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An Application of Differential Subordination to Analytic Functions

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Abstract—In the present paper, using the technique of differential subordination, we obtain certain results for analytic functions defined by a multiplier transformation in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$. We claim that our results extend and generalize the existing results in this particular direction.

Keywords—Analytic function, Differential subordination, Multiplier transformation.

I. Introduction

LET $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N} = \{1, 2, 3, \cdots\},$$

which are analytic in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{A}$ is said to be starlike of order $(0 \leq \ < 1)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \ ,\ z \in \mathbb{E}.$$

Let $\mathcal{S}^*(\ )$ denote the class of starlike functions of order . We write $\mathcal{S}^*(0) = \mathcal{S}^*$, therefore, $\mathcal{S}^*$ is the class of starlike functions (w.r.t. origin).

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n,\ )$ as

$$I_p(n,\ )[f](z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\ }{p+\ }\right)^n a_k z^k,\ (\ \geq 0, n \in \mathbb{Z}).$$

The operator $I_p(n,\ )$ has been recently studied by Aghalary et al. [9]. Earlier, the operator $I_1(n,\ )$ was investigated by Cho and Srivastava [7] and Cho and Kim [8], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [1]. $I_1(n, 0)$ is the well-known Sălăgean ([5]) derivative operator D^n, defined as:

$$D^n[f](z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$.

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A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_n(\ )$ if it satisfies the condition

$$\Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \ ,\ z \in \mathbb{E}.$$

In 1989, the class $\mathcal{S}_n(\ )$ has been studied by Owa, Shen and Obradovič [10].

Uralgaddi [2] proved if $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots \in \mathcal{S}_n(0)$ for some $m, n \in \mathbb{N}$, then

$$\Re\left(\frac{D^n[f](z)}{z}\right)^{\frac{1}{n+1}} > \frac{1}{2},\ z \in \mathbb{E}.$$

Recently, Li and Owa [6], proved the following results:

Theorem 1.1: Let $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots$, be analytic in $\mathbb{E}$ and satisfy the condition

$$\Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \frac{2 - m(n+1)}{2},\ z \in \mathbb{E}$$

for some $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then

$$\Re\left(\frac{D^n[f](z)}{z}\right)^{\frac{1}{n+1}} > \frac{1}{2},\ z \in \mathbb{E}.$$

Theorem 1.2: If $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots \in \mathcal{S}_n(\ )$ for some $, 0 \leq \ < 1, n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, then for any $, 0 < \ \leq \frac{m}{2(1 - \ )}$, the sharp estimate is

$$\Re\left(\frac{D^n[f](z)}{z}\right)^{\beta} > 2^{\frac{2\beta(\alpha-1)}{m}},\ z \in \mathbb{E}.$$

The main objective of the present paper is to generalize certain existing results stated above using differential subordination and find the corresponding generalized results for multiplier transformation $I_p(n,\ )$ in the subordination form.

II. Preliminaries

We shall need the following definitions and lemmas to prove our results.

Definition 2.1: Let f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g in $\mathbb{E}$, written as $f(z) \prec g(z)$ in $\mathbb{E}$, if g is univalent in $\mathbb{E}$, $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Definition 2.2: Let $: \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$(p(z), zp'(z); z) \prec h(z),\quad (p(0), 0; 0) = h(0), \quad (1)$$

then p is called a solution of the differential subordination (1). The univalent function q is called a dominant of the differential subordination (1) if $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

Definition 2.3: A function $L(z,t), z \in \mathbb{E}$ and $t \geq 0$ is said to be a subordination chain if $L(.,t)$ is analytic and univalent in $\mathbb{E}$ for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{E}$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 2.4: ([3, page 159]). The function $L(z,t) : \mathbb{E} \times [0,\infty) \to \mathbb{C}$, of the form $L(z,t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t\to\infty} |a_1(t)| = \infty$, is said to be a subordination chain if and only if $\Re\left[\frac{z\ L/\ z}{L/\ t}\right] > 0$ for all $z \in \mathbb{E}$ and $t \geq 0$.

Lemma 2.5: ([12]). Let F be analytic in $\mathbb{E}$ and let G be analytic and univalent in $\overline{\mathbb{E}}$ except for points $\ _0$ such that $\lim_{z\to\ _0} G(z) = \infty$, with $F(0) = G(0)$. If $F \not\prec G$ in $\mathbb{E}$, then there is a point $z_0 \in \mathbb{E}$ and $\ _0 \in\ \mathbb{E}$ (boundary of $\mathbb{E}$) such that $F(|z| < |z_0|) \subset G(\mathbb{E})$, $F(z_0) = G(\ _0)$ and $z_0F'(z_0) = m\ _0G'(\ _0)$ for some $m \geq 1$.

III. Main Results

The following result is essentially due to Miller and Mocanu [13,page 76]. For the completeness of our results, we also prove it here with an alternative proof using subordination chain.

Theorem 3.1: Let q, $q(z) \neq 0, z \in \mathbb{E}$, be a univalent function such that $\frac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$. If an analytic function P, $P(z) \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$\frac{zP'(z)}{P(z)} \prec \frac{zq'(z)}{q(z)} = h(z), \tag{2}$$

then

$$P \prec q = \exp\left[\int_0^z \frac{h(t)}{t}dt\right],$$

and q is the best dominant.

Proof: Let us define h as

$$h(z) = \frac{zq'(z)}{q(z)},\ z \in \mathbb{E}. \tag{3}$$

Since h is starlike and hence univalent in $\mathbb{E}$. The subordination in (2) is, therefore, well-defined in $\mathbb{E}$.
We need to show that $P \prec q$. Suppose to the contrary that $P \not\prec q$ in $\mathbb{E}$. Then by Lemma 2.5, there exist points $z_0 \in \mathbb{E}$ and $\ _0 \in\ \mathbb{E}$ such that $P(z_0) = q(\ _0)$ and $z_0P'(z_0) = m\ _0q'(\ _0)$, $m \geq 1$. Then

$$\frac{z_0P'(z_0)}{P(z_0)} = \frac{m\ _0q'(\ _0)}{q(\ _0)},\ z \in \mathbb{E}. \tag{4}$$

Consider a function

$$L(z,t) = (1+t)\frac{zq'(z)}{q(z)},\ z \in \mathbb{E}. \tag{5}$$

The function $L(z,t)$ is analytic in $\mathbb{E}$ for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{E}$. Now,

$$a_1(t) = \left(\frac{L(z,t)}{z}\right)_{(0,t)} = (1+t)\frac{q'(0)}{q(0)}.$$

As q is univalent in $\mathbb{E}$, so, $q'(0) \neq 0$. Therefore, it follows that $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$\frac{z\ L/\ z}{L/\ t} = (1+t)\frac{zQ'(z)}{Q(z)},\ z \in \mathbb{E},$$

where $Q(z) = \frac{zq'(z)}{q(z)}$. Since Q is starlike in $\mathbb{E}$ and $t \geq 0$. Therefore, we obtain

$$\Re\left(\frac{z\ L/\ z}{L/\ t}\right) > 0,\ z \in \mathbb{E}.$$

Hence, in view of Lemma 2.4, $L(z,t)$ is a subordination chain. Therefore, $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$. From (5), we have $L(z,0) = h(z)$, thus we deduce that $L(\ _0,t) \not\in h(\mathbb{E})$ for $|\ _0| = 1$ and $t \geq 0$. In view of (4) and (5), we can write

$$\frac{z_0P'(z_0)}{P(z_0)} = L(\ _0, m-1) \not\in h(\mathbb{E}),$$

where $z_0 \in \mathbb{E}$, $|\ _0| = 1$ and $m \geq 1$, which is a contradiction to (2). Hence,

$$P \prec q = \exp\left[\int_0^z \frac{h(t)}{t}dt\right].$$

This completes the proof of the theorem. ■

Theorem 3.2: Let h be starlike univalent in $\mathbb{E}$ with $h(0) = 0$. Let $f \in \mathcal{A}_p$ satisfy

$$\frac{I_p(n+1,\ )[f](z)}{I_p(n,\ )[f](z)} - 1 \prec h(z),\ z \in \mathbb{E}, \tag{6}$$

then

$$\left(\frac{I_p(n,\ )[f](z)}{z^p}\right)^\beta \prec q(z) = \exp\left[\ (\ \ +p)\int_0^z \frac{h(t)}{t}dt\right],$$

for $z \in \mathbb{E}$, $\ > 0$. The function q is the best dominant.

Proof: Let us write

$$\left(\frac{I_p(n,\ )[f](z)}{z^p}\right)^\beta = r(z),\ z \in \mathbb{E}. \tag{7}$$

Differentiate (7) logarithmically, we obtain

$$\frac{zI'_p(n,\ )[f](z)}{I_p(n,\ )[f](z)} - p = \frac{zr'(z)}{r(z)},\ z \in \mathbb{E}. \tag{8}$$

A little calculation yields the following equality

$$zI'_p(n,\ )[f](z) = (p+\ )I_p(n+1,\ )[f](z) -\ I_p(n,\ )[f](z), \tag{9}$$

By making use of (9), from (6) and (8), we have

$$\frac{I_p(n+1,\ )[f](z)}{I_p(n,\ )[f](z)} - 1 = \frac{1}{(\ \ +p)}\frac{zr'(z)}{r(z)} \prec h(z).$$

As $p \in \mathbb{N}$, ≥ 0 and by our assumption, > 0. Therefore, we have $(+ p) > 0$. Now in view of Theorem 3.1, we obtain

$$\left(\frac{I_p(n, \ )[f](z)}{z^p}\right)^{\beta} = r(z) \prec q(z),\ z \in \mathbb{E},$$

where $q(z) = \exp\left[(+ p) \int_0^z \frac{h(t)}{t} dt \right]$, it completes the proof. ∎

IV. Applications to Analytic Functions

For $h(z) = \frac{2(1 - \ )z}{1 - z}$, where $\neq 1$, is a real number. It is easy to check that h is starlike in $\mathbb{E}$. When we make this selection of h in Theorem 3.2, we get the following result.

Corollary 4.1: If $f \in \mathcal{A}_p$ satisfies

$$\frac{I_p(n+1, \ )[f](z)}{I_p(n, \ )[f](z)} \prec \frac{1 + (1 - 2 \ )z}{1 - z},\ z \in \mathbb{E},$$

then

$$\left(\frac{I_p(n, \ )[f](z)}{z^p}\right)^{\beta} \prec (1 - z)^{2\beta(\alpha-1)(\lambda+p)},\ z \in \mathbb{E},$$

where $\neq 1$, > 0, are real numbers.

If we put $p = 1$, $= 0$ in Corollary 4.1, we have the following result.

Corollary 4.2: If $f \in \mathcal{A}$ satisfies

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec \frac{1 + (1 - 2 \ )z}{1 - z},\ z \in \mathbb{E},$$

then

$$\left(\frac{D^n[f](z)}{z}\right)^{\beta} \prec (1 - z)^{2\beta(\alpha-1)},\ z \in \mathbb{E},$$

where $\neq 1$, > 0, are real numbers.

Remark 4.3: The result in Corollary 4.2, is a generalization of the above stated Theorem 1.2, for $m = 1$, due to Li and Owa [6].

Remark 4.4: For $= \frac{1}{n+1}$ and $= \frac{1-n}{2}$ in Corollary 4.2, we obtain the above stated Theorem 1.1, for $m = 1$, of Li and Owa [6] in subordination form which is more general than its existing form.

When we select $= \frac{1}{n+1}$, in Corollary 4.2, we obtain the following result.

Corollary 4.5: If $f \in S_n(\ )$, then

$$\left(\frac{D^n[f](z)}{z}\right)^{\frac{1}{n+1}} \prec (1 - z)^{\frac{2(\alpha-1)}{n+1}},\ z \in \mathbb{E},$$

where $\neq 1$, is real number.

Remark 4.6: The result in Corollary 4.5, sharpens the result of Uralegaddi [2] and generalizes the result of Li and Owa [6]. For $= 0$, in Corollary 4.5, we obtain the Corollary 1, due to Li and Owa [6] for $m = 1$, in subordination form which is more general than its existing form.

If we select, $n = 0$ in Corollary 4.2, we have the following result.

Corollary 4.7: If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2 \ )z}{1 - z},\ z \in \mathbb{E},$$

then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec (1 - z)^{2\beta(\alpha-1)},\ z \in \mathbb{E},$$

where $\neq 1$, > 0, are real numbers.

Remark 4.8: The result in Corollary 4.7, is more general than the result due Miller and Mocanu [11], Golusin [4] and Li and Owa [6], which can be obtained by selecting $= 0$ and $= \frac{1}{2}$.

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