

Existence and uniqueness of periodic solution for a discrete-time SIR epidemic model with time delays and impulses

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Abstract—In this paper, a discrete-time SIR epidemic model with nonlinear incidence rate, time delays and impulses is investigated. Sufficient conditions for the existence and uniqueness of periodic solutions are obtained by using contraction theorem and inequality techniques. An example is employed to illustrate our results.

Keywords—Discrete-time SIR epidemic model, Time delay, Non-linear incidence rate, Impulse.

I. INTRODUCTION

EPIDEMICS play a key role in the development and survival of biological species, not in the least of our own human species. Their study has thus attracted interest since ancient times. In 1927, Kermack and McKendrick [1] constructed a famous SIR epidemic model to study the law of the black plague. In 1932, an SIS mathematical model was constructed, and the "threshold theory" was established [2]. From then on, the theory and application of epidemic dynamics was developed greatly.

The basic and important concern for mathematical models in epidemiology is qualitative analysis; the persistence, permanence, asymptotic stability, and the existence and uniqueness of solutions for the model. Many influential results related in these topics have been established and can be found in many articles and books (see [3-15]).

Incidence rate plays an important role in the modelling of epidemic dynamics. In many epidemic dynamic models, the bilinear incidence rate and the standard incidence rate are frequently used. It has been suggested by several authors that the disease transmission process may have a nonlinear incidence rate (see [11, 13-15]).

Recently, much attention has been given to the persistence and global stability of epidemic model with time delays ([4-8, 14-17]).

However, to the best of our knowledge, discrete-time models are seldom considered. In a previous work we studied such a SIR discrete-time model. The advantages of a discrete-time approach are multiple. First, difference models are more realistic than differential ones since the epidemic statistics are compiled from given time intervals and not continuously (a fact that can be of importance for fast-spreading epidemics).

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Second, the discrete-time models can provide natural simulators for the continuous cases (i.e., differential models). One can thus not only study with good accuracy the behaviour of the continuous-time model, but also assess the effect of larger time steps. Finally, the use of discrete-time models makes it possible to use the entire arsenal of methods recently developed for the study of mappings and lattice equations, either from the integrability and/or chaos points of view.

Motivated by the above discussions, in this paper, we investigate the existence of periodic solution for a discrete-time SIR epidemic model with time delay and impulses of the following form:

$$\left\{ \begin{array}{l} S(n+1) - S(n) \\ = B - \mu_1 S(n) \\ - \frac{\beta S(n) I(n - \tau(n))}{1 + \alpha I(n - \tau(n))}, \\ I(n+1) - I(n) \\ = \frac{\beta S(n) I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \\ - (\mu_2 + \gamma) I(n), \\ R(n+1) - R(n) \\ = \gamma I(n) - \mu_3 R(n), \end{array} \right. \quad n = 0, 1, \dots, n \neq n_k, \quad (1)$$

$$\left. \begin{array}{l} S(n_k + 1) - S(n_k) = b_{1k} S(n_k), \\ I(n_k + 1) - I(n_k) = b_{2k} I(n_k), \\ R(n_k + 1) - R(n_k) = b_{3k} R(n_k), \end{array} \right\} \quad k = 1, 2, \dots,$$

where $S(n)$ denotes the number of members of a population susceptible to the disease, $I(n)$ the number of infective members and $R(n)$ the number of members who have been removed from the possibility of infection through full immunity. The parameters μ_1, μ_2, μ_3 are positive constants representing the death rates of susceptibles, infectives, and recovered, respectively. It is natural biologically to assume that $\mu_1 \leq \min\{\mu_2, \mu_3\}$. The parameters B and γ are positive constants representing the birth rate of the population and the recovery rate of infectives, respectively.

For convenience, we use the following notations

$$b_{ik}^+ = \max\{0, b_{ik}\}, i = 1, 2, 3.$$

We denote the product of $1 + b_{ik}$ when $n_k \in [a, b]$ by $\prod_{a \leq n_k < b} (1 + b_{ik})$ with the understanding that $\prod_{a \leq n_k < b} (1 + b_{ik}) = 1$ for all $a \geq b$.

Throughout this paper, we assume that

(H₁) $\tau(n) \geq 0$ is an ω -periodic function, i.e., $\tau(n + \omega) = \tau(n)$, $\{b_{ik}\}$ and $\{n_k\}$ real ω -periodic sequence, where ω is an positive integer with $\omega \geq 1$, i.e., there exists a $q \in N$ such that $n_{k+q} = n_k + \omega$, $b_{i(k+q)} = b_{ik}$, $b_{ik} > -1$, $0 < n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$, $k = 1, 2, 3, \dots$, $i = 1, 2, 3$.

(H₂) $\prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{\substack{u \neq n_k, 0 \leq u < \omega \\ u \neq n_k, 0 \leq u < \omega}} (1 - \mu_1) \neq 1$,
 $\prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{\substack{u \neq n_k, 0 \leq u < \omega \\ u \neq n_k, 0 \leq u < \omega}} (1 - \mu_2 - \gamma) \neq 1$,
 $\prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{\substack{u \neq n_k, 0 \leq u < \omega \\ u \neq n_k, 0 \leq u < \omega}} (1 - \mu_3) \neq 1$.

(H₃) There exist two positive numbers r, L such that

$$0 < r = \max\{r_1, r_2, r_3\}, 0 < L = \max\{r, r_4\} < 1,$$

where

$$r_1 = \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b_{1k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_1) \right| \alpha},$$

$$r_2 = \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b_{2k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \right| \alpha},$$

$$r_3 = \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_3) \prod_{0 \leq n_k < \omega} (1 + b_{3k}^+) \gamma}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_3) \right|},$$

$$r_4 = \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b_{2k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \right|} \times \frac{M}{1 - r}$$

and

$$M = \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b_{1k}^+) B}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{u \neq n_k, 0 \leq u < \omega} (1 - \mu_1) \right|}.$$

The remaining parts of this paper are organized as follows. In Section 2, we shall derive sufficient conditions for the existence of unique periodic solution for system (1) by using contraction theorem. We then, in Section 3, give an example to illustrate the new results of this paper.

II. EXISTENCE OF UNIQUE PERIODIC SOLUTION

Consider the following non-impulsive equation:

$$\begin{cases} \tilde{S}(n+1) - \tilde{S}(n) \\ = -\tilde{\mu}_1 \tilde{S}(n) + \prod_{0 \leq n_k < n} (1 + b_{1k})^{-1} \left[\tilde{B} \right. \\ \left. - \frac{\tilde{\beta} \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})}{1 + \alpha \tilde{I}(n - \tau(n)) \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})} \right], \\ \tilde{I}(n+1) - \tilde{I}(n) \\ = -(\tilde{\mu}_2 + \tilde{\gamma}) \tilde{I}(n) + \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1} \\ \times \frac{\tilde{\beta} \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})}{1 + \alpha \tilde{I}(n - \tau(n)) \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k})}, \\ \tilde{R}(n+1) - \tilde{R}(n) \\ = -\tilde{\mu}_3 \tilde{R}(n) \\ + \prod_{0 \leq n_k < n} (1 + b_{3k})^{-1} \tilde{\gamma} \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}), \end{cases} \quad (2)$$

where

$$\tilde{B} = \begin{cases} B, & n \neq n_k, \\ 0, & n = n_k, \end{cases} \quad \tilde{\beta} = \begin{cases} \beta, & n \neq n_k, \\ 0, & n = n_k, \end{cases}$$

$$\tilde{\gamma} = \begin{cases} \gamma, & n \neq n_k, \\ 0, & n = n_k, \end{cases} \quad \tilde{\mu}_i = \begin{cases} \mu_i, & n \neq n_k, i = 1, 2, 3, \\ 0, & n = n_k. \end{cases}$$

Lemma 1. Assume that (H₁) holds. Then

(i) if $\tilde{x}(n) = (\tilde{S}(n), \tilde{I}(n), \tilde{R}(n))^T$ is a solution of (2), then

$$x(n) = \left(\tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}), \right. \\ \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}), \\ \left. \tilde{R}(n) \prod_{0 \leq n_k < n} (1 + b_{3k}) \right)^T \quad (3)$$

is a solution of (1);

(ii) if $x(n) = (S(n), I(n), R(n))^T$ is a solution of (1), then

$$\tilde{x}(n) = \left(S(n) \prod_{0 \leq n_k < n} (1 + b_{1k})^{-1}, \right. \\ I(n) \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1}, \\ \left. R(n) \prod_{0 \leq n_k < n} (1 + b_{3k})^{-1} \right)^T \quad (4)$$

is a solution of (2).

Proof: First, we prove (i). When $n = 0, 1, 2, \dots, n \neq n_k, k = 1, 2, \dots$, we have

$$S(n+1) - (1 - \mu_1)S(n) - B + \frac{\beta S(n)I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \\ = \tilde{S}(n+1) \prod_{0 \leq n_k < n+1} (1 + b_{1k}) \\ - (1 - \mu_1) \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) - B$$

$$\begin{aligned}
 & + \left(\beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \right. \\
 & \times \left. \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right) \left(1 + \alpha \tilde{I}(n - \tau(n)) \right. \\
 & \times \left. \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right)^{-\frac{1}{2}} \\
 = & \prod_{0 \leq n_k < n} (1 + b_{1k}) \left\{ \tilde{S}(n + 1) - (1 - \mu_1) \tilde{S}(n) \right. \\
 & - \prod_{0 \leq n_k < n} (1 + b_{1k})^{-1} \left[B \right. \\
 & - \left. \left(\beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \right. \right. \\
 & \times \left. \left. \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right) \left(1 + \alpha \tilde{I}(n - \tau(n)) \right. \right. \\
 & \left. \left. \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right)^{-\frac{1}{2}} \right] \left. \right\} \\
 = & 0, \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 & I(n + 1) - (1 - \mu_2 - \gamma)I(n) - \frac{\beta S(n)I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \\
 = & \tilde{I}(n + 1) \prod_{0 \leq n_k < n + 1} (1 + b_{2k}) \\
 & - (1 - \mu_2 - \gamma) \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}) \\
 & - \left(\beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \right. \\
 & \times \left. \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right) \\
 & \times \left(1 + \alpha \tilde{I}(n - \tau(n)) \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right)^{-\frac{1}{2}} \\
 = & \prod_{0 \leq n_k < n} (1 + b_{2k}) \left[\tilde{I}(n + 1) - (1 - \mu_2 - \gamma) \tilde{I}(n) \right. \\
 & - \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1} \\
 & \times \left(\beta \tilde{S}(n) \prod_{0 \leq n_k < n} (1 + b_{1k}) \tilde{I}(n - \tau(n)) \right. \\
 & \times \left. \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right) \left(1 + \alpha \tilde{I}(n - \tau(n)) \right. \\
 & \times \left. \prod_{0 \leq n_k < n - \tau(n)} (1 + b_{2k}) \right)^{-\frac{1}{2}} \left. \right] \\
 = & 0, \tag{6}
 \end{aligned}$$

$$R(n + 1) - (1 - \mu_3)R(n) - \gamma I(n)$$

$$\begin{aligned}
 = & \tilde{R}(n + 1) \prod_{0 \leq n_k < n + 1} (1 + b_{3k}) \\
 & - (1 - \mu_3) \tilde{R}(n) \prod_{0 \leq n_k < n} (1 + b_{3k}) \\
 & - \gamma \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}) \\
 = & \prod_{0 \leq n_k < n} (1 + b_{3k}) \left[\tilde{R}(n + 1) - (1 - \mu_3) \tilde{R}(n) \right. \\
 & \left. - \prod_{0 \leq n_k < n} (1 + b_{3k})^{-1} \gamma \tilde{I}(n) \prod_{0 \leq n_k < n} (1 + b_{2k}) \right] \\
 = & 0. \tag{7}
 \end{aligned}$$

On the other hand, for every $n = n_k, k = 1, 2, \dots$, from (2), we get $\tilde{S}(n_k + 1) = \tilde{S}(n_k), \tilde{I}(n_k + 1) = \tilde{I}(n_k), \tilde{R}(n_k + 1) = \tilde{R}(n_k)$, so

$$\begin{aligned}
 S(n_k + 1) & = \tilde{S}(n_k + 1) \prod_{0 \leq n_s < n_k + 1} (1 + b_{1s}) \\
 & = (1 + b_{1k}) \tilde{S}(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{1s}) \\
 & = (1 + b_{1k}) S(n_k), \tag{8}
 \end{aligned}$$

$$\begin{aligned}
 I(n_k + 1) & = \tilde{I}(n_k + 1) \prod_{0 \leq n_s < n_k + 1} (1 + b_{2s}) \\
 & = (1 + b_{2k}) \tilde{I}(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{2s}) \\
 & = (1 + b_{2k}) I(n_k), \tag{9}
 \end{aligned}$$

$$\begin{aligned}
 R(n_k + 1) & = \tilde{R}(n_k + 1) \prod_{0 \leq n_s < n_k + 1} (1 + b_{3s}) \\
 & = (1 + b_{3k}) \tilde{R}(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{3s}) \\
 & = (1 + b_{3k}) R(n_k). \tag{10}
 \end{aligned}$$

It follows from (5)-(10) that (3) is the solution of (1).

Next, we prove (ii). When $n = n_k, k = 1, 2, \dots$, from (1) we obtain

$$\begin{aligned}
 \tilde{S}(n_k + 1) & = S(n_k + 1) \prod_{0 \leq n_s < n_k + 1} (1 + b_{1s})^{-1} \\
 & = (1 + b_{1k}) S(n_k) \prod_{0 \leq n_s < n_k + 1} (1 + b_{1s})^{-1} \\
 & = S(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{1s})^{-1} \\
 & = \tilde{S}(n_k), \\
 \tilde{I}(n_k + 1) & = I(n_k + 1) \prod_{0 \leq n_s < n_k + 1} (1 + b_{2s})^{-1} \\
 & = (1 + b_{2k}) I(n_k) \prod_{0 \leq n_s < n_k + 1} (1 + b_{2s})^{-1} \\
 & = I(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{2s})^{-1} \\
 & = \tilde{I}(n_k),
 \end{aligned}$$

$$\begin{aligned} \tilde{R}(n_k + 1) &= R(n_k + 1) \prod_{0 \leq n_s < n_k + 1} (1 + b_{3s})^{-1} \\ &= (1 + b_{3k})R(n_k) \prod_{0 \leq n_s < n_k + 1} (1 + b_{3s})^{-1} \\ &= R(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{3s})^{-1} \\ &= \tilde{R}(n_k). \end{aligned}$$

Hence,

$$\begin{aligned} &\tilde{S}(n_k + 1) - \tilde{S}(n_k) \\ &= -\tilde{\mu}_1 \tilde{S}(n_k) + \prod_{0 \leq n_s < n_k} (1 + b_{1k})^{-1} \left[\tilde{B} \right. \\ &\quad - \left(\tilde{\beta} \tilde{S}(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{1k}) \tilde{I}(n_k - \tau(n_k)) \right. \\ &\quad \times \left. \prod_{0 \leq n_s < n_k - \tau(n_k)} (1 + b_{2k}) \right) \left(1 + \alpha \tilde{I}(n_k - \tau(n_k)) \right. \\ &\quad \times \left. \left. \prod_{0 \leq n_s < n_k - \tau(n_k)} (1 + b_{2k}) \right)^{-\frac{1}{2}} \right], \\ &\tilde{I}(n_k + 1) - \tilde{I}(n_k) \\ &= -(\tilde{\mu}_2 + \tilde{\gamma}) \tilde{I}(n_k) + \prod_{0 \leq n_s < n_k} (1 + b_{2k})^{-1} \\ &\quad \times \left(\tilde{\beta} \tilde{S}(n_k) \prod_{0 \leq n_s < n_k} (1 + b_{1k}) \tilde{I}(n_k - \tau(n_k)) \right. \\ &\quad \times \left. \prod_{0 \leq n_s < n_k - \tau(n_k)} (1 + b_{2k}) \right) \left(1 + \alpha \tilde{I}(n_k - \tau(n_k)) \right. \\ &\quad \times \left. \left. \prod_{0 \leq n_s < n_k - \tau(n_k)} (1 + b_{2k}) \right)^{-\frac{1}{2}}, \\ &\tilde{R}(n_k + 1) - \tilde{R}(n_k) \\ &= -\tilde{\mu}_3 \tilde{R}(n_k) + \prod_{0 \leq n_s < n_k} (1 + b_{3k})^{-1} \tilde{\gamma} \tilde{I}(n_k) \\ &\quad \times \prod_{0 \leq n_s < n_k} (1 + b_{2k}). \end{aligned}$$

When $n \geq 0, n \neq n_k, k = 1, 2, \dots$, it is easy to verify that $(\tilde{S}(n), \tilde{I}(n), \tilde{R}(n))^T$, also satisfies (2). Therefore, (4) is the solution of (2). This completes the proof. ■

Lemma 2. Suppose that (H_1) and (H_2) hold. Let $x(n) = (S(n), I(n), R(n))^T$ be an ω -periodic solution of (1). Then

$$\begin{aligned} S(n) &= \sum_{s=n}^{n+\omega-1} H_1 \prod_{s \leq n_k < n+\omega} (1 + b_{1k}) \\ &\quad \times \left[\tilde{B} - \frac{\tilde{\beta} S(s) I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right], \end{aligned} \quad (11)$$

$$\begin{aligned} I(n) &= \sum_{s=n}^{n+\omega-1} H_2 \prod_{s \leq n_k < n+\omega} (1 + b_{2k}) \\ &\quad \times \frac{\tilde{\beta} S(s) I(s - \tau(s))}{1 + \alpha I(s - \tau(s))}, \end{aligned} \quad (12)$$

$$R(n) = \sum_{s=n}^{n+\omega-1} H_3 \prod_{s \leq n_k < n+\omega} (1 + b_{3k}) \tilde{\gamma} I(s), \quad (13)$$

where

$$\begin{aligned} H_1 &= \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_1)}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_1)}, \\ H_2 &= \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_2 - \tilde{\gamma})}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_1 - \tilde{\gamma})}, \\ H_3 &= \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_3)}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_3)}. \end{aligned}$$

Proof: By Lemma 1, that $\tilde{x}(n) = (\tilde{S}, \tilde{I}, \tilde{R})^T$ is a solution of (2) is equivalent to that (3) is a solution of (1). From (2) we have

$$\begin{aligned} &\tilde{S}(n + 1) - (1 - \tilde{\mu}_1) \tilde{S}(n) \\ &= \prod_{0 \leq n_k < n} (1 + b_{1k})^{-1} \left[\tilde{B} - \frac{\tilde{\beta} S(n) I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \right], \end{aligned} \quad (14)$$

$$\begin{aligned} &\tilde{I}(n + 1) - (1 - \tilde{\mu}_2 - \tilde{\gamma}) \tilde{I}(n) \\ &= \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1} \frac{\tilde{\beta} S(n) I(n - \tau(n))}{1 + \alpha I(n - \tau(n))}, \end{aligned} \quad (15)$$

$$\begin{aligned} &\tilde{R}(n + 1) - (1 - \tilde{\mu}_3) \tilde{R}(n) \\ &= \prod_{0 \leq n_k < n} (1 + b_{23k})^{-1} \tilde{\gamma} \tilde{I}(n). \end{aligned} \quad (16)$$

Multiplying both sides of the above equation (14) by $\prod_{0 \leq s < n+1} (1 - \tilde{\mu}_1)^{-1}$, the equation (15) by $\prod_{0 \leq s < n+1} (1 - \tilde{\mu}_2 - \tilde{\gamma})^{-1}$, the equations (16) by $\prod_{0 \leq s < n+1} (1 - \tilde{\mu}_3)^{-1}$, respectively, we get

$$\begin{aligned} &S(n + 1) \prod_{0 \leq n_k < n+1} (1 + b_{1k})^{-1} \prod_{0 \leq s < n+1} (1 - \tilde{\mu}_1)^{-1} \\ &- S(n) \prod_{0 \leq n_k < n} (1 + b_{1k})^{-1} \prod_{0 \leq s < n} (1 - \tilde{\mu}_1)^{-1} \\ &= \prod_{0 \leq n_k < n} (1 + b_{1k})^{-1} \prod_{0 \leq s < n+1} (1 - \tilde{\mu}_1)^{-1} \\ &\quad \times \left[\tilde{B} - \frac{\tilde{\beta} S(n) I(n - \tau(n))}{1 + \alpha I(n - \tau(n))} \right], \end{aligned} \quad (17)$$

$$\begin{aligned} &I(n + 1) \prod_{0 \leq n_k < n+1} (1 + b_{2k})^{-1} \prod_{0 \leq s < n+1} (1 - \tilde{\mu}_2 - \tilde{\gamma})^{-1} \\ &- I(n) \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1} \prod_{0 \leq s < n} (1 - \tilde{\mu}_2 - \tilde{\gamma})^{-1} \\ &= \prod_{0 \leq n_k < n} (1 + b_{2k})^{-1} \end{aligned}$$

$$\times \prod_{0 \leq s < n+1} (1 - \tilde{\mu}_2 - \tilde{\gamma})^{-1} \frac{\tilde{\beta}S(n)I(n - \tau(n))}{1 + \alpha I(n - \tau(n))}, \quad (18)$$

$$\begin{aligned} & R(n+1) \prod_{0 \leq n_k < n+1} (1 + b_{3k})^{-1} \prod_{0 \leq s < n+1} (1 - \tilde{\mu}_3)^{-1} \\ & - R(n) \prod_{0 \leq n_k < n} (1 + b_{3k})^{-1} \prod_{0 \leq s < n} (1 - \tilde{\mu}_3)^{-1} \\ & = \prod_{0 \leq n_k < n} (1 + b_{3k})^{-1} \prod_{0 \leq s < n+1} (1 - \tilde{\mu}_3)^{-1} \tilde{\gamma} I(n). \end{aligned} \quad (19)$$

Summing both sides of the above equations (17), (18), (19) from n to $n + \omega - 1$, noticing that $S(n + \omega) = S(n)$, $I(n + \omega) = I(n)$, $R(n + \omega) = R(n)$ and $\prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{u \neq n_k, 0 \leq u < \omega} (1 - \mu_1) = \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{0 \leq u < \omega} (1 - \mu_1) \neq 1$, $\prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{u \neq n_k, 0 \leq u < \omega} (1 - \mu_2 - \gamma) = \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{0 \leq u < \omega} (1 - \mu_2 - \gamma) \neq 1$, $\prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{u \neq n_k, 0 \leq u < \omega} (1 - \mu_3) = \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{0 \leq u < \omega} (1 - \mu_3) \neq 1$, we obtain (11), (12) and (13). This completes the proof. ■

Now, we take $X = \{x(n) | x(n) = (S(n), I(n), R(n))^T, S(n + \omega) = S(n), I(n + \omega) = I(n), R(n + \omega) = R(n)\}$. Then X is a Banach space with the norm

$$\|x\| = \max\{|S|_0, |I|_0, |R|_0\}, |S|_0 = \max_{0 \leq n < \omega} |S(n)|,$$

$$|I|_0 = \max_{0 \leq n < \omega} |I(n)|, |R|_0 = \max_{0 \leq n < \omega} |R(n)|.$$

Set a mapping $\Phi : X \rightarrow X$ by setting

$$\Phi x(n) = (\Phi S(n), \Phi I(n), \Phi R(n))^T,$$

where

$$\begin{aligned} \Phi S(n) &= \sum_{s=n}^{n+\omega-1} H_1 \prod_{s \leq n_k < n+\omega} (1 + b_{1k}) \\ &\times \left[\tilde{B} - \frac{\tilde{\beta}S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right], \\ \Phi I(n) &= \sum_{s=n}^{n+\omega-1} H_2 \prod_{s \leq n_k < n+\omega} (1 + b_{2k}) \frac{\tilde{\beta}S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))}, \\ \Phi R(n) &= \sum_{s=n}^{n+\omega-1} H_3 \prod_{s \leq n_k < n+\omega} (1 + b_{3k}) \tilde{\gamma} I(s). \end{aligned}$$

It is easy to know the fact that the existence of ω -periodic solution of (1) is equivalent to the existence of fixed point of the mapping Φ in X .

Theorem 1. Suppose that (H_1) - (H_3) hold. Then, there exists a unique ω -periodic solution of system (1) in the region $X^* = \{x | x \in X : \|x - x_0\| \leq \frac{Mr}{1-r}\}$, where

$$x_0(n) = \left(\sum_{s=n}^{n+\omega-1} H_1 \prod_{s \leq n_k < n+\omega} (1 + b_{1k}) \tilde{B}, 0, 0 \right).$$

Proof: It is easy to see that X^* is a closed convex subset of X . According to the definition of the norm of Banach space X , we get

$$\begin{aligned} & \|x_0\| \\ &= \max \left\{ \max_{0 \leq n < \omega} \left| \sum_{s=n}^{n+\omega-1} H_1 \prod_{s \leq n_k < n+\omega} (1 + b_{1k}) \tilde{B} \right|, 0, 0 \right\} \\ &= \max_{0 \leq n < \omega} \left| \sum_{s=n}^{n+\omega-1} \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_1) \prod_{s \leq n_k < n+\omega} (1 + b_{1k})}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_1)} \tilde{B} \right| \\ &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b_{1k}^+) B}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{u \neq n_k, 0 \leq u < \omega} (1 - \mu_1) \right|} =: M. \end{aligned}$$

Therefore, for $\forall x \in X^*$, we have

$$\|x\| \leq \|x - x_0\| + \|x_0\| \leq \frac{Mr}{1-r} + M = \frac{M}{1-r}. \quad (20)$$

Now, we prove that the mapping Φ is a self-mapping from X^* to X^* . In fact, for $\forall x \in X^*$, noticing (20), we obtain

$$\begin{aligned} & |\Phi S - S_0|_0 \\ &= \max_{0 \leq n < \omega} |\Phi S - S_0| \\ &= \max_{0 \leq n < \omega} \left| \sum_{s=n}^{n+\omega-1} H_1 \prod_{s \leq n_k < n+\omega} (1 + b_{1k}) \frac{\tilde{\beta}S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right| \\ &\leq \max_{0 \leq n < \omega} \sum_{s=n}^{n+\omega-1} \left| \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_1)}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_1)} \right| \\ &\times \prod_{s \leq n_k < n+\omega} (1 + b_{1k}^+) \left| \frac{\tilde{\beta}S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right| \\ &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b_{1k}^+) B}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \right|} |S|_0 \\ &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b_{1k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \right| \alpha} |S|_0 \\ &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b_{1k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{n \neq n_k, 0 \leq n_k < \omega} (1 - \mu_1) \right| \alpha} \|x\| \\ &\leq \frac{Mr_1}{1-r} \\ &\leq \frac{Mr}{1-r}, \\ &|\Phi I - I_0|_0 \\ &= \max_{0 \leq n < \omega} |\Phi I - I_0| \\ &= \max_{0 \leq n < \omega} \left| \sum_{s=n}^{n+\omega-1} H_2 \prod_{s \leq n_k < n+\omega} (1 + b_{2k}) \frac{\tilde{\beta}S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \max_{0 \leq n < \omega} \sum_{s=n}^{n+\omega-1} \left| \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_2 - \tilde{\gamma})}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_2 - \tilde{\gamma})} \right| \\
 &\quad \times \prod_{s \leq n_k < n+\omega} (1 + b_{2k}) \left| \frac{\tilde{\beta}S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right| \\
 &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b_{2k}^+)}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \right|} \\
 &\quad \times \max_{0 \leq n < \omega} \left| \frac{\tilde{\beta}S(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right| \\
 &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b_{2k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \right|} \alpha |S|_0 \\
 &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b_{2k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \right|} |x| \\
 &\leq \frac{Mr_2}{1-r} \\
 &\leq \frac{Mr}{1-r}, \\
 &|\Phi R - R_0|_0 \\
 &= \max_{0 \leq n < \omega} |\Phi R - R_0| \\
 &= \max_{0 \leq n < \omega} \left| \sum_{s=n}^{n+\omega-1} H_3 \prod_{s \leq n_k < n+\omega} (1 + b_{3k}) \tilde{\gamma} I(s) \right| \\
 &\leq \max_{0 \leq n < \omega} \sum_{s=n}^{n+\omega-1} \left| \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_3)}{1 - \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_3)} \right| \\
 &\quad \times \prod_{s \leq n_k < n+\omega} (1 + b_{3k}) \tilde{\gamma} |I(s)| \\
 &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_3) \prod_{0 \leq n_k < \omega} (1 + b_{3k}^+) \gamma}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_3) \right|} |I|_0 \\
 &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_3) \prod_{0 \leq n_k < \omega} (1 + b_{3k}^+) \gamma}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{3k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_3) \right|} |x| \\
 &\leq \frac{Mr_3}{1-r} \\
 &\leq \frac{Mr}{1-r}.
 \end{aligned}$$

Then

$$\begin{aligned}
 \|\Phi x - x_0\| &= \max \left\{ |\Phi S - S_0|_0, |\Phi I - I_0|_0, |\Phi R - R_0|_0 \right\} \\
 &\leq \frac{Mr}{1-r}.
 \end{aligned}$$

Next, we prove that the mapping Φ is a contraction mapping of the X^* . In fact, for $\forall x = (S_1, I_1, R_1), y = (S_2, I_2, R_2) \in$

X^* and

$$\|x - y\| = \max\{|S_1 - S_2|_0, |I_1 - I_2|_0, |R_1 - R_2|_0\},$$

we have

$$\begin{aligned}
 &|\Phi S_1 - \Phi S_2|_0 \\
 &= \max_{0 \leq n < \omega} |\Phi S_1(n) - \Phi S_2(n)| \\
 &= \max_{0 \leq n < \omega} \left| \sum_{s=n}^{n+\omega-1} H_1 \prod_{s \leq n_k < n+\omega} (1 + b_{1k}) \right. \\
 &\quad \times \left. \left[\frac{\tilde{\beta}S_2(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} - \frac{\tilde{\beta}S_1(s)I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right] \right| \\
 &\leq \max_{0 \leq n < \omega} \sum_{s=n}^{n+\omega-1} \frac{\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_1) \prod_{s \leq n_k < n+\omega} (1 + b_{1k})}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_1) \right|} \\
 &\quad \times \left| \frac{\tilde{\beta}I(s - \tau(s))}{1 + \alpha I(s - \tau(s))} \right| |S_2(s) - S_1(s)| \\
 &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_1) \prod_{0 \leq n_k < \omega} (1 + b_{1k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{1k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_1) \right|} \alpha \\
 &\quad \times \max_{0 \leq n < \omega} |S_1(n) - S_2(n)| \\
 &= r_1 |S_1 - S_2|_0 \\
 &\leq r_1 \|x - y\| \\
 &\leq L \|x - y\|, \\
 &|\Phi I_1 - \Phi I_2|_0 \\
 &= \max_{0 \leq n < \omega} |\Phi I_1(n) - \Phi I_2(n)| \\
 &= \max_{0 \leq n < \omega} \left| \sum_{s=n}^{n+\omega-1} H_2 \prod_{s \leq n_k < n+\omega} (1 + b_{2k}) \right. \\
 &\quad \times \left. \left[\frac{\tilde{\beta}S(s)I_1(s - \tau(s))}{1 + \alpha I_1(s - \tau(s))} - \frac{\tilde{\beta}S(s)I_2(s - \tau(s))}{1 + \alpha I_2(s - \tau(s))} \right] \right| \\
 &\leq \max_{0 \leq n < \omega} \sum_{s=n}^{n+\omega-1} \left(\prod_{s+1 \leq u < n+\omega} (1 - \tilde{\mu}_2 - \tilde{\gamma}) \right. \\
 &\quad \times \prod_{s \leq n_k < n+\omega} (1 + b_{2k}) \beta \left. \right) \left(\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \right. \right. \\
 &\quad \times \left. \left. \prod_{0 \leq u < \omega} (1 - \tilde{\mu}_2 - \tilde{\gamma}) \right| \right)^{-\frac{1}{2}} \\
 &\quad \times |S(s)| |I_1(s - \tau(s)) - I_2(s - \tau(s))| \\
 &\leq \frac{\omega \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \gamma) \prod_{0 \leq n_k < \omega} (1 + b_{2k}^+) \beta}{\left| 1 - \prod_{0 \leq n_k < \omega} (1 + b_{2k}) \prod_{n \neq n_k, 0 \leq n < \omega} (1 - \mu_2 - \delta) \right|} \\
 &\quad \times \frac{M}{1-r} |I_1 - I_2|_0 \\
 &\leq r_4 \|x - y\| \\
 &\leq L \|x - y\|, \\
 &|\Phi R_1 - \Phi R_2|_0 \\
 &= \max_{0 \leq n < \omega} |\Phi R_1(n) - \Phi R_2(n)| = 0 \leq L \|x - y\|.
 \end{aligned}$$

Then

$$\begin{aligned} & \|\Phi x - \Phi y\| \\ &= \max\{|\Phi S_1 - \Phi S_2|_0, |\Phi I_1 - \Phi I_2|_0, |\Phi R_1 - \Phi R_2|_0\} \\ &\leq L\|x - y\|. \end{aligned}$$

Notice that $L < 1$, it is clear that Φ is a contraction. Therefore, the mapping Φ possesses a unique fixed point $x^* \in X^*$ such that $\Phi x^* = x^*$, which implies (1) has a unique ω -periodic solution. This completes the proof. ■

III. AN EXAMPLE

In this section, we give an example to illustrate that our results are feasible. Consider the following discrete-time SIR epidemic model with time delays and impulses:

$$\left. \begin{aligned} & \left. \begin{aligned} & S(n+1) - S(n) \\ &= 0.05 - 0.06S(n) \\ &\quad - \frac{0.02S(n)I(n - \sin(\frac{n\pi}{4}))}{1 + I(n - \sin(\frac{n\pi}{4}))}, \\ & I(n+1) - I(n) \\ &= \frac{0.02S(n)I(n - \sin(\frac{n\pi}{4}))}{1 + I(n - \sin(\frac{n\pi}{4}))} \\ &\quad - (0.06 + 0.02)I(n), \\ & R(n+1) - R(n) \\ &= 0.02I(n) - 0.06R(n), \end{aligned} \right\} n = 0, 1, \dots, n \neq n_k, \\ & \left. \begin{aligned} & S(n_k+1) - S(n_k) = 0.1S(n_k), \\ & I(n_k+1) - I(n_k) = 0.1I(n_k), \\ & R(n_k+1) - R(n_k) = 0.1R(n_k), \end{aligned} \right\} k = 1, 2, \dots, \end{aligned} \right\} \quad (21)$$

where $n_k = 4k + 1$.

Noticing that $\omega = 8, \mu_1 = \mu_2 = \mu_3 = 0.06, \gamma = 0.02, B = 0.05, \beta = 0.02, \alpha = 1$, it is easy to check that $\prod_{0 \leq n_k < 8} (1 + b_{1k}) = \prod_{0 \leq n_k < 8} (1 + b_{2k}) = \prod_{0 \leq n_k < 8} (1 + b_{3k}) = 1.21$ and $r_1 = 0.8082, r_2 = 0.4408, r_3 = 0.8082$, therefore $r = \max\{r_1, r_2, r_3\} = 0.8082$. At the moment $r_4 = 0.8906$, then $0 < L = \max\{r, r_4\} = 0.8906 < 1$. Hence, $(H_1) - H_3$ are all satisfied. By Theorem 1, (21) has a unique 8-periodic solution in the region $X^* = \{x | x \in X, \|x - x_0\| \leq 4.6435\}$.

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