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Strong Law of Large Numbers for *- Mixing Sequence

Bainian Li, Kongsheng Zhang

Abstract—Strong law of large numbers and complete convergence for sequences of *-mixing random variables are investigated. In particular, Teicher's strong law of large numbers for independent random variables are generalized to the case of *-mixing random sequences and extended to independent and identically distributed Marcinkiewicz Law of large numbers for *-mixing.

Keywords—*-mixing squences; strong law of large numbers; martingale differences; Lacunary System

I. Introduction

LET (Ω, F, P) be a probability space and let $\{X_n, n = 1, 2, ...\}$ be a sequence of real-valued random variables defined on (Ω, F, P). For each positive integer n, let $F_n$be the smallest σ-algebra with respect to which X_n is measurable and for $n \le m$, let F_n^m be the smallest σ-algebra with respect to which $X_n, ..., X_m$ are jointly measurable.

Definition 1.1. Let $\{X_n, n \ge 1\}$ be a sequence of random variable, X_n is named *-mixing if there exists a positive integer N and function f such that $f \downarrow 0$ and for all $n \ge N, m \ge 1, A \in F_1^m, B \in F_{m+n}^\infty$,

$$|P(AB) - P(A)P(B)| \le f(n)P(A)P(B). \tag{1}$$

Evidently, inequality (1) is equivalent to the condition for all $B \in F_{m+n}^\infty$,

$$|P(B|F_1^m) - P(B)| \le f(n)P(B) \ a.s. \tag{2}$$

It follows that X_n is integrable, and

$$|E(X_{m+n}|F_1^m) - E(X_{m+n})| \le f(n)E|X_{m+n}| \tag{3}$$

The following strong law for *-mixing sequences can be found in Blum[1].

Theorem A. Let $\{X_n, n \ge 1\}$ be a *-mixing sequence such that $EX_n = 0$, and $EX_n^2 < \infty, n \ge 1$, and $\sum_{i=1}^\infty EX_i^2/i^2 < \infty$, then

$$\sum_{i=1}^n X_i/n \longrightarrow 0, a.s. \tag{4}$$

In this paper we shall further generalize Theorem A.

II. Main Results

Theorem 2.1. Let $\{Y_n, n \ge 1\}$ be a nonnegative *-mixing sequence such that $EY_i = \mu_i \le K < \infty$, for all i, and $\sum_{i=1}^\infty EY_i^2/i^2 < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (Y_i - \mu_i) = 0, a.s. \tag{5}$$

Bainian Li and Kongsheng Zhang are with the School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu, 233030 PR China, email: libainian49@163.com, zks155@163.com

To complete the proof, we need the following lemma

Lemma 2.2. ([2]) Suppose $\{Y_n, n \ge 1\}$ is a *-mixing sequence, $EY_i < \infty, i \ge 1,$ for any $\sigma-$field $B \in F_1^m, m \ge M$, then

$$|E(Y_{m+n}|B) - E(Y_{m+n})| \le f(m)E|Y_{m+n}|. \tag{6}$$

Proof of Theorem 2.1. Let $X_i = Y_i - \mu_i$, then $EX_i = 0, E|X_i| \le 2K$, it suffices to prove

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i = 0, a.s. \tag{7}$$

By lemma 2.2, for any $\varepsilon > 0$, there exists $M' > 0$ for all $n \ge 2$, we get

$$|E(X_{nM'+R}|F_{M'+R}^{(n-1)M'+R}) | \le f(M')E|X_{nM'+R}| \le 2K\varepsilon, \tag{8}$$

where R is a nonnegative integer, and $0 \le R \le M' - 1$. For $R = 0, 1, ..., M' - 1$, we prove

$$\sum_{n=2}^N X_{nM'+R}/N \to 0 (N \to \infty). \tag{9}$$

Let $H_0 = F_0 = (\Omega, \Phi)$, $H_n = F_{M'+R}^{nM'+R}, n \ge 2.$
Clearly $H_n \uparrow$, for fixed R, let

$$Z_n = X_{nM'+R} - E(X_{nM'+R}|H_{n-1}), n \ge 2.$$

Obviously, $\{Z_n, H_n, n \ge 2\}$ is a martingale difference. By virtue of condition expectation, we obtain $E\{(E(X_{nM'+R}|H_{n-1}))^2\}$

$$\begin{aligned}
&= E\{E(X_{nM'+R}|H_{n-1}) \cdot E(X_{nM'+R}|H_{n-1})\} \\
&= E\{E(X_{nM'+R} \cdot E(X_{nM'+R}|H_{n-1})|H_{n-1})\} \\
&= E\{X_{nM'+R}E(X_{nM'+R}|H_{n-1})\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
EZ_n^2 &= EX_{nM'+R}^2 - 2E(X_{nM'+R}E(X_{nM'+R}|H_{n-1})) \\
&\quad + E\{(E(X_{nM'+R}|H_{n-1}))^2\} \\
&= EX_{nM'+R}^2 - E\{(E(X_{nM'+R}|H_{n-1}))^2\} \\
&\le EX_{nM'+R}^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
\sum_{n=2}^\infty EZ_n^2/n^2 &\le \sum_{n=2}^\infty EX_{nM'+R}^2/n^2 \\
&= \sum_{n=2}^\infty EX_{nM'+R}^2/(nM'+R)^2 (\frac{nM'+R}{n})^2 \\
&\le 4(M')^2 \sum_{i=1}^\infty EX_i^2/i^2,
\end{aligned} \tag{10}$$

since $EX_i^2 \leq EY_i^2 + K^2$.

Combined with (10) and $\sum_{i=1}^{\infty} EY_i^2/i^2 < \infty$, we deduce that $\sum_{n=2}^{\infty} EZ_n^2/n^2 < \infty$.

By using condition expectation again, one concludes

$$\sum_{n=2}^{\infty} E(Z_n^2|H_{n-1})/n^2 < \infty. \quad (11)$$

From (11) and Theorem 8.1(see Chow[3]), we have

$$\sum_{n=2}^{N} Z_n/N \longrightarrow 0 \ a.s. \quad (12)$$

Since (8) implies

$$|E(X_{nM'+R}|H_{n-1})| \leq 2K\varepsilon, n \geq 2,$$

hence

$$|\sum_{n=2}^{N} E(X_{nM'+R}|H_{n-1})|/N \leq 2K\varepsilon. \quad (13)$$

Combined with (12) and (13), this yields (9). For $R = 0, 1, ..., M' - 1$, one has

$$(X_{2M'} + X_{3M'} + ... + X_{NM'})/N \to 0, (N \to \infty) \ a.s.$$

$$(X_{2M'+1} + X_{3M'+1} + ... + X_{NM'+1})/N \to 0, (N \to \infty) \ a.s.$$

$$\frac{1}{N}(X_{2M'+(M'-1)} + X_{3M'+(M'-1)} + \dots + X_{NM'+(M'-1)}) \to 0, (N \to \infty) \ a.s.$$

From the above results, one has

$$\sum_{i=2M'}^{(N+1)M'-1} X_i/N \to 0, (N \to \infty) \ a.s., \quad (14)$$

which deduces (7) and completes the proof.

Theorem 2.3. ([3]) Let $\{X_n, n \geq 1\}$ be a *-mixing sequence such that $EX_n = 0, EX_n^2 < \infty, n \geq 1$. Suppose that $\sum_{n=1}^{\infty} a_n^{-2} EX_n^2 < \infty$ and $\sup_n a_n^{-1} \sum_{i=1}^{n} E|X_i| < \infty$, where$\{a_n\}$ is a sequence of positive constants increasing to ∞. Then

$$a_n^{-1} \sum_{i=1}^{n} X_i \longrightarrow 0, a.s. \quad (15)$$

Proof. Given $\varepsilon > 0$, choose $n_0 \geq N$ so large that $f(n_0) < \varepsilon$.

From Lemma 2.2 we deduce that for all positive integers i and j,

$$\begin{aligned} &|E(X_{in_0+j}|X_{n_0+j}, X_{2n_0+j}, ..., X_{(i-1)n_0+j})| \\ &= |E[E(X_{in_0+j}|X_1, X_2, ..., X_{(i-1)n_0+j}) \\ &\quad |X_{n_0+j}, X_{2n_0+j}, ..., X_{(i-1)n_0+j}]| \\ &\leq f(n_0)E|X_{in_0+j}| \ a.s. \end{aligned}$$

If $n \geq n_0$, choose nonnegative integers q and r such that $0 \leq r \leq n_0 - 1$ and $n = qn_0 + r$. Then

$$\begin{aligned} a_n^{-1} \sum_{i=1}^{n} X_i &= a_n^{-1} \sum_{i=1}^{n_0} X_i + a_n^{-1} \sum_{i=1}^{q-1} \sum_{i=1}^{q-1} X_{in_0+j} \\ &\quad + a_n^{-1} \sum_{j=1}^{r} X_{qn_0+j} \\ &= I_1 + I_2, \end{aligned} \quad (16)$$

where $I_1 = a_n^{-1} \sum_{i=1}^{n_0} X_i, I_2 = a_n^{-1} \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} X_{in_0+j} + a_n^{-1} \sum_{j=1}^{r} X_{qn_0+j}$.

Obviously, $I_1 \to 0, a.s.(n \to \infty), I_2$ is dominated by

$$\begin{aligned} I_2 \quad = \quad & \sum_{j=1}^{q-1} a_n^{-1} \sum_{i=1}^{q-1} |[X_{in_0+j} \\ & -E(X_{in_0+j}|X_{n_0+j}, X_{2n_0+j}, \dots, X_{(i-1)n_0+j})]| \\ & + \sum_{j=1}^{r} a_n^{-1} |X_{qn_0+j} - E(X_{qn_0+j}|X_{n_0+j}, X_{2n_0+j}, \dots, \\ & X_{(q-1)n_0+j})| + f(n_0) a_n^{-1} \sum_{i=n_0+1}^{n} E|X_i|. \end{aligned}$$

Based on the fact $\sum_{n=1}^{\infty} a_n^{-2} EX_n^2 < \infty$ and Theorem 2.18([3]), we see that the first two terms here converge a.s. to zero. The second term is convergent to zero since r is fixed, and by $\sup_n a_n^{-1} \sum_{i=1}^{n} E|X_i| < \infty$, the last term also converges a.s. to zero. We deduce that for all $\varepsilon > 0$,

$$\limsup_{n} |b_n^{-1} \sum_{i=1}^{n} X_i| < \varepsilon(\sup_n b_n^{-1} \sum_{i=1}^{n} |X_i|) \quad a.s.,$$

which completes the proof.

Lemma 2.4. Let $\{X_n, n \geq 1\}$ be a sequence of *-mixing random variables satisfying $\sum_{n=1}^{\infty} f(n) < \infty, p \geq 2$. Assume that $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Then there exists a constant C depending only on p and f such that

$$E\left(\max_{1<j<n} |\sum_{i=a+1}^{a+j} X_i|^p\right) \leq C\left[\sum_{i=a+1}^{a+j} E|X_i|^p + \left(\sum_{i=a+1}^{a+j} EX_i^2\right)^{p/2}\right],$$

for every $a \geq 0$ and $n \geq 1$. In particular, we have

$$E\left(\max_{1<j<n} |\sum_{i=1}^{j} X_i|^p\right) \leq C\left[\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right],$$

for every $n \geq 1$.

Proof.

$$\begin{aligned} E\left(\sum_{i=a+1}^{a+j} X_i\right)^2 \quad &= \quad \sum_{i=a+1}^{a+j} EX_i^2 + 2 \sum_{a+1 \leq i < j \leq a+n} E(X_i X_j) \\ &\leq \quad \sum_{i=a+1}^{a+j} EX_i^2 + 2 \cdot \\ &\quad \sum_{a+1 \leq i < j \leq a+n} f(j-i) E|X_i| E|X_j| \\ &\leq \quad \sum_{i=a+1}^{a+j} EX_i^2 + 2 \cdot \\ &\quad \sum_{a+1 \leq i < j \leq a+n} f(j-i) E(X_i^2)^{1/2} E(X_j^2)^{1/2} \\ &\leq \quad \sum_{i=a+1}^{a+j} EX_i^2 + \end{aligned}$$

$$\begin{aligned}&\sum_{k=1}^{n-1}\sum_{i=a+1}^{a+n-k} f(k)(EX_i^2+EX_{k+i}^2)\\ \le&\left(1+2\sum_{k=1}^{\infty}f(k)\right)\sum_{i=a+1}^{a+j}EX_i^2\\ =&C_1\sum_{i=a+1}^{a+j}EX_i^2\end{aligned}$$

It is well known that *-mixing is also φ-mixing. Therefore, by [4, Lemma 2.2]) we can immediately complete the proof of Lemma 2.4.

Lemma 2.5. Let $\{X_n, n \ge 1\}$ be a zero-mean *-mixing and $\sum_{k=1}^{\infty} f(k) < \infty$, for some $p \ge 2, \sup_i E|X_i|^p < \infty$. Then there exists constant $C > 0$ depending only on p for any real-valued sequence $\{a_{ni}\}$, such that

$$E|\sum_{i=1}^{n} a_{ni}X_i|^p \le C(\sum_{i=1}^{n} a_{ni}^2)^{p/2}.$$

Proof. Let $a_{ni} = 0, i > n$, since $\sum_{k=1}^{\infty} f(k) < \infty$, $\sup_i E|X_i|^p < \infty$. By Lemma 2.4, we have

$$\begin{aligned}E(|\sum_{i=1}^{n}a_{ni}X_i|^p &\le C\left[\sum_{i=1}^{n}E|a_{ni}X_i|^p+\left(\sum_{i=1}^{n}Ea_{ni}X_i^2\right)^{p/2}\right]\\ &\le C[\sum_{i=1}^{n}|a_{ni}|^p+(\sum_{i=1}^{n}a_{ni}^2)^{p/2}].\end{aligned}$$

Since $p \ge 2$, it follows that

$(\sum_{i=1}^{n}|a_{ni}|^p)^{1/p} \le (\sum_{i=1}^{n} a_{ni}^2)^{1/2}$

$\Leftrightarrow (\sum_{i=1}^{n}|a_{ni}|^p) \le (\sum_{i=1}^{n} a_{ni}^2)^{p/2}$, which proves the statement.

Remark 1.

(1) Lemma 2.5 implies that *-mixing is a Lacunary System.

(2) If $a_{ni} = 1$, we have

$$E(|\sum_{i=1}^{n} X_i|)^p \le cn^{p/2}.$$

III. Larger deviations for *-mixing

Theorem 3.1. Let $\{Xn, n \ge 1\}$ be a zero-mean *-mixing, $\sum_{k=1}^{\infty} f(k) < \infty$, for some $p > 2, E|X_i|^p < \infty$. If there exists $1/2 < r \le 1, \theta = 2r - 1$ and positive constant K such that $\sum_{i=1}^{n} a_{ni}^2 \le Kn^{\theta}, (i = 1, 2, ..., n)$, then

$$n^{-r}\sum_{i=1}^{n} a_{ni}X_i \longrightarrow 0, \quad a.s. \tag{17}$$

Proof. Denote $S_n = \sum_{i=1}^{n} a_{ni}X_i$, by Markov's inequality, we have

$$P\{S_n \ge n^r x\} \le \frac{E|S_n|^p}{x^p n^{pr}}.$$

From lemma 2.5, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}P\{|S_n|\ge n^r x\} &\le \sum_{n=1}^{\infty}\frac{E|S_n|^p}{x^p n^{pr}}\\ &\le \sum_{n=1}^{\infty}\frac{C(\sum_{i=1}^{n}a_{ni}^2)^{p/2}}{x^p n^{pr}}\\ &\le \sum_{n=1}^{\infty}\frac{CK}{x^p n^{p/2}}<\infty.\end{aligned}$$

Inequality (17) follows from Borel-Cantelli lemma.

Remark 2. Marcinkiewicz Law of large numbers of independent and identically distributed variables has been extended to the case of *-mixing.

Theorem 3.2. Let $\{X_n, n \ge 1\}$ be a zero-mean *-mixing, $\sum_{k=1}^{\infty} f(k) < \infty$, for some $p > 2, E|X_i|^p < \infty$. If there exists $1/2 < r \le 1, \theta = 1 - 2/p$ and positive constant K such that $\sum_{i=1}^{n} a_{ni}^2 \le Kn^{\theta}, (i = 1, 2, ..., n)$, then

$$\frac{\sum_{i=1}^{n} a_{ni}X_i}{\sqrt{n\log n}} \longrightarrow 0, a.s. \tag{18}$$

Proof. By Markov's inequality and lemma 2.5, we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}P\{|\sum_{i=1}^{n}a_{ni}X_i/\sqrt{n\log n}|\ge x\}\\ &\le \sum_{n=1}^{\infty}\frac{E|S_n|^p}{x^p n^{p/2}(\log n)^{p/2}}\\ &\le \sum_{n=1}^{\infty}\frac{C(\sum_{i=1}^{n}a_{ni}^2)^{p/2}}{x^p n^{p/2}(\log n)^{p/2}}\\ &\le \sum_{n=1}^{\infty}\frac{CKn^{p/2-1}}{x^p n^{p/2}(\log n)^{p/2}}\\ &= \sum_{n=1}^{\infty}\frac{CK}{x^p n(\log n)^{p/2}}\\ &<\infty.\end{aligned}$$

Therefore, inequality (18) follows from Borel-Cantelli lemma.

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