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Using the OWA Operator in the Minkowski Distance

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***Abstract*—**We study different types of aggregation operators such as the ordered weighted averaging (OWA) operator and the generalized OWA (GOWA) operator. We analyze the use of OWA operators in the Minkowski distance. We will call these new distance aggregation operator the Minkowski ordered weighted averaging distance (MOWAD) operator. We give a general overview of this type of generalization and study some of their main properties. We also analyze a wide range of particular cases found in this generalization such as the ordered weighted averaging distance (OWAD) operator, the Euclidean ordered weighted averaging distance (EOWAD) operator, the normalized Minkowski distance, etc. Finally, we give an illustrative example of the new approach where we can see the different results obtained by using different aggregation operators.

***Keywords*—**Aggregation operators, Minkowski distance, OWA operators, Selection of strategies.

I. Introduction

THE distance measures are very useful techniques that have been used in a wide range of applications such as fuzzy set theory, multicriteria decision making, business decisions, etc. Among the great variety of distances we can find in the literature, the Minkowski distance represents a generalization to a wide range of them such as the Hamming distance, the Euclidean distance, the geometric distance and the harmonic distance.

Often, when calculating distances, we want an average result of all the individual distances. We call this the normalization process. In the literature, we find principally two types of normalized distances. The first type is the case when we normalize the distance giving the same weight to all the individual distances. The second type is the case when we normalize the distance giving different weights to the individual distances. Then, assuming that we are using the Minkowski distance, for the first type we will obtain the normalized Minkowski distance and for the second type the weighted Minkowski distance.

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Sometimes, when calculating the normalized distance, it would be interesting to consider the attitudinal character of the decision maker. A very useful technique for the aggregation of the information considering the attitudinal character of the decision maker is the ordered weighted averaging (OWA) operator introduced by Yager in [1]. The OWA operator provides a parameterized family of aggregation operators that include the maximum, the minimum and the average criteria. It has been used in a wide range of applications such as [2]–[21].

In this paper, we suggest a new type of distance measure consisting in normalize the Minkowski distance with the OWA operator. Then, the normalization developed will reflect the attitudinal character of the decision maker and it will provide a parameterized family of distance operators that include the maximum distance, the minimum distance and the average distance. We will call this generalization as the Minkowski ordered weighted averaging distance (MOWAD) operator. By studying special cases of the MOWAD operator, we will be able to develop a wide range of distance operators such as the Hamming ordered weighted averaging distance (HOWAD) operator, the Euclidean ordered weighted averaging distance (EOWAD) operator, the ordered weighted geometric averaging distance (OWGAD) operator and the ordered weighted harmonic averaging distance (OWHAD) operator. We should note that some considerations about using OWA operators in distance measures have been studied in [21].

This paper is organized as follows. In Section II, we briefly describe some aggregation operators such as the OWA operator, the generalized OWA operator and the Minkowski distance. In Section III, we develop the MOWAD operator. In Section IV, we study different families of MOWAD operators and in Section V we present an illustrative example of the new approach. Finally, in Section VI, we summarize the main conclusions found in the paper.

II. AGGREGATION OPERATORS

In this Section, we briefly describe the OWA operator, the generalized OWA (GOWA) operator and the normalized Minkowski distance.

A. OWA Operator

The OWA operator was introduced by Yager in [1] and it provides a parameterized family of aggregation operators that include the maximum, the minimum and the arithmetic mean. It can be defined as follows.

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Definition 1. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$OWA(a_1, a_2,\ldots, a_n) = \sum_{j=1}^{n} w_j b_j \quad (1)$$

where b_j is the jth largest of the a_i.

From a generalized perspective of the reordering step, we have to distinguish between the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator [12]. Note that this distinction in the reordering step is relevant in order to distinguish between situations where the highest argument is the best result and situations where the lowest argument is the best result [25].

B. GOWA Operator

The GOWA operator [17] is a generalization of the OWA operator by using generalized means. The generalized mean was introduced in [26]–[27] and it represents a generalization to a wide range of mean aggregations. It can be defined as follows.

Definition 2. A generalized mean of dimension n is a mapping GM: $R^n \rightarrow R$ such that:

$$GM(a_1, a_2,\ldots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^{\lambda} \right)^{1/\lambda} \quad (2)$$

where a_i is the argument variable and λ is a parameter such that $\lambda \in (-\infty, \infty)$. Note that depending on the value of the parameter λ, we obtain different types of means. When $\lambda = \infty$, we obtain the maximum. When $\lambda = 1$, the arithmetic mean. When $\lambda = 0$, the geometric mean. When $\lambda = -1$, the harmonic mean. When $\lambda = 2$, the quadratic mean. When $\lambda = -\infty$, the minimum.

Note that if the arguments have different weights, then, the generalized mean is transformed in the weighted generalized mean. With this information, we can define the GOWA operator as follows.

Definition 3. A GOWA operator of dimension n is a mapping $GOWA$: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$GOWA(a_1, a_2,\ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \quad (3)$$

where b_j is the jth largest of the a_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

From a generalized perspective of the reordering step, we can distinguish between the descending generalized OWA (DGOWA) operator and the ascending generalized OWA (AGOWA) operator.

It can be demonstrated that the GOWA operator generalizes a wide range of aggregation operators [17] such as the maximum or the minimum.

Other special cases obtained with the weighting vector of the GOWA operator [17] are the generalized mean and the weighted generalized mean. Then, the GOWA operator also includes the particular cases of the generalized mean such as the arithmetic mean, the geometric mean, the harmonic mean and the quadratic mean, and the particular cases of the weighted generalized mean such as the weighted average, the weighted geometric mean, the weighted harmonic mean and the weighted quadratic mean.

If we analyze the parameter λ, we can also obtain another group of special cases such as the usual OWA operator [1], the ordered weighted geometric (OWG) operator [28]–[30], the ordered weighted harmonic averaging (OWHA) operator [17] and the ordered weighted quadratic averaging (OWQA) operator [17]. Note that this group of particular cases can be constructed with a descending or an ascending order.

C. Normalized Minkowski Distance

The normalized Minkowski distance is a distance measure that generalizes a wide range of distances such as the normalized Hamming distance, the normalized Euclidean distance, the normalized geometric distance and the normalized harmonic distance. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. It can be formulated for two sets A and B as follows.

Definition 4. A normalized Minkowski distance of dimension n is a mapping d_m: $R^n \times R^n \rightarrow R$ such that:

$$d_m(A,B) = \left(\frac{1}{n} \sum_{i=1}^{n} | a_i - b_i |^{\lambda} \right)^{1/\lambda} \quad (4)$$

where a_i and b_i are the ith arguments of the sets A and B and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

If we give different values to the parameter λ, we can obtain a wide range of special cases. For example, if $\lambda = 1$, we obtain the normalized Hamming distance. If $\lambda = 2$, the normalized Euclidean distance. If $\lambda = 0$, the normalized geometric distance. If $\lambda = -1$, the normalized harmonic distance. Note that the formulation shown above is the general expression. For the formulation used in fuzzy set theory see for example [31]–[33].

Sometimes, when normalizing the Minkowski distance, we prefer to give different weights to each individual distance.

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Then, the distance is known as the weighted Minkowski distance. It can be defined as follows.

Definition 5. A weighted Minkowski distance of dimension n is a mapping d_{wm}: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$. Then:

$$d_{wm}(A,B) = \left(\sum_{i=1}^{n} w_i \mid a_i - b_i \mid^{\lambda} \right)^{1/\lambda} \quad (5)$$

where a_i and b_i are the ith arguments of the sets A and B and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

In this case, we can also obtain a wide range of special cases by using different values in the parameter λ. For example, if $\lambda = 1$, we obtain the weighted Hamming distance. If $\lambda = 2$, the weighted Euclidean distance. If $\lambda = 0$, the weighted geometric distance. If $\lambda = -1$, the weighted harmonic distance.

III. THE MINKOWSKI ORDERED WEIGHTED AVERAGING DISTANCE OPERATOR

The Minkowski OWAD (MOWAD) operator represents an extension of the traditional normalized Minkowski distance by using OWA operators. The difference is that we reorder the arguments of the individual distances according to their values. Then, we can calculate the distance between two elements, two sets, two fuzzy sets, etc., modifying the results according to the attitudinal character of the decision maker. For example, this type of distance is useful when a decision maker wants to compare two fuzzy subsets but he wants to give more importance to the highest individual distance because he believes that it will be more significant in the analysis. Note that this type of normalized distance operator can be constructed by mixing the Minkowski distance with OWA operators, by mixing the Hamming distance with GOWA operators or by mixing the Hamming OWAD operator with generalized means. It can be defined as follows.

Definition 6. A Minkowski OWAD operator of dimension n is a mapping $MOWAD$: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$. Then, the distance between two sets A and B is:

$$MOWAD(d_1, d_2, \ldots, d_n) = \left(\sum_{j=1}^{n} w_j D_j^{\lambda} \right)^{1/\lambda} \quad (6)$$

where D_j is the jth largest of the d_i and d_i is the individual distance between A and B. That is, $d_i = |a_i - b_i|$. λ is a parameter such that $\lambda \in (-\infty, \infty)$. As we can see, we adapt the characteristics of the Minkowski distance to the characteristics of the OWA operator. Note that different notations are possible in order to formulate this type of aggregation such as: *MOWAD(A,B)*.

A fundamental aspect of the MOWAD operator is the reordering of the arguments based upon their values. That is, the weights rather than being associated with a specific argument, as in the case with the usual Minkowski distance, are associated with a particular position in the ordering. This reordering introduces nonlinearity into an otherwise linear process.

If D is a vector corresponding to the ordered arguments D_j^{λ}, we shall call this the ordered argument vector, and W^T is the transpose of the weighting vector, then the MOWAD aggregation can be expressed as:

$$MOWAD(d_1, d_2, \ldots, d_n) = \left(W^T D \right)^{1/\lambda} \quad (7)$$

Note that from a generalized perspective of the reordering step, we can distinguish between the descending Minkowski OWAD (DMOWAD) and the ascending Minkowski OWAD (AMOWAD) operators. Note also that it is possible to use them in situations where the highest value is the best result and in situations where the lowest value is the best result. But in a more efficient way, it is better to use one of them for one situation and the other one for the other situation, as it is explained in [12], [25] for the OWA operator. The DMOWAD operator has the same definition than the MOWAD operator.

Definition 7. An AMOWAD operator of dimension n is a mapping $AMOWAD$: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$. Then, the distance between two sets A and B is:

$$AMOWAD(d_1, d_2, \ldots, d_n) = \left(\sum_{j=1}^{n} w_j D_j^{\lambda} \right)^{1/\lambda} \quad (8)$$

where D_j is the jth lowest of the d_i and d_i is the individual distance between A and B. That is, $d_i = |a_i - b_i|$. λ is a parameter such that $\lambda \in (-\infty, \infty)$. As we can see, the elements D_j (j= 1, 2, …, n) are ordered in an increasing way: $D_1 \leq D_2 \leq \ldots \leq D_n$. Then, it is possible to see that the weights of the DMOWAD are related to those of the AMOWAD by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DMOWAD and w^*_{n-j+1} the jth weight of the AMOWAD operator.

The MOWAD operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent for both the DMOWAD and the AMOWAD operator. It is commutative because any permutation of the arguments has the same evaluation. That is, $MOWAD(d_1,d_2,\ldots,d_n) = MOWAD(e_1,e_2,\ldots,e_n)$, where $(e_1,\ldots,e_n)$ is any permutation of the arguments $(d_1,\ldots,d_n)$. It is monotonic because if $d_i \geq e_i$, for all d_i, then, $MOWAD(d_1,d_2,\ldots,d_n) \geq MOWAD(e_1,e_2,\ldots,e_n)$. It is bounded because the MOWAD

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aggregation is delimitated by the minimum and the maximum. That is, Min$\{d_i\}$ ≤ $MOWAD(d_1,d_2,\ldots,d_n)$ ≤ Max$\{d_i\}$. It is idempotent because if $d_i = d$, for all d_i, then, $MOWAD(d_1,d_2,\ldots,d_n) = d$.

Another interesting issue to analyze is the attitudinal character of the MOWAD operator. Based on the measure developed for the GOWA operators in [17], it can be formulated in two different forms depending on the type of ordering used. For the first form we get the following:

$$\alpha(W) = \left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right)^{\lambda}\right)^{1/\lambda} \tag{9}$$

And for the second, we get:

$$\alpha(W) = \left(\sum_{j=1}^{n} w_j \left(\frac{j-1}{n-1}\right)^{\lambda}\right)^{1/\lambda} \tag{10}$$

Note that we will also select one of these two equations according to the problem analyzed. That is, our selection will be different depending on if we are in a situation where the highest argument is the best result or in a situation where the lowest value is the best result.

IV. Families of MOWAD Operators

A. *Analysing the Weighting Vector W*

By choosing a different manifestation of the weighting vector in the MOWAD operator, we are able to obtain different types of aggregation operators. For example, we can obtain the maximum distance, the minimum distance, the normalized Minkowski distance and the weighted Minkowski distance.

For the DMOWAD operator, the maximum distance is obtained when $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. The minimum distance is found when $w_n = 1$ and $w_j = 0$, for all $j \neq n$. And for the AMOWAD operator, the maximum distance is found when $w_n = 1$ and $w_j = 0$, for all $j \neq n$, and the minimum distance is found when $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. As we can see, the maximum and the minimum distances are obtained independently of the value of the parameter λ. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get for any λ, $MOWAD(d_1, d_2,\ldots, d_n) = D_k$, where D_k is the kth largest or lowest of the arguments d_i.

The normalized Minkowski distance and the weighted Minkowski distance are also particular cases of the MOWAD operator. The normalized Minkowski distance is obtained when $w_j = 1/n$, for all j. The weighted Minkowski distance is obtained when $j = i$, for all i and j, where j is the jth argument of D_j and i is the ith argument of d_i.

Remark 1: Other families of aggregation operators could be obtained by choosing a different manifestation in the weighting vector. For example, the Hurwicz MOWAD criteria is obtained when $w_1 = \alpha$, $w_n = 1 - \alpha$, $w_j = 0$, for all $j \neq 1,n$, then, $MOWAD(d_1, d_2,\ldots, d_n) = \alpha$ Max$\{d_i\}$ + $(1 - \alpha)$ Min$\{d_i\}$. Note that if $\alpha = 1$, the Hurwicz MOWAD criteria becomes the maximum distance and if $\alpha = 0$, it becomes the minimum distance.

Remark 2: When $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$, we are using the window-MOWAD operator that it is based on the window-OWA operator [13]. Note that k and m must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, then, the window-MOWAD is transformed in the maximum. If $m = 1$, $k = n$, the window-MOWAD becomes the minimum. And if $m = n$ and $k = 1$, the window-MOWAD is transformed in the normalized Minkowski distance.

Remark 3: If $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$, we are using the olympic-MOWAD operator that it is based on the olympic average [16]. Note that if $n = 3$ or $n = 4$, the olympic-MOWAD average is transformed in the MOWAD median and if $m = n - 2$ and $k = 2$, the window-MOWAD is transformed in the olympic-MOWAD operator.

Remark 4: The median and the weighted median can also be used as MOWAD operators. For the MOWAD median, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and this affects the $[(n + 1)/2]$th largest argument d_i. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and this affects the arguments with the $(n/2)$th and $[(n/2) + 1]$th largest d_i. For the weighted MOWAD median, we select the argument that has the kth largest d_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

Remark 5: Another type of aggregation that could be used is the E-Z MOWAD weights that it is based on the E-Z OWA weights [18]. In this case, we should distinguish between two classes. In the first class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n, and in the second class we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$. Note that for the first class, the maximum distance is obtained if $k = 1$ and b_1 = Max$\{a_i\}$, and the normalized Minkowski distance if $k = n$. In the second class, the minimum distance is obtained if $k = 1$ and b_n = Min$\{a_i\}$, and the normalized Minkowski distance if $k = n$.

In [4], Filev and Yager suggested two methods for obtaining the OWA weights. Following their methodology we can apply these methods for the MOWAD weights as follows. For the first method, the weights can be expressed as $w_1 = \alpha$, $w_n = w_{n-1}(1 - w_1)/w_1$, and $w_j = w_{j-1}(1 - w_1)$ for $j = 2$ to $n - 1$. For the second method, the weights are obtained as $w_n = 1 - \alpha$, $w_1 = w_2(1 - w_n)/w_n$, and $w_j = w_j(1 - w_n)$ for $j = 2$ to $n - 1$.

Remark 6: Another useful approach for obtaining the weights is the functional method introduced by Yager [16] for the OWA operator. For the MOWAD operator, it can be summarized as follows. Let f be a function f: [0, 1] → [0, 1] such that $f(0) = f(1)$ and $f(x) \geq f(y)$ for $x > y$. We call this function a basic unit interval monotonic function (BUM). Using this BUM function we obtain the MOWAD weights w_j for $j = 1$ to n as

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \quad (11)$$

It can easily be shown that using this method, the w_j satisfy that the sum of the weights is 1 and $w_j \in [0,1]$.

Remark 7: By using the orness or attitudinal character and the dispersion measure it is also possible to obtain the weights of the MOWAD operator. For example, following [8] we could develop the maximal entropy MOWAD (MEMOWAD) as follows

$$\text{Maximize: } -\sum_{j=1}^{n} w_j \ln w_j \quad (12)$$

$$\text{Subject to: } \left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right)^2\right)^{1/2} = \alpha(W) \quad (13)$$

where $\alpha \in [0, 1]$, $w_j \in [0,1]$, and the sum of the weights is 1. Note that other methods similar to the MEMOWAD could be developed for obtaining the MOWAD weights following the same methodologies than [5]–[6], [9]–[10]. Then, we could obtain for example, the maximal renyi entropy MOWAD weights, the minimal variability MOWAD weights, the minimax disparity MOWAD weights, etc.

Remark 8: Other families of MOWAD operators could be obtained such as the weights that depend on the aggregated objects [13]. Note that in the MOWAD operator, the aggregated objects are individual distances. Then, the weights depend on the distances between the elements of the different sets. For example, we could develop the BADD-MOWAD operator that it is based on the OWA version developed in [13].

$$w_j = \frac{b_j^{\alpha}}{\sum_{j=1}^{n} b_j^{\alpha}} \quad (14)$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments d_i, that is, the individual distances. Note that the sum of the weights is 1 and $w_j \in [0,1]$. Also note that if $\alpha = 0$, we get the normalized Minkowski distance and if $\alpha = \infty$, we get the maximum distance. Another family of MOWAD operator that depends on the aggregated objects is

$$w_j = \frac{(1-b_j)^{\alpha}}{\sum_{j=1}^{n} (1-b_j)^{\alpha}} \quad (15)$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments d_i. Note that in this case if $\alpha = 0$, we also get the normalized Minkowski distance and if $\alpha = \infty$, we get the minimum distance. A third family of MOWAD operator that depends on the aggregated objects is

$$w_j = \frac{(1/b_j)^{\alpha}}{\sum_{j=1}^{n} (1/b_j)^{\alpha}} \quad (16)$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments d_i. In this case, we also get the normalized Minkowski distance if $\alpha = 0$ and if $\alpha = \infty$, we get the minimum distance.

Remark 9: Another interesting family is the S-MOWAD operator based on the S-OWA operator [13], [15]. It can be subdivided in three classes, the "orlike", the "andlike" and the generalized S-MOWAD operator. The "orlike" S-MOWAD operator is found when $w_1 = (1/n)(1 - \alpha) + \alpha$, and $w_j = (1/n)(1 - \alpha)$ for $j = 2$ to n with $\alpha \in [0, 1]$. Note that if $\alpha = 0$, we get the arithmetic mean and if $\alpha = 1$, we get the maximum. The "andlike" S-MOWAD operator is found when $w_n = (1/n)(1 - \beta) + \beta$ and $w_j = (1/n)(1 - \beta)$ for $j = 1$ to $n - 1$ with $\beta \in [0, 1]$. Note that in this class, if $\beta = 0$ we get the average and if $\beta = 1$, we get the minimum. Finally, the generalized S-MOWAD operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-MOWAD operator becomes the "andlike" S-MOWAD operator and if $\beta = 0$, it becomes the "orlike" S-MOWAD operator. Also note that if $\alpha + \beta = 1$, the generalized S-MOWAD operator becomes the Hurwicz generalized distance criteria.

Remark 10: A further type of aggregation operators that could be used in the MOWAD operator is the centered-OWA operator [19]. Following the same methodology, we could say that a MOWAD operator is a centered aggregation operator if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. We shall refer to this as softly decaying centered-MOWAD operator. Note that the normalized Minkowski distance is an example of this particular case of centered-MOWAD operator. Another particular situation of the centered-MOWAD operator appears if we remove the third condition. We shall refer to it as a non-inclusive centered-MOWAD operator. For this situation, we find the median MOWAD as a particular case.

Remark 11: A special type of centered-MOWAD operator is the Gaussian MOWAD weights which follows the same methodology than the Gaussian OWA weights suggested by Xu [11]. In order to define it, we have to consider a Gaussian distribution $\eta(\mu, \sigma)$ where

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n} j = \frac{n+1}{2} \quad (17)$$

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$$\sigma_n = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(j-\mu_n)^2} \qquad (18)$$

Assuming that

$$\eta(j) = \frac{1}{\sqrt{2\Pi\sigma_n}} e^{-(j-\mu_n)^2/2\sigma_n^2} \qquad (19)$$

we can define the MOWAD weights as

$$w_j = \frac{\eta_j}{\sum_{j=}^{n}\eta(j)} = \frac{e^{-(j-\mu_n)^2/2\sigma_n^2}}{\sum_{j=1}^{n} e^{-(j-\mu_n)^2/2\sigma_n^2}} \qquad (20)$$

Note that the sum of the weights is 1 and $w_j \in [0,1]$.

B. Analysing the Parameter λ

If we analyze different values of the parameter λ, we obtain another group of particular cases such as the Hamming ordered weighted averaging distance (HOWAD) operator, the Euclidean ordered weighted averaging distance (EOWAD) operator, the ordered weighted geometric averaging distance (OWGAD) operator and the ordered weighted harmonic averaging distance (OWHAD) operator.

Remark 12: The Hamming OWAD operator or simply OWAD operator [22] is found when the parameter $\lambda = 1$. In this type of distance, we introduce a reordering in the individual distances in order to aggregate them in the most efficient way according to the interests of the decision maker. It can be constructed as a particular case of the MOWAD operator, but it is also possible to construct it by mixing the OWA operator with the Hamming distance.

$$HOWAD(d_1, d_2,\ldots, d_n) = \sum_{j=1}^{n} w_j D_j \qquad (21)$$

In this case it is possible to distinguish between descending (DHOWAD or DOWAD) and ascending (AHOWAD or AOWAD) orders.

With the HOWAD operator it is also possible to obtain another parameterized family of aggregation operators such as the maximum distance, the minimum distance, the normalized Hamming distance and the weighted Hamming distance. The maximum and the minimum distances are obtained as it has been explained with the MOWAD operator. The normalized Hamming distance is found when $w_j = 1/n$, for all j. The weighted Hamming distance is obtained when $j = i$, for all i and j, where j is the jth argument of D_j and i is the ith argument of d_i.

Remark 13: The Euclidean OWAD operator [21], [23] or also the ordered weighted quadratic averaging distance (OWQAD) operator is found when the parameter $\lambda = 2$. Note that it can be constructed as a particular case of the MOWAD operator or by mixing the Euclidean distance with the OWA operator or by mixing the Hamming distance with the OWQA operator.

$$EOWAD(d_1, d_2,\ldots, d_n) = \left(\sum_{j=1}^{n} w_j D_j^2\right)^{1/2} \qquad (22)$$

From a generalized perspective of the reordering step, we can distinguish between the descending EOWAD (DEOWAD) operator and the ascending EOWAD (AEOWAD) operator.

With the EOWAD operator it is also possible to obtain another parameterized family of aggregation operators that include for example, the maximum distance, the minimum distance, the normalized Euclidean distance and the weighted Euclidean distance.

Remark 14: Another particular case obtained with the MOWAD operator is the OWGAD operator [24]. This case is found when $\lambda = 0$. Note that it is possible to construct it in another way such as by mixing the Hamming distance with the OWGA operator or by mixing the geometric distance with the OWA operator.

$$OWGAD(d_1, d_2,\ldots, d_n) = \sum_{j=1}^{n} D_j^{\,w_j} \qquad (23)$$

In this case, we can distinguish between the descending OWGAD (DOWGAD) operator and the ascending OWGAD (AOWGAD) operator. Note that the geometric operators cannot aggregate negative numbers and the value zero. Therefore, this distance aggregation operator is only useful in some special situations. Note also that it is possible to transform this operator, so it can deal with zero or negative numbers [34].

Note that it is also possible to obtain another parameterized family of aggregation operators. With the OWGAD operator, we can obtain among others the maximum distance, the minimum distance, the normalized geometric distance and the weighted geometric distance.

Remark 15: Another special case found in the MOWAD operator is the OWHAD operator. In this case, $\lambda = -1$. Note that the OWHAD operator can also be constructed by mixing the harmonic distance with the OWA operator or by mixing the Hamming distance with the OWHA operator.

$$OWHAD(d_1, d_2,\ldots, d_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{D_j}} \qquad (24)$$

From a generalized perspective of the reordering step, we can distinguish between the descending OWHAD (DOWHAD) operator and the ascending OWHAD (AOWHAD) operator.

With the OWHAD operator it is also possible to obtain another parameterized family of aggregation operators. We

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can obtain among others the maximum distance, the minimum distance, the normalized harmonic distance and the weighted harmonic distance. The maximum distance is obtained when $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$, and the minimum distance when $w_n = 1$ and $w_j = 0$, for all $j \neq n$. The normalized harmonic distance is found when $w_j = 1/n$, for all j. The weighted harmonic distance is obtained when $j = i$, for all i and j, where j is the jth argument of D_j and i is the ith argument of d_i.

V. MOWAD OPERATOR IN THE SELECTION OF STRATEGIES

In the following, we are going to develop an illustrative example in order to see the results obtained in the aggregation by using different types of MOWAD operators. We will analyze the selection of strategies the decision maker needs to find the best strategy according to his interests. Note that other selection problems could be developed such as the selection of human resources, the selection of financial products, the selection of investments, etc. [7], [23]–[25], [34]–[37].

Assume that an enterprise is considering its global strategy for the next year and they are thinking in some changes in order to obtain more benefits. In order to do so, the board of directors has established five possible strategies S_i that the enterprise could develop in the future.

(1) S_1 consists in implement strategy 1.
(2) S_2 consists in implement strategy 2.
(3) S_3 consists in implement strategy 3.
(4) S_4 consists in implement strategy 4.
(5) S_5 consists in implement strategy 5.

After careful review of the information, the experts have given the following general information. They have summarized the information of the strategies in five main characteristics C_i with the following results. Note that the results are valuations between 0 and 1.

TABLE I
CHARACTERISTICS OF THE STRATEGIES

	C_1	C_2	C_3	C_4	C_5
S_1	0.5	0.7	0.8	0.6	0.5
S_2	0.8	0.9	0.2	0.4	0.5
S_3	0.5	0.7	0.6	0.3	0.7
S_4	0.7	0.9	0.6	0.2	0.6
S_5	0.2	0.7	0.8	0.7	0.5

According to the objectives and policies of the enterprise, the experts have established the ideal strategy for the company independently of the strategies available. They have established the following valuations for it.

TABLE II
CHARACTERISTICS OF THE IDEAL STRATEGY

	C_1	C_2	C_3	C_4	C_5
Ideal	0.9	1	0.9	0.9	0.8

With this information we can develop different aggregation methods in order to select a strategy. First, we are going to consider four basic aggregations that are particular cases of the MOWAD operator such as the normalized Hamming distance, the normalized Euclidean distance, the weighted Hamming distance and the weighted Euclidean distance. Note that we will use the following weighting vector $W = (0.1, 0.2, 0.2, 0.2, 0.3)$, when necessary. The results are the following.

TABLE III
AGGREGATED RESULTS

	NHD	*NED*	*WHD*	*WED*
S_1	0.28	0.29	0.27	0.28
S_2	0.34	0.41	0.36	0.42
S_3	0.34	0.37	0.31	0.35
S_4	0.3	0.36	0.3	0.36
S_5	0.32	0.38	0.28	0.32

As we can see, the best alternative according to these four cases is the strategy S_1 because it has the lowest distance.

Now, we are going to study the results obtained by using the OWAD operator, the AOWAD operator, the EOWAD operator and the AEOWAD operator.

TABLE IV
AGGREGATED RESULTS 2

	OWAD	*AOWAD*	*EOWAD*	*AEOWAD*
S_1	0.25	0.31	0.27	0.32
S_2	0.28	0.4	0.349	0.46
S_3	0.29	0.39	0.327	0.42
S_4	0.24	0.36	0.293	0.42
S_5	0.26	0.38	0.309	0.43

As we can see, we will select a different strategy depending on the particular case of MOWAD operator used in the aggregation. If we use the AOWAD operator, the EOWAD operator or the AEOWAD operator, the optimal choice will be the strategy S_1. And if we use the OWAD operator, then the best alternative is the strategy S_4.

If we try to order the strategies, a typical situation when we want to consider more than one alternative, we can see that each distance aggregation operator gives us a different result leading to different decisions.

TABLE V
ORDERING OF THE STRATEGIES

NHD	$S_1 \succ S_4 \succ S_5 \succ S_2 = S_3$	*OWAD*	$S_4 \succ S_1 \succ S_5 \succ S_2 \succ S_3$
NED	$S_1 \succ S_4 \succ S_3 \succ S_5 \succ S_2$	*AOWAD*	$S_1 \succ S_4 \succ S_5 \succ S_3 \succ S_2$
WHD	$S_1 \succ S_5 \succ S_4 \succ S_3 \succ S_2$	*EOWAD*	$S_1 \succ S_4 \succ S_5 \succ S_3 \succ S_2$
WED	$S_1 \succ S_5 \succ S_3 \succ S_4 \succ S_2$	*AEOWAD*	$S_1 \succ S_3 = S_4 \succ S_5 \succ S_2$

As a general conclusion for the example, we can see that depending on the method used in the selection process, our

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decision will be different. Note that the method used has to be in accordance with the interests of the decision maker.

VI. Conclusion

We have studied different types of distance measures. First, we have reviewed some basic aggregation operators such as the OWA operator, the GOWA operator and the normalized Minkowski distance. With this initial information, we have introduced the MOWAD operator. We have considered some of its main properties such as the distinction between descending and ascending orders. Next, we have developed a wide range of particular cases of the MOWAD operator such as the HOWAD operator, the EOWAD operator, the OWGAD operator and the OWHAD operator. We have seen that these special cases also provide a parameterized family of aggregation operators with similar properties than the MOWAD operator. We have also considered the usual families found in the weighting vector such as the window-MOWAD, the olympic-MOWAD, the MOWAD median, the S-MOWAD, the centered-MOWAD, etc. Finally, we have presented an illustrative example of the new approach where we have seen that depending on the distance aggregation operator used, the result is completely different.

This work represents an extension about the possibility of using OWA operators in the Minkowski distance which has been applied in strategic management. In future research, we expect to develop other extensions to the Minkowski distance by using different types of OWA operators and we will apply it in other decision making problems.

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