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Traveling Wave Solutions For The Sawada-Kotera-Kadomtsev-Petviashivili Equation And The Bogoyavlensky-Konoplechenko Equation By $\left(\frac{G'}{G}\right)$-Expansion Method

Nisha Goyal and R.K. Gupta

Abstract—This paper presents a new function expansion method for finding traveling wave solutions of a nonlinear equations and calls it the $\left(\frac{G'}{G}\right)$-expansion method, given by Wang et al recently. As an application of this new method, we study the well-known Sawada-Kotera-Kadomtsev-Petviashivili equation and Bogoyavlensky-Konoplechenko equation. With two new expansions, general types of soliton solutions and periodic solutions for these two equations are obtained.

Keywords—Sawada-Kotera-Kadomtsev-Petviashivili equation, Bogoyavlensky-Konoplechenko equation, $\left(\frac{G'}{G}\right)$-expansion method, Exact solutions.

I. Introduction

THE nonlinear phenomena exist in all the fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. It is well known that many non-linear evolution equations (NLEEs) are widely used to describe these complex phenomena. Research on solutions of NLEEs is popular. So, the powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Some of these approaches are the homogeneous balance method [1,2], the hyperbolic tangent expansion method [3,4], the tanh-method [5], the inverse scattering transform [6], the Bäcklund transform [7], the Hirota bilinear method [8,9], the generalized Riccati equation [10,11], the Weierstrass elliptic function method [12,13], the sine-cosine method [14,15], the Jacobi elliptic function expansion [16,17], the truncated Painleve expansion [18], Lie Classical method [19] and so on.

Among the possible exact solutions of NLEEs, certain solutions for special form may depend only on a single combination of variables such as traveling wave variables. Also there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Recently a so-called $\left(\frac{G'}{G}\right)$-expansion method has drawn a lot of attention. The method was presented by Mingliang Wang in [20] at first. The main merits of the $\left(\frac{G'}{G}\right)$-expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. The method was soon been applied to other non-linear problems by several authors [21,22,23].

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In this paper, we pay attention to the analytical method for getting the exact solution of some NLEES. Among the possible exact solutions of NLEEs, certain solutions for special form may depend only on a single combination of variables such as traveling wave variables. Our main goal in this study is to present the $\left(\frac{G'}{G}\right)$-expansion method for constructing the traveling wave solutions.

In section II, we describe the $\left(\frac{G'}{G}\right)$-expansion method. In section III, in order to illustrate the method we apply the method to two physically important nonlinear evolution equations, namely, the Sawada-Kotera-Kadomtsev-Petviashivili equation and the Bogoyavlensky-Konoplechenko equation and abundant exact solutions are obtained which included the hyperbolic functions, the trigonometric functions and rational functions. Finally, we record some concluding remarks.

II. $\left(\frac{G'}{G}\right)$-Expansion Method

We assume the given nonlinear partial differential equation for $u(x,y,t)$ to be in the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xt}, ...) = 0, \tag{1}$$

where P is a polynomial in its arguments. The essence of the $\left(\frac{G'}{G}\right)$-expansion method can be presented in the following steps:

Step 1. Find traveling wave solutions of equation (1) by taking $u(x,y,t) = u(\xi), \xi = x + y - kt$ and transform equation (1) to the ordinary differential equation

$$Q(u, u', u'', ...) = 0, \tag{2}$$

where prime denotes the derivative with respect to ξ .

Step 2. If possible, integrate equation (2) term by term one or more times. This yields constants of integration. For simplicity, the integration constants can be set to zero.

Step 3. Introduce the solution $u(\xi)$ of equation (2) in the finite series form

$$u(\xi)=\sum_{i=0}^{N} a_i\left(\frac{G'(\xi)}{G(\xi)}\right), \tag{3}$$

where a_i are real constants with $a_N \neq 0$ to be determined, N is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0, \tag{4}$$

where λ and μ are real constants to be determined.

Step 4. Determine N. This, usually, can be accomplished by balancing the linear terms of highest order with the highest order nonlinear terms in equation (2).

Step 5. Substituting (3) together with (4) into equation (2) yields an algebraic equation involving powers of $\left(\frac{G'}{G}\right)$. Equating the coefficients of each power of $\left(\frac{G'}{G}\right)$ to zero gives a system of algebraic equations for a_i, λ, μ and k. Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant $D=\lambda^2-4\mu$ the solutions of equation (4) are well known for us. So, we can obtain exact solutions of equation (1).

III. Applications of $\left(\frac{G'}{G}\right)$ Method

In this section, we apply the $\left(\frac{G'}{G}\right)$-expansion method to solve the Sawada-Kotera-Kadomtsev-Petviashivili equation and Bogoyavlensky-Konoplechenko equation.

III.1 Sawada-Kotera-Kadomtsev-Petviashivili (SKKP) Equation

The Sawada-Kotera-Kadomtsev-Petviashivili (SKKP) equation is

$$(u_t+15uu_{xxx}+15u_xu_{xx}+45u^2u_x+u_{xxxxx})_x+u_{yy}=0, \tag{5}$$

where u is function of x, y and t.
Equation (5) can be written as

$$\begin{aligned}&u_{tx}+15u_xu_{xxx}+15uu_{xxxx}+15u_{xx}^2+15u_xu_{xxx}\\&+45u^2u_{xx}+90uu_x^2+u_{xxxxxx}+u_{yy}=0.\end{aligned} \tag{6}$$

According to the method described above in section 2, we make the transformation $u(x,y,t)=u(\xi), \xi=x+y-kt$. Then we get

$$\begin{aligned}&-ku''+15u'u'''+15uu''''+15u''^2+15u'u'''+45u^2u''\\&+90uu'^2+u''''''+u''=0,\end{aligned} \tag{7}$$

where prime denotes the derivative with respect to ξ.
Now, balancing uu'''' with u'''''' gives $N=2$. Therefore, we can write the solution of equation (7) in the form

$$u(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2, \tag{8}$$

where $a_2 \neq 0$ and $G=G(\xi)$. From equations (4) and (8), we derive

$$\begin{aligned}u'(\xi)=&-2a_2\left(\tfrac{G'}{G}\right)^3-(a_1+2a_2\lambda)\left(\tfrac{G'}{G}\right)^2-(a_1\lambda+2a_2\mu)\left(\tfrac{G'}{G}\right)\\&-a_1\mu,\end{aligned} \tag{9}$$

$$\begin{aligned}u''(\xi)=&\,6a_2\left(\tfrac{G'}{G}\right)^4+(10a_2\lambda+2a_1)\left(\tfrac{G'}{G}\right)^3\\&+(4a_2\lambda^2+8a_2\mu+3a_1\lambda)\left(\tfrac{G'}{G}\right)^2+(a_1\lambda^2+2a_1\mu+6a_2\lambda\mu)\left(\tfrac{G'}{G}\right)\\&+a_1\lambda\mu,\end{aligned} \tag{10}$$

$$\begin{aligned}u'''(\xi)=&-24a_2\left(\tfrac{G'}{G}\right)^5-(6a_1+54a_2\lambda)\left(\tfrac{G'}{G}\right)^4\\&-(40a_2\mu+38a_2\lambda^2+12a_1\lambda)\left(\tfrac{G'}{G}\right)^3\\&-(52a_2\lambda\mu+8a_2\lambda^3+7a_1\lambda^2+8a_1\mu)\left(\tfrac{G'}{G}\right)^2\\&-(8a_1\lambda\mu+14a_2\lambda^2\mu+16a_2\mu^2+a_1\lambda^3)\left(\tfrac{G'}{G}\right)\\&-6a_2\lambda\mu^2-2a_1\mu^2-a_1\lambda^2\mu,\end{aligned} \tag{11}$$

$$\begin{aligned}u''''(\xi)=&\,120a_2\left(\tfrac{G'}{G}\right)^6+(336a_2\lambda+24a_1)\left(\tfrac{G'}{G}\right)^5\\&+(330a_2\lambda^2+240a_2\mu+60a_1\lambda)\left(\tfrac{G'}{G}\right)^4\\&+(50a_1\lambda^2+130a_2\lambda^3+40a_1\mu+440a_2\lambda\mu)\left(\tfrac{G'}{G}\right)^3\\&+(15a_1\lambda^3+16a_2\lambda^4+60a_1\lambda\mu+232a_2\lambda^2\mu+136a_2\mu^2)\left(\tfrac{G'}{G}\right)^2\\&+(a_1\lambda^4+22a_1\lambda^2\mu+120a_2\lambda\mu^2+16a_1\mu^2+30a_2\lambda^3\mu)\left(\tfrac{G'}{G}\right)\\&+16a_2\mu^3+14a_2\lambda^2\mu^2+a_1\lambda^3\mu+8a_1\lambda\mu^2,\end{aligned} \tag{12}$$

$$\begin{aligned}u''''''(\xi)=&\,5040a_2\left(\tfrac{G'}{G}\right)^8+(19440a_2\lambda+720a_1)\left(\tfrac{G'}{G}\right)^7\\&(2520a_1\lambda+29400a_2\lambda^2+13440a_2\mu)\left(\tfrac{G'}{G}\right)^6\\&+(21840a_2\lambda^3+38640a_2\lambda\mu+1680a_1\mu\\&+3360a_1\lambda^2)\left(\tfrac{G'}{G}\right)^5+(40152a_2\lambda^2\mu+8106a_2\lambda^4+4200a_1\lambda\mu\\&+12096a_2\mu^2+2100a_1\lambda^3)\left(\tfrac{G'}{G}\right)^4+(1232a_1\mu^2+17920a_2\lambda^3\mu\\&+22960a_2\lambda\mu^2+602a_1\lambda^4+1330a_2\lambda^5+3584a_1\lambda^2\mu)\left(\tfrac{G'}{G}\right)^3\\&+(3968a_2\mu^2+63a_1\lambda^5+3096a_2\lambda^4\mu+1176a_1\lambda^3\mu+13320a_2\lambda^2\mu^2\\&+64a_2\lambda^6+1848a_1\lambda\mu^2)\left(\tfrac{G'}{G}\right)^2+(2352a_2\lambda^3\mu^2+a_1\lambda^6+720a_1\lambda^2\mu^2\\&+3696a_2\lambda\mu^3+272a_1\mu^3+114a_1\lambda^4\mu+126\lambda^5\mu)\left(\tfrac{G'}{G}\right)\\&+272a_2\mu^4+52a_1\lambda^3\mu^2+62a_2\lambda^4\mu^2+a_1\lambda^5\mu+584a_2\lambda^2\mu^3\\&+136a_1\lambda\mu^3,\end{aligned} \tag{13}$$

Substituting equations (9-13) into equation (7), setting the coefficients of $\left(\frac{G'}{G}\right)^i, (i=0,1,2,3,4,5,6,7,8)$ to zero, we obtain a system of algebraic equations for $a_0, a_1, a_2, k, \lambda$ and

μ as follows:

$$
\begin{aligned}
&\left(\tfrac{G'}{G}\right)^8 : -3780\,{a_2}^2 - 630\,{a_2}^3 - 5040\,a_2 = 0,\\
&\left(\tfrac{G'}{G}\right)^7 : -720\,a_1 - 1170\,{a_2}^3\lambda - 3600\,a_1a_2 - 19440\,a_2\lambda\\
&-11520\,{a_2}^2\lambda - 1350\,a_1{a_2}^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^6 : -13440\,a_2\mu - 29400\,a_2\lambda^2 - 10500\,a_1a_2\lambda\\
&-540\,{a_2}^3\lambda^2 - 12690\,{a_2}^2\lambda^2 - 2520\,a_1\lambda - 900\,a_0{a_2}^2\\
&-2475\,a_1{a_2}^2\lambda - 8880\,{a_2}^2\mu - 1800\,a_0a_2 - 1080\,{a_2}^3\mu\\
&-600\,{a_1}^2 - 900\,{a_1}^2a_2 = 0,\\
&\left(\tfrac{G'}{G}\right)^5 : -180\,{a_1}^3 - 1080\,a_0a_1a_2 - 360\,a_0a_1 - 990\,{a_2}^3\lambda\,\mu\\
&-3360\,a_1\lambda^2 - 1620\,{a_1}^2\lambda - 2250\,a_1{a_2}^2\mu - 10920\,a_1\lambda^2a_2\\
&-1680\,a_1\mu - 21840\,a_2\lambda^3 - 1620\,{a_1}^2a_2\lambda - 1620\,a_0{a_2}^2\lambda\\
&-18840\,{a_2}^2\lambda\,\mu - 5040\,a_0a_2\lambda - 38640\,a_2\lambda\,\mu - 7800\,a_1\mu\,a_2\\
&-5910\,{a_2}^2\lambda^3 - 1125\,a_1{a_2}^2\lambda^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^4 : -6\,a_2 - 1890\,a_0a_1a_2\lambda - 2025\,a_1{a_2}^2\lambda\,\mu\\
&-15420\,a_1\lambda\,\mu\,a_2 - 4695\,a_1\lambda^3a_2 - 4950\,a_0a_2\lambda^2\\
&-1440\,{a_1}^2a_2\mu - 1440\,a_0{a_2}^2\mu - 4200\,a_1\lambda\,\mu - 270\,{a_0}^2a_2\\
&-960\,{a_2}^2\lambda^4 - 12096\,a_2\mu^2 - 2100\,a_1\lambda^3 - 1515\,{a_1}^2\lambda^2\\
&-450\,{a_2}^3\mu^2 - 6720\,{a_2}^2\mu^2 + 6\,ka_2 - 315\,{a_1}^3\lambda - 1140\,{a_1}^2\mu\\
&-270\,a_0{a_1}^2 - 8106\,a_2\lambda^4 - 900\,a_0a_1\lambda - 720\,a_0{a_2}^2\lambda^2\\
&-40152\,a_2\lambda^2\mu - 720\,{a_1}^2a_2\lambda^2 - 3600\,a_0a_2\mu\\
&-12480\,{a_2}^2\lambda^2\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^3 : -10\,a_2\lambda - 2\,a_1 - 1980\,{a_1}^2\lambda\,\mu - 1950\,a_0a_2\lambda^3\\
&-8280\,{a_2}^2\lambda\,\mu^2 - 750\,a_0a_1\lambda^2 - 3584\,a_1\lambda^2\mu - 2490\,{a_2}^2\lambda^3\mu\\
&-900\,a_1{a_2}^2\mu^2 - 450\,a_0{a_1}^2\lambda - 675\,a_1\lambda^4a_2 - 600\,a_0a_1\mu\\
&-22960\,a_2\lambda\,\mu^2 - 450\,{a_0}^2a_2\lambda - 1232\,a_1\mu^2 - 602\,a_1\lambda^4\\
&-9210\,a_1\lambda^2a_2\mu - 6600\,a_0a_2\lambda\,\mu - 810\,a_0a_1a_2\lambda^2\\
&-1620\,a_0a_1a_2\mu - 1260\,a_0{a_2}^2\lambda\,\mu - 1260\,{a_1}^2a_2\lambda\,\mu\\
&-17920\,a_2\lambda^3\mu - 5160\,a_1\mu^2a_2 + 10\,ka_2\lambda - 135\,{a_1}^3\lambda^2\\
&-1330\,a_2\lambda^5 - 90\,{a_0}^2a_1 - 555\,{a_1}^2\lambda^3 + 2\,ka_1-\\
&270\,{a_1}^3\mu = 0,\\
&\left(\tfrac{G'}{G}\right)^2 : -1848\,a_1\lambda\,\mu^2 - 135\,{a_0}^2a_1\lambda - 180\,a_0{a_1}^2\lambda^2\\
&-240\,a_0a_2\lambda^4 - 13320\,a_2\lambda^2\mu^2 - 1176\,a_1\lambda^3\mu - 540\,{a_1}^2a_2\mu^2\\
&+8\,ka_2\mu - 360\,a_0{a_1}^2\mu - 2040\,a_0a_2\mu^2 - 360\,{a_0}^2a_2\mu\\
&-180\,{a_0}^2a_2\lambda^2 + 3\,ka_1\lambda - 3096\,a_2\lambda^4\mu - 960\,{a_1}^2\lambda^2\mu\\
&-540\,a_0{a_2}^2\mu^2 - 2190\,{a_2}^2\lambda^2\mu^2 + 4\,ka_2\lambda^2 - 225\,{a_1}^3\lambda\,\mu\\
&-1350\,a_0a_1a_2\lambda\,\mu - 225\,a_0a_1\lambda^3 - 60\,{a_1}^2\lambda^4 - 64\,a_2\lambda^6\\
&-3968\,a_2\mu^3 - 600\,{a_1}^2\mu^2 - 1680\,{a_2}^2\mu^3 - 63\,a_1\lambda^5\\
&-3480\,a_0a_2\lambda^2\mu - 900\,a_0a_1\lambda\,\mu - 5520\,a_1\lambda\,\mu^2a_2\\
&-1545\,a_1\lambda^3a_2\mu - 3\,a_1\lambda - 8\,a_2\mu - 4\,a_2\lambda^2 = 0,\\
&\left(\tfrac{G'}{G}\right) : -450\,a_0a_2\lambda^3\mu - 15\,a_0a_1\lambda^4 - 240\,a_0a_1\mu^2\\
&-90\,{a_0}^2a_1\mu - 720\,a_1\lambda^2\mu^2 - 272\,a_1\mu^3 - 960\,a_1\mu^3a_2\\
&-3696\,a_2\lambda\,\mu^3 + 2\,ka_1\mu - 114\,a_1\lambda^4\mu - 126\,a_2\lambda^5\mu - a_1\lambda^2\\
&-6\,a_2\lambda\,\mu - 2\,a_1\mu - a_1\lambda^6 - 90\,{a_1}^3\mu^2 - 105\,{a_1}^2\lambda^3\mu\\
&+ka_1\lambda^2 - 720\,{a_2}^2\lambda\,\mu^3 - 2352\,a_2\lambda^3\mu^2 - 480\,{a_1}^2\lambda\,\mu^2\\
&-270\,{a_0}^2a_2\lambda\,\mu - 270\,a_0{a_1}^2\lambda\,\mu - 540\,a_0a_1a_2\mu^2 + 6\,ka_2\lambda\,\mu\\
&-1110\,a_1\lambda^2a_2\mu^2 - 330\,a_0a_1\lambda^2\mu - 1800\,a_0a_2\lambda\,\mu^2\\
&-45\,{a_0}^2a_1\lambda^2 = 0,\\
&\left(\tfrac{G'}{G}\right)^0 : -62\,a_2\lambda^4\mu^2 - a_1\lambda\,\mu - 210\,a_0a_2\lambda^2\mu^2 - 60\,{a_2}^2\mu^4\\
&-60\,{a_1}^2\mu^3 - 15\,a_0a_1\lambda^3\mu - 240\,a_1\lambda\,\mu^3a_2 + ka_1\lambda\,\mu\\
&+2\,ka_2\mu^2 - 45\,{a_1}^2\lambda^2\mu^2 - 120\,a_0a_1\lambda\,\mu^2 - 90\,a_0{a_1}^2\mu^2\\
&-90\,{a_0}^2a_2\mu^2 - 272\,a_2\mu^4 - 52\,a_1\lambda^3\mu^2 - 45\,{a_0}^2a_1\lambda\,\mu\\
&-2\,a_2\mu^2 - 240\,a_0a_2\mu^3 - a_1\lambda^5\mu - 136\,a_1\lambda\,\mu^3\\
&-584\,a_2\lambda^2\mu^3 = 0,
\end{aligned}
\tag{14}
$$

Solving these systems of algebraic equations by Maple gives

Case 1.

$$
\begin{aligned}
&k = 1 + \lambda^4 + 76\mu^2 + 22\lambda^2\mu + 120a_0\mu + 15a_0\lambda^2 + 45a_0^2,\\
&a_1 = -2\lambda, a_2 = -2,
\end{aligned}
\tag{15}
$$

and μ, λ and a_0 arbitrary constants.

Case 2.

$$
\begin{aligned}
&k = 1 + \tfrac{15}{4}a_0a_1^2 + \tfrac{1}{16}a_1^4 + 45a_0^2,\\
&\lambda = -\tfrac{1}{2}a_1, a_2 = -2, \mu = 0,
\end{aligned}
\tag{16}
$$

and a_0 and a_1 are arbitrary constants.

For Case 1, Substituting the solution set (15) and the corresponding solutions of (4) into (8), we have the solutions of equation (7) as follows:
When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$
\begin{aligned}
&u_{11}(\xi) =\\
&a_0 - 2\lambda\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right)\\
&-2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right)^2.
\end{aligned}
\tag{17}
$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$
\begin{aligned}
&u_{12}(\xi) =\\
&a_0 - 2\lambda\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right)\\
&-2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right)^2.
\end{aligned}
\tag{18}
$$

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions

$$
u_{13}(\xi) = \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2},
\tag{19}
$$

where $\xi = x + y - (1 + \lambda^4 + 76\mu^2 + 22\lambda^2\mu + 120a_0\mu + 15a_0\lambda^2 + 45a_0^2)t$.

For Case 2, Substituting the solution set (16) and the corresponding solutions of (4) into (8), we have the solutions of equation (7) as follows:
When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$
\begin{aligned}
&u_{21}(\xi) = a_0\\
&+a_1\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right)\\
&-2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right)^2.
\end{aligned}
\tag{20}
$$

When $\lambda^2-4\mu<0$, we obtain the trigonometric function traveling wave solutions

$$u_{22}(\xi)=a_0 +a_1\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right) -2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right)^2. \tag{21}$$

When $\lambda^2-4\mu=0$, we obtain the rational function solutions

$$u_{23}(\xi)=\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}, \tag{22}$$

where $\lambda=-\frac{1}{2}a_1, \mu=0$
and $\xi=x+y-(1+\frac{15}{4}a_0a_1^2+\frac{1}{16}a_1^4+45a_0^2)t$.

III.2 Bogoyavlensky-Konoplechenko Equation

Now, let us consider the following Bogoyavlensky-Konoplechenko equation in the form

$$u_{xt}+\alpha u_{xxxx}+\beta u_{xxxy}+6\alpha u_{xx}u_x+4\beta u_{xy}u_x+4\beta u_{xx}u_y=0, \tag{23}$$

where u is function of x, y and t.
We make the transformation $u(x,y,t)=u(\xi), \xi=x+y-kt$. Then we get

$$-ku''+\alpha u''''+\beta u''''+6\alpha u''u'+8\beta u'u''=0, \tag{24}$$

where prime denotes differentiation w.r.t ξ.
Balancing $u'u''$ with u'''' gives N = 1.
Therefore, we can write the solution of equation (24) in the form

$$u(\xi)=a_0+a_1\left(\frac{G'}{G}\right), a_1\neq 0 \tag{25}$$

By using equations (4) and (25) we have

$$\begin{aligned}
&u'(\xi)=-a_1\left(\tfrac{G'}{G}\right)^2-a_1\lambda\left(\tfrac{G'}{G}\right)-a_1\mu,\\
&u''(\xi)=2a_1\left(\tfrac{G'}{G}\right)^3+3a_1\left(\tfrac{G'}{G}\right)^2+(a_1\lambda^2+2a_1\mu)\left(\tfrac{G'}{G}\right)\\
&+a_1\lambda\mu,\\
&u'''(\xi)=-6a_1\left(\tfrac{G'}{G}\right)^4-12a_1\lambda\left(\tfrac{G'}{G}\right)^3-a_1(7\lambda^2+8\mu)\left(\tfrac{G'}{G}\right)^2\\
&-a_1(\lambda^3+8\lambda\mu)\left(\tfrac{G'}{G}\right)-a_1(\lambda^2\mu+2\mu^2),\\
&u''''(\xi)=24a_1\left(\tfrac{G'}{G}\right)^5+60a_1\lambda\left(\tfrac{G'}{G}\right)^4+a_1(50\lambda^2+40\mu)\\
&\left(\tfrac{G'}{G}\right)^3+a_1(15\lambda^3+60\lambda\mu)\left(\tfrac{G'}{G}\right)^2+a_1(\lambda^4+22\lambda^2\mu+16\mu^2)\\
&\left(\tfrac{G'}{G}\right)+a_1(\lambda^3\mu+8\lambda\mu^2),
\end{aligned} \tag{26}$$

Substituting equations (26) into (24), setting coefficients of $\left(\frac{G'}{G}\right)^i, (i=0,1,2,3,4,5)$ to zero, we obtain a system of nonlinear algebraic equations a_0, a_1, k, λ and μ as follows:

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^5: 12\,aa_1-24\,a-24\,b+16\,ba_1=0,\\
&\left(\tfrac{G'}{G}\right)^4: -60\,a\lambda-60\,b\lambda+30\,aa_1\lambda+40\,ba_1\lambda=0,\\
&\left(\tfrac{G'}{G}\right)^3: 2\,k+32\,ba_1\lambda^2-40\,b\mu-40\,a\mu+24\,aa_1\mu+32\,ba_1\mu\\
&+24\,aa_1\lambda^2-50\,b\lambda^2-50\,a\lambda^2=0,\\
&\left(\tfrac{G'}{G}\right)^2: -60\,a\lambda\,\mu+6\,aa_1\lambda^3+8\,ba_1\lambda^3+48\,ba_1\lambda\,\mu-15\,a\lambda^3\\
&-60\,b\lambda\,\mu+3\,k\lambda-15\,b\lambda^3+36\,aa_1\lambda\,\mu=0,\\
&\left(\tfrac{G'}{G}\right): 16\,ba_1\lambda^2\mu-22\,b\lambda^2\mu-a\lambda^4+12\,aa_1\mu^2+16\,ba_1\mu^2\\
&-22\,a\lambda^2\mu+12\,aa_1\lambda^2\mu-16\,a\mu^2-b\lambda^4+k\lambda^2+2\,k\mu-16\,b\mu^2=0,\\
&\left(\tfrac{G'}{G}\right)^0: -8\,a\lambda\,\mu^2+8\,ba_1\lambda\,\mu^2-b\lambda^3\mu+k\lambda\,\mu+6\,aa_1\lambda\,\mu^2-a\lambda^3\mu\\
&-8\,b\lambda\,\mu^2=0.
\end{aligned} \tag{27}$$

Solving this system by Maple gives

$$\begin{aligned}
\alpha&=-\tfrac{2\beta(-3+2a_1)}{3(a_1-2)}, a_1=1,\\
k&=\tfrac{\beta a_1(-\lambda^2+4\mu)}{3(a_1-2)}.
\end{aligned} \tag{28}$$

Substituting the solution set (28) and the corresponding solutions of (4) into (25), we have the solutions of equation (24) as follows:
When $\lambda^2-4\mu>0$, we obtain the hyperbolic function traveling wave solutions

$$u_{11}(\xi)=a_0+ a_1\lambda\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right). \tag{29}$$

When $\lambda^2-4\mu<0$, we obtain the trigonometric function traveling wave solutions

$$u_{12}(\xi)=a_0 +a_1\lambda\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}-\frac{\lambda}{2}\right)\right). \tag{30}$$

When $\lambda^2-4\mu=0$, we obtain the rational function solutions

$$u_{13}(\xi)=\frac{C_2}{C_1+C_2\xi}-\frac{\lambda}{2}, \tag{31}$$

where $\xi=x+y-\left(\frac{\beta a_1(-\lambda^2+4\mu)}{3(a_1-2)}\right)t$.

IV. Discussion and Concluding Remarks

In this paper, an implementation of the $\left(\frac{G'}{G}\right)$-expansion method is given by applying it to three nonlinear equations to illustrate the validity and advantages of the method. As a result, hyperbolic function solutions, trigonometric function solutions and rational function solutions with parameters are obtained. The $\left(\frac{G'}{G}\right)$-expansion method is direct, concise and effective. The performance of this method is reliable, simple and gives many new exact solutions. The obtained solutions with free parameters may be important to explain some physical phenomena. The paper shows that the devised algorithm is effective and can be used for many other NPDEs in mathematical physics.

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