671134.

II. PRELIMINARIES

In this section, we state notation and basic facts. For a set A, $|A|$ ($\overline{A}$, resp.) denote its cardinality (closure, resp.). For each $n \geq 1$, A^n denotes the set of all n-sequences of elements of A and $A^{<\omega}$=$\{\sigma \subset A : |\sigma| < \omega\}$. If s=$(s_0, s_1, \cdots, s_{n-1}) \in A^{<\omega}$ and $a \in A$, then $s \oplus a$ denotes the sequence $(s_0, \cdots, s_{n-1}, a)$, and $s\,|_n$=$(s_0, s_1, \cdots, s_{n-1})$ if $s \in \Omega^\omega$. For a space X, $\mathcal{P}(X)$ ($\mathcal{K}(X)$, $\mathcal{N}(X)$, 2^X, resp.) denote a collection of all subsets (compact subsets, nbd, closed subsets, resp.) of it. We say that a collection $\mathcal{V}$ of subsets of X is a refinement of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$ and $\cup\mathcal{V}$=$\cup\mathcal{U}$.

Definition 2.1. [1] A space X is called *strongly screenable* if each open cover of X has a σ-discrete, open refinement. A collection $\mathcal{U}$ of subsets of a topological space X is called *discrete* if the closures of the elements of $\mathcal{U}$ are disjoint, and if every subcollection of these closures has a closed union. Note that a collection $\mathcal{U}$ of subsets of a topological space X is discrete if and only if for every $x \in X$ there is a nbd (=neighborhood) O of x such that the cardinality of $\{U \in \mathcal{U} : O \cap U \neq \emptyset\}$ is at most one, see [5].

The descriptions and the details of topological game $G(\mathcal{DC}, X)$ are introduced in [3]. Some of the following Lemmas will be used in the latter sections.

Lemma 2.1. [7] Player I has a winning strategy in $G(\mathcal{DC}, X)$ if and only if there is a function s from 2^X into $2^X \cap \mathcal{DC}$ satisfying:

(a) $s(F) \subset F$ for each $F \in 2^X$;

(b) if $\{F_i : i \in \omega\}$ is a decreasing sequence of closed subsets of X such that $s(F_i) \cap F_{i+1}$=$\emptyset$ for each $i \in \omega$, then $\cap_{i\in\omega} F_i$=$\emptyset$.

Recall that a space X is called $\mathcal{DC}$-like if Player I has a winning strategy in $G(\mathcal{DC}, X)$.

Lemma 2.2. [3] If a space X has a countable closed cover by $\mathcal{DC}$-like sets, then X is a $\mathcal{DC}$-like space.

Definition 2.2. [2] A space X is called *scattered* if each nonempty closed subset A of X has an isolated point of A. A space X is called *C-scattered* if each nonempty closed subspace A has a point $a \in A$ with a compact nbd in A.

Notice that scattered spaces and locally compact Hausdorff spaces are C-scattered. For a space X, pick $J \in 2^X$. Put $J^{(1)}$=$\{x \in J : x$ has a compact nbd in $J\}$. Let $X^{(0)}$=X. If an ordinal α=$\beta + 1$, let $X^{(\alpha)}$=$(X^{(\beta)})^{(1)}$. If α is a limit ordinal, let $X^{(\alpha)}$=$\cap_{\beta<\alpha} X^{(\beta)}$. Clearly, a space X is C-scattered if and only if $X^{(\alpha)}$=$\emptyset$. If X is regular C-scattered and A is closed in X, then A is C-scattered (see, [2]). A space X is called σ-scattered if X is the union of countably many closed scattered subspaces.

Lemma 2.3. [3] (a) If a space X has a σ-closure-preserving cover by compact sets, then X is a $\mathcal{DC}$-like space.

(b) If X is a regular subparacompact, σ-scattered space, then X is a $\mathcal{DC}$-like space.

Lemma 2.4. [11] Let Z, X be spaces and $\mathcal{O}$ be an open cover of $Z \times X^\omega$, which is closed under finite unions. For $(z,x) \in Z\times X^\omega$, we have obtained a sequence $\{z_j : j \in \omega\}$ of points of Z, a sequence $\{K_j=\prod_{i\in\omega} K_{j,i} : j \in \omega\}$ of elements of $\mathcal{K}(Z \times X^\omega)$, a sequence $\{A_j : j \in \omega\}$ of finite subsets of ω and a decreasing sequence $\{B_j=U_j \times \prod_{i\in\omega} B_{j,i} : j \in \omega\}$ of elements of $\mathcal{B}$ such that: For $j \in \omega$,

(a) $(z,x) \in B_j$ and $K_{j+1,i} \subset \overline{B}_{j,i}$;

(b) $n(z_j, K_j) \leq n(z, K_j)$ and $n(B_j) < n(B_{j+1})$;

(c) $A_j \subset \{0, 1, \cdots, n(B_j)\}$ and if $n(z_{j+1}, K_{j+1}) \leq n(B_j)$, then there is an $i \in A_{j+1}$ with $i < n(z_{j+1}, K_{j+1})$;

(d) for each $i \leq n(B_j)$,

(d-1) if $i \in A_{j+1}$, then $K_{j+1,i} \cap \overline{B}_{j+1,i}=\emptyset$;

(d-2) if $i \notin A_{j+1}$, then $K_{j+1,i} \subset \overline{B}_{j+1,i}$, and if, in addition, $i \leq n(B_{j-1})$ and $i \notin A_j$, then $K_{j+1,i}=K_{j,i} \subset \overline{B}_{j+1,i}$, where $B_{-1}=Z \times X^\omega$.

Then there is an $i \in \omega$ such that $|\{j \in \omega : i \in A_j\}|=\omega$.

III. TThe Strongly Screenableness of Finite Products

Now, we state our main result in this paper.

Theorem 3.1. If X is a regular strongly screenable $\mathcal{DC}$-like space, then $X\times Y$ is strongly screenable for each strongly screenable space Y.

Proof: Let $\mathcal{G}$ be an open cover of X. Assume that $\mathcal{G}$ is closed under finite unions. Let D=$D_X \times D_Y$ be an open rectangle in $X \times Y$. Moreover, let $s : 2^X \to 2^X \cap \mathcal{DC}$ be a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$. Then there is a discrete collection $\mathcal{C}$ of compact sets in D_X such that $s(\overline{D_X})$=$\cup\mathcal{C}$. Since X is a regular strongly screenable, we can take a sequence $\{\mathcal{U}_m : m \in \omega\}$, where for each $m \in \omega$, $\mathcal{U}_m$=$\{U_\lambda : \lambda \in \Lambda_m\}$, of collections of open subsets in D_X such that:

(1) $\cup_{m\in\omega}\mathcal{U}_m$ covers D_X;

(2) each $\mathcal{U}_m$ is discrete;

(3) for each $m \in \omega$ and $\lambda \in \Lambda_m$, $\overline{U_\lambda}$ meets at most one element of $\mathcal{C}$.

Now, we define collections $\mathcal{H}_{m,n}(D)$ and $\mathcal{R}_{m,n}(D)$ of open rectangle in D for each $m, n \in \omega$ as follows.

Fix $\lambda \in \cup_{m\in\omega}\Lambda_m$. Put K_λ=$\overline{U_\lambda} \cap (\cup\mathcal{C})$. Then it is compact in X. For each $\lambda \in \cup_{m\in\omega}\Lambda_m$ and $y \in Y$, since $K_\lambda \times \{y\}$ is compact in $X \times Y$, there is a $G(y)$ such that $K_\lambda \times \{y\} \subset G(y)$. It follows from Wallace's theorem in [9] that there are $V^0_{\lambda,y} \in \mathcal{N}(y)$ and $E_{\lambda,y} \in \mathcal{N}(K_\lambda)$ such that $K_\lambda \times \{y\} \subset E_{\lambda,y} \times V^0_{\lambda,y} \subset G(y)$. Since Y is strongly screenableness, there is a σ-discrete open refinement $\cup_{n\in\omega}\{V^0_{\lambda,\xi} : \xi \in \Xi_n(\lambda)\}$ of $\{V^0_{\lambda,y} : y \in Y\}$ such that there is a $y(\xi) \in Y$ such that $V^0_{\lambda,\xi} \subset V^0_{\lambda,y(\xi)}$ for each $\xi \in \Xi_n(\lambda)$. Let $V_{\lambda,\xi}$=$V^0_{\lambda,\xi} \cap D_Y$ for each $\xi \in \Xi_n(\lambda)$ and each $\lambda \in \Lambda_m$. Then, $\{V_{\lambda,\xi} : \xi \in \Xi_n(\lambda)\}$ is a σ-discrete open cover of Y. Let $E_{\lambda,\xi}$=$E_{\lambda,y(\xi)}$ for each $\lambda \in \cup_{m\in\omega}\Lambda_m$ and each $\xi \in \Xi_n(\lambda)$. By the regularity of X, there is a $F_{\lambda,\xi} \in \mathcal{N}(K_\lambda)$ such $F_{\lambda,\xi} \subset \overline{F_{\lambda,\xi}} \subset E_{\lambda,\xi}$. Hence, the collections $\mathcal{V}_n(\lambda)$=$\{V_{\lambda,\xi} : \xi \in \Xi_n(\lambda)\}$ and two collections $\{E_{\lambda,\xi} : \xi \in \Xi_n(\lambda)\}$ and $\{F_{\lambda,\xi} : \xi \in \Xi_n(\lambda)\}$ satisfying

(4-1) $K_\lambda \subset F_{\lambda,\xi} \subset \overline{F_{\lambda,\xi}} \subset E_{\lambda,\xi}$ for each $\xi \in \Xi_n(\lambda)$;

(4-2) For each $\xi \in \Xi_n(\lambda)$, $E_{\lambda,\xi} \times V_{\lambda,\xi} \subset G$ for some $G \in \mathcal{G}$;

(4-3) $\mathcal{V}_n(\lambda)$ is discrete and $\{\cup\mathcal{V}_n(\lambda) : n \in \omega\}$ covers D_Y.

Define $H_{\lambda,\xi}$=$(E_{\lambda,\xi} \cap U_\lambda \cap D_X) \times V_{\lambda,\xi}$, and $R_{\lambda,\xi}$=$(U_\lambda \cap D_X \setminus \overline{F_{\lambda,\xi}}) \times V_{\lambda,\xi}$. Furthermore, let $\mathcal{H}_{m,n}(D)$=$\{H_{\lambda,\xi} : \xi \in \Xi_n(\lambda)$ and $\lambda \in \Lambda_m\}$, $\mathcal{R}_{m,n}(D)$=$\{R_{\lambda,\xi} : \xi \in \Xi_n(\lambda)$ and $\lambda \in \Lambda_m\}$ for each $m, n \in \omega$.

Observe that both $\mathcal{H}_{m,n}(D)$ and $\mathcal{R}_{m,n}(D)$ are discrete collections of open subrectangles by (2) and (4-3). Moreover, it follows from (1), (4-2) and (4-3) that:

(5-1) each member of $\cup_{m,n\in\omega}\mathcal{H}_{m,n}(D)$ is contained in some member of $\mathcal{G}$;

(5-2) $\cup_{m,n\in\omega}(\mathcal{H}_{m,n}(D) \cup \mathcal{R}_{m,n}(D))$ covers D.

Fix $m, n \in \omega$. For each $Z \in \mathcal{R}_{m,n}(D)$, we assert that

(6) $\overline{Z_0} \cap s(\overline{D_0})$=$\emptyset$.

Let $Z \in \mathcal{R}_{m,n}(D)$. Then, we can choose some $\lambda \in \Lambda_m$ and $\xi \in \Xi_n(\lambda)$ such that $\overline{Z_0} \cap s(\overline{D_0}) \subset K_\lambda \setminus E_{\lambda,\xi}$=$\emptyset$.

For each $\sigma \in (\omega \times \omega)^{<\omega}$ with $\sigma \neq \emptyset$, we define collections $\mathcal{H}_\sigma$ and $\mathcal{R}_\sigma$ of open sets of $X \times Y$. Fix $m, n \in \omega$. Let $\mathcal{H}_{m,n}$=$\mathcal{H}_{m,n}(X \times Y)$, $\mathcal{R}_{m,n}$=$\mathcal{R}_{m,n}(X \times Y)$. Assume that for $\varrho \in (\omega \times \omega)^{<\omega}$ with $\varrho \neq \emptyset$, we have already defined $\mathcal{H}_\varrho$ and $\mathcal{R}_\varrho$. Fix $\varrho \in (\omega \times \omega)^{<\omega}$ and $m, n \in \omega$. Let $\mathcal{H}_{\varrho\oplus(m,n)}$=$\cup\{\mathcal{H}_{m,n}(D) : D \in \mathcal{R}_\varrho\}$, $\mathcal{R}_{\varrho\oplus(m,n)}$=$\cup\{\mathcal{R}_{m,n}(D) : D \in \mathcal{R}_\varrho\}$.

Observe that, by (5-1), (5-2) and the induction, each $\mathcal{H}_{\varrho\oplus(m,n)}$ is discrete in $X \times Y$ and refines $\mathcal{G}$ partly. Hence, our proof is complete if we have

(7) $\{\cup\mathcal{H}_\sigma : \sigma \in (\omega \times \omega)^{<\omega} \setminus \{\emptyset\}\}$ covers $X \times Y$.

To show this, we assume that there is a $z \in X \times Y$ such that $z \notin \cup\mathcal{H}_\sigma$ for each $\sigma \in (\omega \times \omega)^{<\omega} \setminus \{\emptyset\}$. By (5-2), take a $D(0) \in \mathcal{R}_{\sigma_0}$ such that $z \in D(0)$ for some Σ_0=$(m_0, n_0) \in \omega \times \omega$. By (5-2) again, there is a $D(1) \in \mathcal{R}_{(m_1,n_1)}(D(0))$ such that $z \in D(1)$ for some $(m_1, n_1) \in \omega\times\omega$. Let σ_1=$((m_0, n_0), (m_1, n_1))$. Then, $D(1) \in \mathcal{R}_{\sigma_1}$ since $\mathcal{R}_{(m_1,n_1)}(D(0)) \subset \mathcal{R}_{\sigma_1}$. Continuing this matter, we can take some f=$((m_0, n_0), (m_1, n_1), \cdots) \in (\omega \times \omega)^\omega$ such that $z \in D(k) \in \mathcal{R}_{(m_k,n_k)}(D(k-1)) \subset \mathcal{R}_{f|_k}$ for each $k \in \omega$, where D_{-1}=$X \times Y$. Since $D(k+1) \subset D(k)$, we have $\overline{D_0(k+1)} \subset \overline{D_0(k)}$ for each $k \in \omega$. Furthermore, for each $k \in \omega$, $\overline{D_0(k+1)} \cap s(\overline{D_0(k)})$=$\emptyset$ by (6). It follows that $\cap_{k\in\omega}\overline{D_0(k)}$=$\emptyset$. This implies that $\cap_{k\in\omega}\overline{D(k)}$=$\emptyset$, which is a contradiction. Therefore, we can take some $k \in \omega$ such that $z \in \cup\mathcal{H}_{f|_k}$. Hence, $X \times Y$ is strongly screenable. □

As a consequence of Lemma 2.4 and Theorem 3.1, we have

Corollary 3.1. If X is a regular strongly screenable with a σ-closure-preserving cover by compact sets space, then $X\times Y$ is strongly screenable for each strongly screenable space Y.

Lemma 3.1. Let X be a regular strongly screenable $\mathcal{C}$-scattered space. Then

(a) if $X^{(\alpha+1)}$=$\emptyset$, there is a σ-discrete open cover $\mathcal{V}$=$\cup_{n\in\omega}\mathcal{V}_n$ of X such that $\overline{V}^{(\alpha)}$ is compact for each $V \in \mathcal{V}$;

(b) if $X^{(\alpha)}$=$\emptyset$, and α is limit. There is a σ-discrete open cover $\mathcal{V}$=$\cup_{n\in\omega}\mathcal{V}_n$ of X such that there is some ordinal $\beta < \alpha$

such that $\overline{V}^{(\beta)}=\emptyset$ for each $V \in \mathcal{V}$.

The proof is similar to that of [2, Theorem 1.6].

Lemma 3.2. Let $\mathcal{V}=\cup_{n\in\omega}\mathcal{V}_n$ be a σ-discrete open cover of space X. If $\overline{V} \times Y$ is strongly screenable for each $V \in \mathcal{V}$, then so is $X \times Y$.

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. Fix $V \in \mathcal{V}_n$, $n \in \omega$. Note that $\overline{V} \times Y$ is closed in $X \times Y$ and hence strongly screenable. Then there is a σ-discrete open refinement $\cup_{m\in\omega}\mathcal{W}_{V,m}$ of $\mathcal{U}|_{\overline{V}\times Y}$. For each $m, n \in \omega$, let $\mathcal{W}_{m,n}=\{W \cap (V \times Y) : W \in \mathcal{W}_{V,m}$ and $V \in \mathcal{V}_n\}$. Observe that $\cap_{m,n\in\omega}\mathcal{W}_{m,n}$ is a σ-discrete open refinement of $\mathcal{U}$.

Since the product $X \times Y$ of a compact space X and a strongly screenable space Y is strongly screenable, from Lemma 3.2, we get

Corollary 3.2. The product $X \times Y$ of a locally compact strongly screenable space X and a space strongly screenable Y is strongly screenable.

Theorem 3.2. If X is a regular strongly screenable $\mathcal{C}$-scattered space, then $X \times Y$ is strongly screenable for each strongly screenable space Y.

Proof. Since X is $\mathcal{C}$-scattered, $\alpha=inf\{\beta : X^{(\beta)}=\emptyset\}$ is an ordinal number. This proof proceeds by transfinite induction on α.

If α=1, then $X \times Y$ is strongly screenable by Corollary 3.2. For some ordinal α, we assume that the product $X \times Y$ of a regular strongly screenable $\mathcal{C}$-scattered space X and a strongly screenable space Y is strongly screenable if $\beta < \alpha$ and $X^{(\beta)}=\emptyset$. Our proof is complete if we show the assumption holds on ordinal α.

(1) Assume that $X^{(\alpha)}=\emptyset$, and α is limit. From Lemma 3.1 (2) and 3.2 and inductions, $X \times Y$ is strongly screenable.

(2) Assume that $X^{(\alpha+1)}=\emptyset$. It follows from Lemma 3.1 (1) that there is a σ-discrete open cover $\mathcal{V}=\cup_{n\in\omega}\mathcal{V}_n$ of X such that $\overline{V}^{(\alpha)}$ is compact for each $V \in \mathcal{V}$. Put $T=\overline{V}$. We claim that $T \times Y$ is strongly screenable.

Let $\mathcal{U}$ be an open cover of $T \times Y$. Assume without loss of generality that $\mathcal{U}$ is closed under finite unions. Since $\overline{T}^{(\alpha)}$ is regular compact subspace of X, we can take two sequences $\{R_\xi(0) \times V_\xi : \xi \in \Xi_n\}$ and $\{R_\xi(1) \times V_\xi : \xi \in \Xi_n\}$, $n \in \omega$, of collections by open rectangles in $X \times Y$ satisfying

(3) $\overline{T}^{(\alpha)} \subset R_\xi(0) \subset \overline{R_\xi(0)} \subset R_\xi(1)$;

(4) $R_\xi(1) \times V_\xi$ is contained in some member of $\mathcal{U}$;

(5) $\cup_{n\in\omega}\{V_\xi : \xi \in \Xi_n\}$ covers Y and each $\{V_\xi : \xi \in \Xi_n\}$ is discrete.

Here notice that $(T \setminus R_\xi(0))^{(\alpha)}=\emptyset$ and $(T \setminus R_\xi(0))$ closed in X. By the inductive assumption, $(T \setminus R_\xi(0)) \times Y$ is strongly screenable. Then, there is a σ-discrete open refinements $\cup_{m\in\omega}\mathcal{W}_{\xi,m}$ of $\mathcal{U}\left|_{(T\setminus R_\xi(0))\times Y}\right.$ such that it covers $(T \setminus R_\xi(0)) \times Y$. For each $m, n \in \omega$, let $\mathcal{H}_{m,n}=\{R_\xi(1) \times V_\xi : \xi \in \Xi_n\} \cup \{W \cap [(T \setminus \overline{R_\xi(0)}) \times V_\xi] : W \in \mathcal{W}_{\xi,m}, \xi \in \Xi_n\}$. Then, it is easy to check that $\cup_{m,n\in\omega}\mathcal{H}_{m,n}$ is a σ-discrete open refinements of $\mathcal{U}$, which witnesses the strongly screenableness of $T \times Y$.

Hence, it follows from Lemma 3.2 that $X \times Y$ is strongly screenable. □.

IV. The strongly screenableness of countable products

Let $\{Y_i : i \in \omega\}$ be a countable collection of spaces. Let us denote by $\mathcal{B}$ the base of X^ω consisting of sets of the form $D=\prod_{i\in\omega} D_i$. For each $i \le n$, let D_i be an open subset of X. For each $i > n$, let $D_i=X$. Moreover, for each $D \in \mathcal{B}$, let $n(D)=inf\{i \in \omega : D_j=X$ for each $j \ge i\}$, where $n(X^\omega)=0$. Now, we study strongly screenableness of $\prod_{i\in\omega} Y_i$.

Theorem 4.1. If Y_i is a regular strongly screenable $\mathcal{DC}$-like space for each $i \in \omega$, then $\prod_{i\in\omega} Y_i$ is strongly screenable.

Proof. Let $X=\oplus_{i\in\omega}Y_i$. The topology of X is as follows: Every Y_i is an open-and-closed subspace of X. By Lemma 2.2, if every Y_i is a regular strongly screenable $\mathcal{DC}$-like space, then so is X. Hence, without loss of generality, we may assume that $Y_i=X$ for each $i \in \omega$. Here notice that $\prod_{i\in\omega} Y_i$ is a closed subspace of X^ω. Therefore, it suffices to prove that X^ω is strongly screenable.

Let $\mathcal{U}$ be an open cover of X^ω. Without loss of generality, it can be assumed that $\mathcal{U}$ is closed under finite unions. Let $\mathcal{U}^*=\{D \in \mathcal{B} : D \subset U$ for some $U \in \mathcal{U}\}$. For each $K \in \mathcal{K}$, by the compactness of K, there is an $U \in \mathcal{U}$ such that $K \subset U$. Moreover, it follows from Wallace's theorem in Engelking [9] that there is a $D \in \mathcal{B}$ such that $K \subset D \subset U$. Clearly, $D \in \mathcal{U}^*$. Furthermore, let $n(K)=inf\{n(U) : U \in \mathcal{U}^*$ and $K \subset U\}$.

Let $s : 2^X \to 2^X \cap \mathcal{DC}$ be a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$. Fix an open set $D=\prod_{i\in\omega} D_i \in \mathcal{B}$. For each $\eta \in \omega^{n(D)+1}$, we shall construct two collections $\mathcal{G}_\eta(D)$ and $\mathcal{D}_\eta(D)$ of open subrectangles of D such that:

(1) $\cup\{\cup(\mathcal{G}_\eta(D) \cup \mathcal{D}_\eta(D)) : \eta \in \omega^{n(D)+1}\}=D$;

(2) Both $\mathcal{G}_\eta(D)$ and $\mathcal{D}_\eta(D)$ are discrete in X^ω;

(3) $\mathcal{G}_\eta(D)$ refines $\mathcal{U}^*$ partly;

(4) The length of nonempty members of $\mathcal{D}_{\eta,\delta}(D)$ is $n(D)+1$.

For each $i \le n(D)$, assume that a compact set $C_{\lambda(D,i)}$ have been defined in $\overline{D_i}$. Note that $C_{\lambda(D,i)}=\emptyset$ may occur for each $i \le n(D)$. Fix $i \le n(D)$. If $C_{\lambda(D,i)} \neq \emptyset$, then let $V_{\gamma(D,i,m)}=D_i$ for each $m \in \omega$. Let us put $\Lambda(D,i)=\{\lambda(D,i)\}$ and $\Gamma(D,i,m)=\{\gamma(D,i,m)\}$ for each $m \in \omega$. Then, we define $\mathcal{C}(D,i)=\{C_\lambda : \lambda \in \Lambda(D,i)\}=\{C_{\lambda(D,i)}\}$ and for each $m \in \omega$, let $\mathcal{V}(D,i,m)=\{V_\gamma : \gamma \in \Gamma(D,i,m)\}=\{V_{\gamma(D,i,m)}\}$. Otherwise, there a discrete collection $\mathcal{C}(D,i)=\{C_\lambda : \lambda \in \Lambda(D,i)\}$ of compact subsets in X such that $s(\overline{D_i})=\cup\mathcal{C}(D,i)$. By the strongly screenableness and regularity of X, we take a sequence $\{\mathcal{V}(D,i,m) : m \in \omega\}$, where for each $m \in \omega$, $\mathcal{V}(D,i,m)=\{V_\gamma : \gamma \in \Gamma(D,i,m)\}$, of collections by open sets in D_i, satisfying

(5) $\mathcal{V}(D,i,m)$ is discrete in X;

(6) $\cup_{m\in\omega}\mathcal{V}(D,i,m)$ covers D_i;

(7) $\overline{V_\gamma}$ meets at most one element of $\mathcal{C}(D,i)$ for each $\gamma \in \cup_{m\in\omega}\Gamma(D,i,m)$.

For $\gamma \in \cup_{m\in\omega}\Gamma(D,i,m)$, let $K_\gamma=\overline{V_\gamma} \cap C_\lambda$ if $\overline{V_\gamma} \cap (\cup\mathcal{C}(D,i)) \neq \emptyset$ and let $K_\gamma=\{s_\gamma\}$ for a point $s_\gamma \in V_\gamma$ otherwise. Hence, $K_{\gamma(D,i,m)}=C_{\lambda(D,i)}$ if $C_{\lambda(D,i)} \neq \emptyset$.

Fix $\eta=(m_0, \cdots, m_{n(D)}) \in \omega^{n(D)+1}$. Let $\Delta_{D,\eta}=\Gamma(D,0,m_0) \times \cdots \times \Gamma(D,n(D),m_{n(D)})$. Fix $\delta=(\gamma(\delta,0), \cdots, \gamma(\delta,n(D))) \in \Delta_{D,\eta}$. Let $K(\delta)=K_{\gamma(\delta,0)} \times \cdots \times K_{\gamma(\delta,n(D))} \times \{s\} \times \cdots \times \{s\} \times \cdots$, and let $r(K(\delta))$=$max\{n(K(\delta)), n(D)\}$. Then, $\mathcal{K}_{D,\eta}$=$\{K(\delta) : \delta \in \Delta_{D,\eta}\} \subset \mathcal{K}$. Take an $U(\delta)$=$\prod_{i\in\omega} U(\delta)_i \in \mathcal{U}^*$ such that $K(\delta) \subset U(\delta)$ and $n(K(\delta))$=$n(U(\delta))$. Furthermore, since X is regular strongly screenable, we can choose an $L(\delta)$=$\prod_{i\in\omega} L(\delta)_i \in \mathcal{B}$ satisfying:

(8) $\prod_{i=0}^{(n(K(\delta))-1)} \overline{L(\delta)_i} \times X \times \cdots \subset U(\delta)$;

(9-1) for i with $n(K(\delta) \le i \le r(K(\delta))$, let $L(\delta)_i$=X;

(9-2) for $i < n(K(\delta)$ with $i \le n(D)$, let $L(\delta)_i$ be an open subset of X such that $K_{\gamma(\delta,i)} \subset L(\delta)_i \subset \overline{L(\delta)_i} \subset U(\delta)_i$;

(9-3) for each i with $n(D) < i < n(K(\delta))$, let $L(\delta)_i$=$\{s\}$;

(9-4) If $r(K(\delta))$=$n(D)$, let $L(\delta)_i$=X for each $i > n(D)$. If $r(K(\delta))$=$n(K(\delta))$ with $\neq n(D)$, let $L(\delta)_i$=X for each $i \ge n(K(\delta))$.

Then we have $K(\delta) \subset L(\delta) \subset \overline{L(\delta)} \subset U(\delta)$. For each $\eta \in \omega^{n(D)+1}$ and $\delta \in \Delta_{D,\eta}$, let $V(\delta)$=$\prod_{i=0}^{n(D)} V_{\gamma(\delta,i)} \times X \times \cdots$. Moreover, let $\mathcal{V}_\eta$=$\{V(\delta) : \delta \in \Delta_{D,\eta}\}$. It follows that each $\mathcal{V}_\eta$ is discrete in X^ω by (5).

For each $A \in \mathcal{P}(\{0, 1, \cdots, n(D)\})$, we define G_δ=$\prod_{i\in\omega} G_{\delta,i}$, and $D_{\delta,A}$=$\prod_{i\in\omega} D_{\delta,A,i}$ as follows.

(10) In case $r(K(\delta))$=$n(D)$. Let $G_{\delta,i}$=$U(\delta)_i \cap V_{\gamma(\delta,i)}$ if $i \le n(D)$, and let $G_{\delta,i}$=X for each $i > n(D)$. In case $r(K(\delta))$=$n(K(\delta)) > n(D)$. Let $G_{\delta,i}$=$\emptyset$ for each $i \in \omega$.

(11) In each case, for each $i \le n(D)$, let $D_{\delta,A,i}$=$V_{\gamma(\delta,i)} \setminus \overline{L(\delta)_i}$ if $i \in A$ and let $D_{\delta,A,i}$=$U(\delta)_i \cap V_{\gamma(\delta,i)}$ otherwise. For each $i > n(D)$, let $D_{\delta,A,i}$=X.

Obviously, $D_{\delta,\emptyset}$=G_δ if $r(K(\delta))$=$n(D)$. Here notice that $D_{\delta,A,i} \subset D_i$ for each $i \in \omega$ and that $n(D_{\delta,A})$=$n(D) + 1$ whenever $D_{\delta,A} \neq \emptyset$. For each $\eta \in \omega^{n(D)+1}$, let us put

$$\mathcal{G}_\eta(D) = \{G_\delta : \delta \in \Delta_{D,\eta}\}.$$

Fix $\eta \in \omega^{n(D)+1}$ and $\delta \in \Delta_{D,\eta}$. In case $r(K(\delta))$=$n(D)$, let $\mathcal{D}_{\eta,\delta}(D)$=$\{D_{\delta,A} : A \in \mathcal{P}(\{0, 1, \cdots, n(D)\}) \setminus \{\emptyset\}\}$. In case $r(K(\delta))$=$n(K(\delta)) > n(D)$, let $\mathcal{D}_{\eta,\delta}(D)$=$\{D_{\delta,A} : A \in \mathcal{P}(\{0, 1, \cdots, n(D)\})\}$. For each $\eta \in \omega^{n(D)+1}$, we set

$$\mathcal{D}_\eta(D) = \{\cup\mathcal{D}_{\eta,\delta}(D) : \delta \in \Delta_{D,\eta}\}.$$

In addition this, for each $\eta \in \omega^{n(D)+1}$ and each $\delta \in \Delta_{D,\eta}$, we have $V(\delta)$=$G_\delta \cup (\cup\mathcal{D}_{\eta,\delta}(D))$. Then, we can check that $\mathcal{G}_\eta(D)$ and $\mathcal{D}_\eta(D)$ are the collections desired before and satisfying the conditions (1)-(4).

For each $\eta \in \omega^{n(D)+1}$, $\delta \in \Delta_{D,\eta}$ and $A \in \mathcal{P}(\{0, 1, \cdots, n(D)\})$, let $D_{\delta,A}$=$\prod_{i\in\omega} D_{\delta,A,i} \in \mathcal{D}_{\eta,\delta}(D)$. Then, for each $i \in A$, we claim that

(12) $K_{\gamma(\delta,i)} \cap \overline{D_{\delta,A,i}}$=$\emptyset$, and $s(\overline{D_i}) \cap \overline{D_{\delta,A,i}}$=$\emptyset$ if $C_{\lambda(D,i)}$=$\emptyset$.

Indeed, $K_{\gamma(\delta,i)} \cap \overline{D_{\delta,A,i}} \subset K_{\gamma(\delta,i)} \cap V_{\gamma(\delta,i)} \cap (X \setminus L(\delta)_i)$=$\emptyset$ since $D_{\delta,A,i}$=$V_{\gamma(\delta,i)} \setminus \overline{L(\delta)_i}$. If $C_{\lambda(D,i)}$=$\emptyset$, then $s(\overline{D_i}) \cap \overline{D_{\delta,A,i}}$=$(\cup\mathcal{C}(D,i)) \cap \overline{(V_{\gamma(\delta,i)} \setminus \overline{L(\delta)_i})} \subset K_{\gamma(\delta,i)} \cap V_{\gamma(\delta,i)} \cap (X \setminus L(\delta)_i)$=$\emptyset$.

Fix $i \le n(D)$. We may assume that a compact subset $K_{\gamma(\delta,i)}$ of $\overline{D_{\delta,A,i}}$ have been defined if $i \notin A$. Let $C_{\lambda(D_{\delta,A},i)}$=$K_{\gamma(\delta,i)}$. And let $C_{\lambda(D_{\delta,A},i)}$=$\emptyset$ if $i \in A$.

For each $t \in \omega$, let Π_t=$\prod_{i=0}^{t} \omega^{i+1}$. We shall inductively define two collections $\mathcal{G}_\tau$ and $\mathcal{D}_\tau$, $\tau \in \Pi_t$, by open rectangles in X^ω, satisfying

(13) for each $t \in \omega$ and $\tau \in \Pi_t$, $\mathcal{G}_\tau$ refines $\mathcal{U}^*$ partly;

(14) for each $t \in \omega$ and $\tau \in \Pi_t$, $\mathcal{G}_\tau$ and $\mathcal{D}_\tau$ are discrete in X^ω.

For each $m \in \Pi_0$=ω, let $\mathcal{G}_m$=$\mathcal{G}_m(X^\omega)$ and $\mathcal{D}_m$=$\mathcal{D}_m(X^\omega)$. Then, by (2), (3) and (4), $\mathcal{G}_m$ and $\mathcal{D}_m$ satisfy the conditions (13) and (14). Assume that for each $t \in \omega$ and $\tau \in \Pi_t$, we have already defined $\mathcal{G}_\sigma$ and $\mathcal{D}_\sigma$ and satisfying the conditions (13) and (14). Let $\tau \in \Pi_{t+1}$ and τ=$\sigma \oplus \eta$, where σ=$\tau_- \in \Pi_t$ and $\eta \in \omega^{t+2}$. Take a $D \in \mathcal{D}_\sigma$. If $D \neq \emptyset$, then we denote $\mathcal{G}_\eta(D)$ and $\mathcal{D}_\eta(D)$ respectively by $\mathcal{G}_\tau(D)$ and $\mathcal{D}_\tau(D)$. And let $\mathcal{G}_\tau(D)$=$\mathcal{D}_\tau(D)$=$\{\emptyset\}$ if D=$\emptyset$. Define $\mathcal{G}_\tau$=$\{\mathcal{G}_\tau(D) : D \in \mathcal{D}_\sigma\}$ and $\mathcal{D}_\tau$=$\{\mathcal{D}_\tau(D) : D \in \mathcal{D}_\sigma\}$. By (1) and inductions, it is easy to check that each member of $\mathcal{G}_\tau$ is contained in some member of $\mathcal{U}^*$. Moreover, it follows from (1), (2) and the inductive assumption that $\mathcal{G}_\tau$ and $\mathcal{D}_\tau$ satisfy the condition (14). Let Π=$\cup_{t\in\omega}\Pi_t$. Clearly, $|\Pi| \le \omega$. Our proof will be complete, if we show

(15) $\cup_{\tau\in\Pi}\mathcal{G}_\tau$ covers X^ω.

Assume the contrary. We can take a point x=$(x_i)_{i\in\omega} \in X^\omega$ such that $x \notin \cup\mathcal{G}_\tau$ for each $\tau \in \Pi$. Since $n(X^\omega)$=0, we can choose a $\tau(0)$=m_0=$\eta(0)$ such that $x \notin \cup\mathcal{G}_{\tau(0)}$. By (1), (4) and inductions, there are a $\delta(0)$=$(\gamma(\delta(0), 0)) \in \Delta_{X^\omega,m(0)}$, and $A(0) \in \mathcal{P}(\{0\})$ such that $x \in D_{\delta(0),A(0)}$ and $n(D_{\delta(0),A(0)})$=1, $D_{\delta(0),A(0)} \in \mathcal{D}_{\tau(0),\delta(0)}(X^\omega)$. For $D_{\delta(0),A(0)}$, we can pick an $\eta(1) \in \omega^2$, $\delta(1)$=$(\gamma(\delta(1), 0), \gamma(\delta(1), 1)) \in \Delta_{D_{\delta(0),A(0)},\eta(1)}$, and $A(1) \in \mathcal{P}(\{0, 1\})$ such that $x \in D_{\delta(1),A(1)}$ and $n(D_{\delta(1),A(1)})$=$n(D_{\delta(0),A(0)}) + 1$, $D_{\delta(1),A(1)} \in \mathcal{D}_{\tau(1),\delta(1)}(D_{\delta(0),A(0)})$. Let $K(1)$= $K(\delta(1)) \in \mathcal{K}_{D_{\delta(0),A(0)},\eta(1)}$. Continuing in this manner, we can choose a sequence $\{\eta(t) : t \in \omega\}$ of elements of $\omega^{<\omega}$, a sequence $\{A(t) : t \in \omega\}$, where $A(t) \in \mathcal{P}(\{0, 1, \cdots, t\})$, a sequence $\{\delta(t) : t \in \omega\}$, where $\delta(t) \in \Delta_{D_{\delta(t-1),A(t-1)},\eta(t)}$, a sequence $\{K(\delta(t)) : t \in \omega\}$ of elements of $\mathcal{K}$, where $K(\delta(t))$=$\prod_{i\in\omega} K(\delta(t))_i \in \mathcal{K}_{D_{\delta(t-1),A(t-1)},\eta(t)}$ and a decreasing sequence $\{D_{\delta(t),A(t)} : t \in \omega\}$ of elements $\mathcal{D}$ such that $D_{\delta(t),A(t)}$ contains x for each $t \in \omega$, where $D_{\delta(t),A(t)} \in \mathcal{B}_{\tau(t),\delta(t)}(D_{\delta(t-1),A(t-1)})$. For each $t \in \omega$, we may assume without loss of generality that $D(t - 1)$=$D_{\delta(t-1),A(t-1)}$, where $D_{\delta(-1),A(-1)}$=$D(-1)$=X^ω. And denote $K(\delta(t))$ by $K(t)$. Furthermore, let $D(t)_i$=$D_{\delta(t),A(t),i}$ for each $i \in \omega$. From above argument, we have

(16) For each $t \in \omega$, $n(D(t))$=$n(D(t - 1)) + 1$=$t + 1$.

(17) If $r(K(t + 1))$=$t + 1$, then there is an $i \in A(t + 1)$ with $i < n(K(t + 1))$.

Observe that $r(K(t + 1))$=$n(D(t))$ if $r(K(t + 1))$=$t + 1$. This implies that $A(t+1) \neq \emptyset$. Assume that $i \ge n(K(t+1))$ for each $i \in A(t + 1)$. By the definition of $L(\delta(t + 1))_i$, then $L(\delta(t + 1))_i$=X. In addition, $D(t + 1)_i$=$V_{\gamma(\delta(t+1),i)} \setminus \overline{L(\delta(t + 1))_i}$=$\emptyset$. But, $x_i \in D(t + 1)_i$. This is a contradiction.

(18) If $i \le t + 1$ with $i \in A(t + 1)$, then $K(t + 1)_i \cap \overline{D(t + 1)_i}$=$\emptyset$. If, in addition, $C_{\lambda(D(t),i)}$=$\emptyset$, then $s(\overline{D(t)_i}) \cap \overline{D(t + 1)_i}$=$\emptyset$.

(19) If $i \le t + 1$ with $i \notin A(t + 1)$, then $K(t + 1)_i \subset \overline{D(t + 1)_i}$. If, in addition, $i \le t$ with $i \notin A(t)$, then $\overline{C_{\lambda(D(t),i)}} \neq \emptyset$ and hence, $K(t)_i$=$K(t + 1)_i$=$C_{\lambda(D(t),i)} \subset \overline{D(t + 1)_i}$.

By Lemma 2.4, we can choose an $i \in \omega$ such that $|\{t \in \omega : i \in A(t)\}|$=$\omega$. We may assume that $\{t \in \omega : i \in A(t)$ and $i \le t + 1\}$=$\{t_k : k \in \omega\}$. Then, $C_{\lambda(D(t_k),i)}$=$\emptyset$

for each $k \in \omega$. In case t_{k+1}=$t_k + 1$. By (18), $s(\overline{D(t_k)_i}) \cap \overline{D(t_{k+1})_i}$=$\emptyset$ clearly. In case $t_{k+1} > t_k + 1$. By (18) and (19), we have $K(t_k + 1)_i = \overline{C_{\lambda(D(t_k+1),i)}} = \overline{C_{\lambda(D(t_{k+1}-1),i)}} = \overline{K(t_k + 1)_i} \subset \overline{L(\delta(t_k + 1))_i}$. Thus $s(\overline{D(t_k)_i}) \cap \overline{D(t_{k+1})_i}$= $\overline{\cup\mathcal{C}(D(t_k), i)} \cap \overline{D(t_k + 1)_i}$= $\overline{\cup\mathcal{C}(D(t_k), i)} \cap \overline{D(t_k + 1)_i} \cap \overline{D(t_{k+1})_i} \subset K(t_k + 1)_i \cap \overline{D(t_{k+1})_i}$=$\emptyset$. Since s is a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$, $\cap_{k\in\omega}\overline{D(t_k)_i}$=$\emptyset$. But $x_i \in \cap_{k\in\omega}D(t_k)_i$, which is a contradiction. Thus there is a $t \in \omega$ such that $x \in \cup\mathcal{G}_{\tau(t)}$, which witnesses the strongly screenableness of X^ω.

Hence, $\prod_{i\in\omega} Y_i$ is strongly screenable. □

Immediately, it follows from Lemma 2.3 (a) and Theorem 4.1, we have

Corollary 4.1 If Y_i is a regular strongly screenable with a σ-closure-preserving cover by compact sets for each $i \in \omega$, then $\prod_{i\in\omega} Y_i$ is strongly screenable.

Similarly, by Lemma 2.4 (b) and Theorem 4.1, we obtain

Corollary 4.2 If Y_i is a regular strongly screenable, σ-scattered space for each $i \in \omega$, then $\prod_{i\in\omega} Y_i$ is strongly screenable.

Remark. It is easy to verify that Theorem 4.1 holds for σ-metacompactness.

V. Example

As is well-known, most covering properties are poorly maintained in respect to products. The following two examples shall show that the condition $\mathcal{DC}$-like can not be omitted in Theorem 3.1 and Theorem 4.1.

Example 5.1. There exists a regular strongly screenable space X such that X^2 is not strongly screenable.

Proof. In [6], Przymusinski have shown that there exists a first countable, separable, Lindelof space X such that X^2 is not subparacompact. It is easy to check that X is regular strongly screenable. Moreover X^2 is regular. Assume that X^2 is strongly screenable. Then, X^2 is subparacompact since regular strongly screenable is paracompact. This is a contradiction. □

Example 5.2. There exists a strongly screenable X such that X^n is strongly screenable for each $n \in \omega$, but X^ω is not strongly screenable.

Proof. Following the argument of [13, Example 3.5], it is easy to check that each X^n is regular Lindelof but X^ω is not screenable. So, X^ω is not strongly screenable. □.

Acknowledgment

The authors would like to express their thanks to the Nature Scientific Fund Committee of China for its subsidy to this subject.

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