

# The Application of HLLC Numerical Solver to the Reduced Multiphase Model

Fatma Ghangir, Andrzej F. Nowakowski, Franck C. G. A. Nicolleau, and Thomas M. Michelitsch

**Abstract**—The performance of high-resolution schemes is investigated for unsteady, inviscid and compressible multiphase flows. An Eulerian diffuse interface approach has been chosen for the simulation of multicomponent flow problems. The reduced five-equation and seven equation models are used with HLL and HLLC approximation. The authors demonstrated the advantages and disadvantages of both seven equations and five equations models studying their performance with HLL and HLLC algorithms on simple test case. The seven equation model is based on two pressure, two velocity concept of Baer–Nunziato [10], while five equation model is based on the mixture velocity and pressure. The numerical evaluations of two variants of Riemann solvers have been conducted for the classical one-dimensional air-water shock tube and compared with analytical solution for error analysis.

**Keywords**—Multiphase flow, gas-liquid flow, Godunov schemes, Riemann solvers, HLL scheme, HLLC scheme.

## I. INTRODUCTION

THE numerical simulation of multiphase or multi-component flows is a challenging research topic with various key applications. One can find such flows in the natural world with popular examples such as raindrops in air and gas bubbles in water. The industrial examples include: bubble columns, reactors, cooling circuits of power plants, carburant injection or spraying of paint. The flow pattern is complex and diverse. Therefore the analysis of multiphase flows can be approached using various levels of complexity. These levels range from the application of empirical correlations to implementation of a complete multidimensional model. For such a model the physical description is based on separate conservation laws for each phase. The presence of a free surface in the flow and the dynamical interaction of two or more fluids in their mutual

interfaces significantly complicate the problem. To deal with problems of practical interests, the two-phase model can be time-averaged [6] or both time and volume averaged to obtain more tractable model. The averaging process removes the interfacial details, but introduces the need for many new closure relationships [7]. As a result, the right hand side terms of balance equations are complemented by additional correlations. Introduction of new closure relationships makes mathematical models for two-phase flow highly phenomenological in nature. The general set of conservation equations after averaging procedures contains two different velocities and pressures for each phase and volume fraction equation.

The averaged flow equations can take several different forms. In this study the five [1] and seven [14] equations model are used. These models were chosen as they can be applied to the situations where two fluids are separated by interface or for the cases where the dispersed and the continuous phase are considered. They do not describe interfaces as sharp (discontinuous) functions but as mathematically continuous change where the transition from one to other medium happens relatively smooth. Numerically this is realized by creation of the artificial mixture zone at the interface. The models can be employed with various equations of state.

The next step is to numerically solve these models. Because the structure of the considered compressible approaches is similar the consistent numerical schemes can be adopted. The common feature of the numerical models discussed further in the subsequent section is that they are of hyperbolic character and they produce correct results even when only one fluid is present. The paper presents the results of the implementation of two numerical schemes for the compressible multiphase problem: the Harten-Lax-van Leer Riemann solver (HLL) [17] and, for the first time, the HLLC Riemann solver (where the last letter “C” means contact discontinuity) [18]. These are Godunov algorithms which present the advantage that they can deal with continuous as well as discontinuous flows. The schemes have been implemented for the one-dimensional test problem of air-water shock tube. For this problem the performance and accuracy of the schemes were investigated by comparing to the analytically obtained solution.

The paper is organized as follows: In the first section, the multiphase seven-equations model and the reduced five-equations model are introduced; In the second section, the numerical solution is discussed and a presentation of HLL and HLLC solvers is made; In the third section, the results of

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numerical simulations for each multiphase model using different solvers are presented; Finally, conclusions are derived based on comparison studies.

## II. PROBLEM STATEMENT

### A. The Multiphase Model

Saurel & Abgrall (1999) [14] proposed a seven equations model. This model consists of seven differential equations for the case when two fluids are concerned and one dimensional flow is taken into account. These equations represent the void fraction evolution and mass, momentum and energy equations applied for individual phase. The main features of this multiphase model are: it is an unconditional hyperbolic model; it uses a modified Godunov scheme with accurate treatment of the non conservative terms; and finally, it is based on relaxation processes for both the pressure and velocity at the interface. In general form it depends on infinite relaxation coefficients. This model is able to capture the interface between fluids as well as is capable of generating and evolution of physical interfaces between phases.

The multiphase model for one-dimensional flow with two phases is given as:

$$\begin{aligned}
 \frac{\partial \alpha_g}{\partial t} + v_i \frac{\partial \alpha_g}{\partial x} &= \mu(P_g - P_l) \\
 \frac{\partial \alpha_g \rho_g}{\partial t} + \frac{\partial \alpha_g \rho_g u_g}{\partial x} &= 0 \\
 \frac{\partial \alpha_g \rho_g u_g}{\partial t} + \frac{\partial (\alpha_g \rho_g u_g^2 + \alpha_g P_g)}{\partial x} &= P_l \frac{\partial \alpha_g}{\partial x} + \lambda(u_l - u_g) \\
 \frac{\partial \alpha_g \rho_g E_g}{\partial t} + \frac{\partial u_g (\alpha_g \rho_g E_g + \alpha_g P_g)}{\partial x} &= P_l v_i \frac{\partial \alpha_g}{\partial x} \\
 &\quad + \lambda v_i (u_l - u_g) + \mu P_l (P_g - P_l) \\
 \frac{\partial \alpha_l \rho_l}{\partial t} + \frac{\partial \alpha_l \rho_l u_l}{\partial x} &= 0 \\
 \frac{\partial \alpha_l \rho_l u_l}{\partial t} + \frac{\partial (\alpha_l \rho_l u_l^2 + \alpha_l P_l)}{\partial x} &= -P_l \frac{\partial \alpha_g}{\partial x} - \lambda(u_l - u_g) \\
 \frac{\partial \alpha_l \rho_l E_l}{\partial t} + \frac{\partial u_l (\alpha_l \rho_l E_l + \alpha_l P_l)}{\partial x} &= -P_l v_i \frac{\partial \alpha_g}{\partial x} \\
 &\quad - \lambda v_i (u_l - u_g) - \mu P_l (P_g - P_l)
 \end{aligned} \tag{1}$$

### B. The Reduced Multiphase Model

Allaire (2000)[1] proposed a five-equation model to deal with a compressible multiphase flow. This model [1] is based on Eulerian method but consists of five equations to consider two fluids. Allaire's model is composed of five equations: the first equation represents the void fraction evolution; the second and third equations express the continuity equation for each fluid; and both of the fluids has the same momentum and energy equations as follows:

$$\begin{aligned}
 \frac{\partial \alpha_g}{\partial t} + u \frac{\partial \alpha_g}{\partial x} &= 0 \\
 \frac{\partial \rho_g \alpha_g}{\partial t} + \frac{\partial \rho_g \alpha_g u}{\partial x} &= 0 \\
 \frac{\partial \rho_l \alpha_l}{\partial t} + \frac{\partial \rho_l \alpha_l u}{\partial x} &= 0 \\
 \frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} &= 0 \\
 \frac{\partial \rho E}{\partial t} + \frac{\partial (u(\rho E + p))}{\partial x} &= 0
 \end{aligned} \tag{2}$$

The seven equation model is based on two pressure, two velocity concepts while five equation model is based on the mixture velocity and pressure.

## III. NUMERICAL SOLUTION

The numerical solution of both the multiphase model (seven equations) and the reduced model (five equations) can be achieved by consequence stages using split Strangé methods as follows:

$$U_i^{n+1} = L_s^{\Delta t} L_h^{\Delta t} U_i^n \tag{3}$$

where,  $L_h^{\Delta t}$  is the hyperbolic operator;  $L_s^{\Delta t}$  is the operator including the source terms and the relaxation processes for the multiphase model, the velocity and pressure relaxation processes are fulfilled instantaneously when the value of  $\lambda$  and  $\mu$  are taken as infinite.

### A. The Hyperbolic Operator

The hyperbolic operator is investigated first which can cause the main numerical problems. The hyperbolic part of the multiphase flow model can be expressed as:

$$\begin{aligned}
 \frac{\partial \alpha_g}{\partial t} + v_i \frac{\partial \alpha_g}{\partial x} &= 0, \\
 \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} &= H \frac{\partial \alpha_g}{\partial x}
 \end{aligned} \tag{4}$$

where

$$U = \begin{bmatrix} \alpha_g \rho_g \\ \alpha_g \rho_g u_g \\ \alpha_g \rho_g E_g \\ \alpha_l \rho_l \\ \alpha_l \rho_l u_l \\ \alpha_l \rho_l E_l \end{bmatrix}, F = \begin{bmatrix} \alpha_g \rho_g u_g \\ \alpha_g \rho_g u_g^2 + \alpha_g P_g \\ u_g (\alpha_g \rho_g E_g + \alpha_g P_g) \\ \alpha_l \rho_l u_l \\ \alpha_l \rho_l u_l^2 + \alpha_l P_l \\ u_l (\alpha_l \rho_l E_l + \alpha_l P_l) \end{bmatrix}, H = \begin{bmatrix} 0 \\ P_l \\ P_l v_i \\ 0 \\ -P_l \\ -P_l v_i \end{bmatrix} \tag{5}$$

Godunov numerical scheme with second order accuracy in time and space is used to discretise the conservative vector and it can be written as:

$$\begin{aligned}
 U_i^{n+1} &= U_i^n - \\
 \frac{\Delta t}{\Delta x} &[F(U^*(\bar{U}_{i+1/2}^-, \bar{U}_{i+1/2}^+)) - F(U^*(\bar{U}_{i-1/2}^-, \bar{U}_{i-1/2}^+))] \\
 &+ \Delta t H(U_i^{n+1/2}) \Delta
 \end{aligned} \tag{6}$$

where,  $\Delta$  represents the discretisation of the space derivative of the void fraction  $\partial \alpha_g / \partial x$ . Both models need the equation of state to calculate all independent variables.

In order to obtain the flux vector  $F(U^*(\bar{U}_{i+1/2}^-, \bar{U}_{i+1/2}^+))$ , HLL (Harten, Lax and van Leer) and HLLC approximate Riemann problems are solved to generate Godunov's scheme.

### 1. The HLL Approximate Riemann Solver

The HLL approximate Riemann solver is defined as:

$$F_{HLL} = \frac{S^+ F_L - S^- F_R + S^+ S^- (U_R - U_L)}{S^+ - S^-} \tag{7}$$

The wave speed can be estimated as:

$$\begin{aligned} S^+ &= \max\{0, \lambda_L^+, \lambda_R^+\}; \\ S^- &= \max\{0, \lambda_L^-, \lambda_R^-\} \end{aligned} \quad (8)$$

According to HLL Riemann solver, the discretisation of the space variation void fraction  $\Delta$  can be written as:

$$\Delta = \frac{1}{\Delta x} \left( \frac{S_{1+1/2}^+ \alpha_{i+1/2,-}^{n+1/2} - S_{1+1/2}^- \alpha_{i+1/2,+}^{n+1/2}}{S_{1+1/2}^+ - S_{1+1/2}^-} - \frac{S_{1-1/2}^+ \alpha_{i-1/2,-}^{n+1/2} - S_{1-1/2}^- \alpha_{i-1/2,+}^{n+1/2}}{S_{1-1/2}^+ - S_{1-1/2}^-} \right) \quad (9)$$

The second order variation of the void fraction with time and space can be determined as:

$$\alpha_i^{n+1} = \alpha_i^n - \frac{\Delta x}{\Delta t} \left[ \frac{u_i^{n+1/2} \frac{S_{1+1/2}^+ \alpha_{i+1/2,-}^{n+1/2} - S_{1+1/2}^- \alpha_{i+1/2,+}^{n+1/2} + S_{1+1/2}^+ S_{1+1/2}^- (\alpha_{i+1/2,-}^{n+1/2} - \alpha_{i+1/2,+}^{n+1/2})}{S_{1+1/2}^+ - S_{1+1/2}^-}}{S_{1+1/2}^+ - S_{1+1/2}^-} - u_i^n \frac{S_{1-1/2}^+ \alpha_{i-1/2,-}^{n+1/2} - S_{1-1/2}^- \alpha_{i-1/2,+}^{n+1/2} + S_{1-1/2}^+ S_{1-1/2}^- (\alpha_{i-1/2,-}^{n+1/2} - \alpha_{i-1/2,+}^{n+1/2})}{S_{1-1/2}^+ - S_{1-1/2}^-} \right] \quad (10)$$

The stability of this method is assured using CFL number (Courant number) and depends on the maximum wave speed. The hyperbolic part of the reduced model can be expressed as:

$$\begin{aligned} \frac{\partial \alpha_g}{\partial t} + v_i \frac{\partial \alpha_g}{\partial x} &= 0, \\ \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} &= 0 \end{aligned} \quad (11)$$

where,  $U = \begin{bmatrix} \alpha_g \rho_g \\ \alpha_l \rho_l \\ \rho u \\ \rho E \end{bmatrix}$  and  $F = \begin{bmatrix} \alpha_g \rho_g u_g \\ \alpha_l \rho_l u_l \\ \rho u^2 + p \\ u(\rho E + p) \end{bmatrix}$  (12)

The conservative vector of the reduced model can be obtained using HLL approximate Riemann solver with second order accuracy as follows:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} [F(U^*(\bar{U}_{i+1/2}^-, \bar{U}_{i+1/2}^+)) - F(U^*(\bar{U}_{i-1/2}^-, \bar{U}_{i-1/2}^+))] \quad (13)$$

The HLL flux vector can be obtained using equation (7).

In terms of the evolution of the void fraction, it can be attained using equation (10).

## 2. The HLLC Approximate Riemann Solver

This solver is proposed firstly by Toro & Roe [18] and it is an adaption of HLL solver. A middle wave speed  $S^*$  is added to the fastest and slowest wave speeds which can produce more information at the interface region.

The flux vector using HLLC scheme can be given as:

$$F_{HLLC} = \begin{cases} F_L & \text{If } 0 \leq S_L \\ F_L^* = F_L + S_L(U_L^* - U_L) & \text{If } S_L \leq 0 \leq S^* \\ F_R^* = F_R + S_R(U_R^* - U_R) & \text{If } S^* \leq 0 \leq S_R \\ F_R & \text{If } 0 \geq S_R \end{cases} \quad (14)$$

where,  $U_L^*$  and  $U_R^*$  for the multiphase model can be obtained as follows [16]:

$$U_K^* = \begin{bmatrix} \alpha_g \rho_g \frac{S_K - u_g^K}{S_K - S^*} \\ \alpha_g \rho_g \frac{S_K - u_g^K}{S_K - S^*} S^* \\ \alpha_g \rho_g \frac{S_K - u_g^K}{S_K - S^*} \\ \left[ E_g^K + (S^* - u_g^K) \left[ S^* + \frac{P_g^K}{\rho_g^K (S_K - u_g^K)} \right] \right] \\ \alpha_l \rho_l \frac{S_K - u_l^K}{S_K - S^*} \\ \alpha_l \rho_l \frac{S_K - u_l^K}{S_K - S^*} S^* \\ \alpha_l \rho_l \frac{S_K - u_l^K}{S_K - S^*} \\ \left[ E_l^K + (S^* - u_l^K) \left[ S^* + \frac{P_l^K}{\rho_l^K (S_K - u_l^K)} \right] \right] \end{bmatrix} \quad (15)$$

For  $K=L$  and  $K=R$ . The middle wave speed  $S^*$  for the multiphase model can be attained using this formula [19]:

$$S^* = \frac{P_m^R - P_m^L + \rho_m^L u_m^L (S_L - u_m^L) - \rho_m^R u_m^R (S_R - u_m^R)}{\rho_m^L (S_L - u_m^L) - \rho_m^R (S_R - u_m^R)} \quad (16)$$

For,

$$\begin{aligned} P_m^K &= \alpha_g^K p_g^K + \alpha_l^K p_l^K \\ \rho_m^K &= \alpha_g^K \rho_g^K + \alpha_l^K \rho_l^K \\ u_m^K &= \frac{\alpha_g^K \rho_g^K u_g^K + \alpha_l^K \rho_l^K u_l^K}{\rho_m^K} \end{aligned} \quad (17)$$

The conservative vector at the middle region for the reduced model can be written as:

$$U_K^* = \begin{bmatrix} \alpha_g \rho_g \frac{S_K - u_K}{S_K - S^*} \\ \alpha_l \rho_l \frac{S_K - u_K}{S_K - S^*} \\ \rho_K \frac{S_K - u_K}{S_K - S^*} S^* \\ \rho_K \frac{S_K - u_K}{S_K - S^*} \\ \left[ E_K + (S^* - u_K) \left[ S^* + \frac{P_K}{\rho_K (S_K - u_K)} \right] \right] \end{bmatrix} \quad (18)$$

The HLLC flux vector is applied on equations (6) and (13) for the multiphase model and the reduced model respectively to obtain the HLLC conservative vector. In multiphase flow model, the discretisation of the space derivative of the void fraction  $\partial \alpha_g / \partial x$  using HLLC scheme can be given as [19]:

$$\Delta = \frac{1}{\Delta x} (\alpha_{HLLC}^{j+1/2} - \alpha_{HLLC}^{j-1/2}) \quad (19)$$

where,

$$\alpha_{HLLC} = \begin{cases} u_L \alpha_L & \text{If } 0 \leq S_L \\ u_L \alpha_L + S_L (\alpha_L^* - \alpha_L) & \text{If } S_L \leq 0 \leq S^* \\ u_R \alpha_R + S_R (\alpha_R^* - \alpha_R) & \text{If } S^* \leq 0 \leq S_R \\ u_R \alpha_R & \text{If } 0 \geq S_R \end{cases} \quad (20)$$

The void fraction evolution is described using HLLC scheme as follows:

$$\alpha_i^{n+1} = \alpha_i^n - \frac{\Delta t}{\Delta x} (\alpha_{HLLC}^{j+1/2} - \alpha_{HLLC}^{j-1/2}) \quad (21)$$

**B. Source Terms**

There are different types of source terms that can be considered such as the gravity and the existence of the area variation along the flow. The gravity roles the main source term in many problems for example the water faucet and the separation problems. This source term can be written for each phase as:

$$\rho_i \alpha_i g \Delta t \tag{22}$$

In order to closure the numerical simulation for the multiphase model the relaxation processes for both the velocity and the pressure are needed using infinite relaxation parameters.

**IV. RESULTS**

*Uniform Water-Air Shock Tube*

There are many standard tests which can be used as a reference to verify the performance of the simulated program, one of these tests is the water-air shock tube.

The water-air shock tube is considered as a tube of 1m length as shown in Fig. 1 filled with nearly pure water in the left hand side at high pressure and nearly pure air in the right hand side with low pressure. There is a strong pressure difference between the sides of the water-air shock tube.

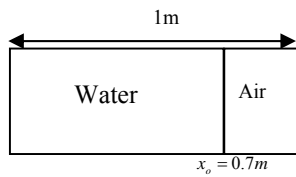


Fig. 1 Schematic diagram of the water-air shock tube

From the numerical point of view, it is difficult to consider the void fraction value as zero. Consequently, the void fraction is assumed to be  $\epsilon = 10^{-8}$  which is a negligible value. Table I demonstrates the initial values of the water-air shock tube as:

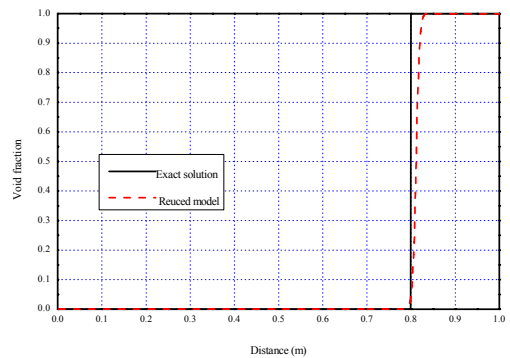
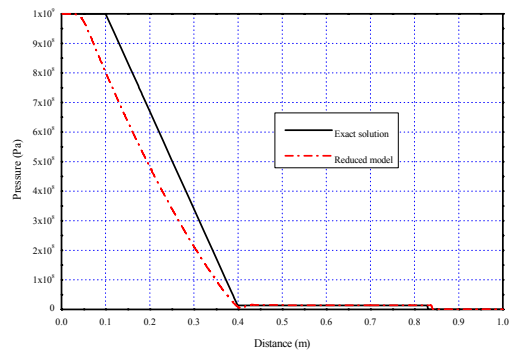
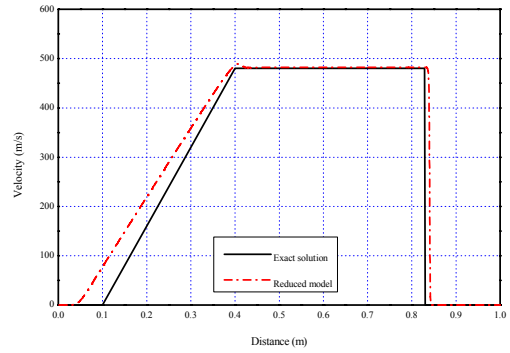
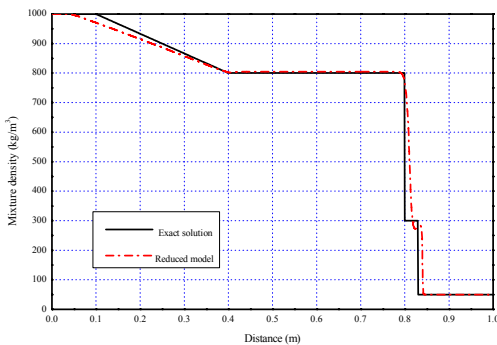
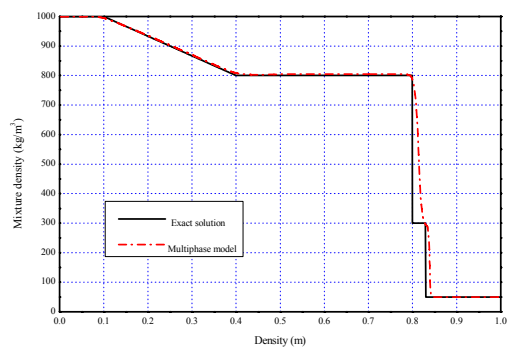


Fig. 2 (a) HLL scheme results (5eq model)



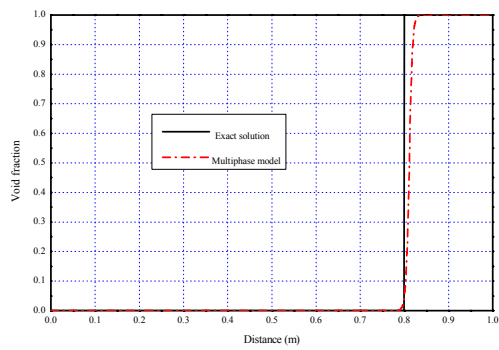
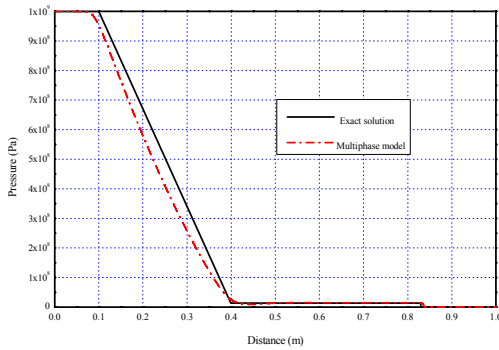
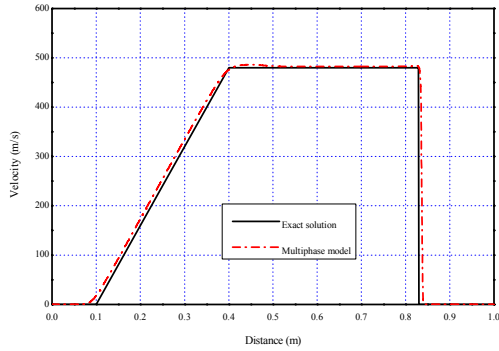


Fig. 2 (b) HLL scheme results (7eq model)

TABLE I  
 INITIAL VALUE OF WATER-AIR SHOCK TUBE

Physical property	$x \leq 0.7$ m		$x > 0.7$ m	
	Air	Water	Air	Water
Density, $\rho$	50	1000	50	1000
Velocity, $u$	0	0	0	0
Pressure, $P$	$10^9$	$10^9$	$10^5$	$10^5$
Void fraction, $\alpha$	$\varepsilon$	$1 - \varepsilon$	$1 - \varepsilon$	$\varepsilon$

Stiffened gas EOS is used for each fluid as:

$$P = (\gamma - 1) \rho e - \gamma P_\infty \quad (23)$$

The parameters of this equation for air and water are tabulated in Table II:

TABLE II  
 PARAMETERS OF THE AIR AND LIQUID FOR STIFFENED GAS EOS

	$P_\infty$	$\gamma$
Air	0	4.1
Liquid	$6 \times 10^8$	4.4

Using HLL scheme the computations have been done with  $CFL$  as 0.6 and the space mesh is divided into 100, 1000 cells then the number of cells are increased to 5000 cells.

The results of both the multiphase model and the reduced model at the water-air shock tube are compared with the analytical solution as shown in Fig. 2, these results are obtained at  $N=1000$  cells and time equal to  $229\mu s$ .

The absolute and relative errors are computed using equation (24) and equation (25) respectively. Different number of the mesh cells (100, 1000 and 5000) is taken into account to compare both the 7-eq and 5-eq models with the analytical solution. Fig. 3(a) indicates the absolute error for  $\alpha = 0$  of the 7-equations and the 5-equations models, the comparison has been done at time equal to  $229\mu s$ .

$$relative\ error = \frac{Numerical\ solution - Analytical\ solution}{Analytical\ solution} \quad (24)$$

$$absolute\ error = |Numerical\ solution - Analytical\ solution| \quad (25)$$

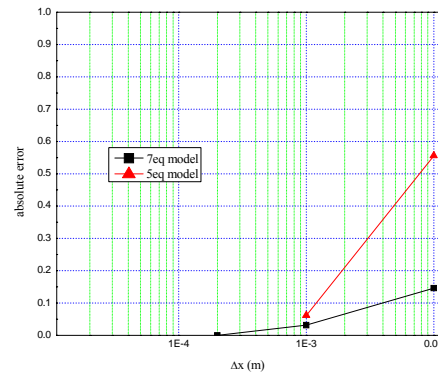


Fig. 3(a) The absolute error of the multiphase and the reduced models for the void fraction

Fig. 3 (b) shows the relative error for the mixture density of the 7-equations and the 5-equations models, different increment spaces have been examined and the computations have been obtained at time equal to  $229\mu s$ .

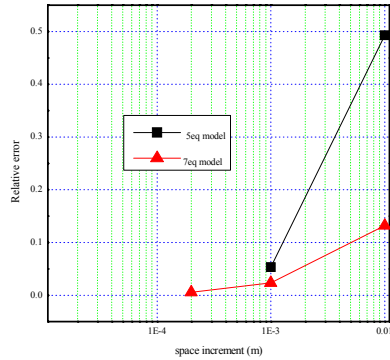


Fig. 3 (b) The relative error of the multiphase and the reduced models for the mixture density

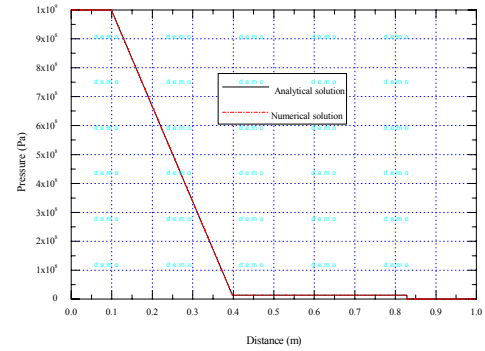


Fig. 4 HLLC scheme results (5eq model)

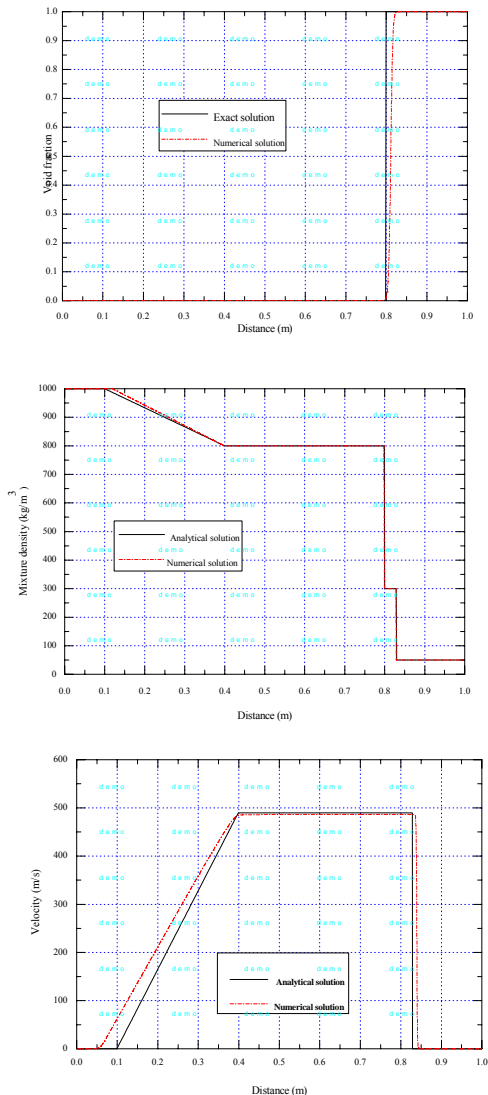
The results of the HLLC scheme for both the multiphase model and the reduced model has been produced with  $CFL$  equal to 0.6, the number of the space increments is equal to 1000 and the calculations are achieved at time is equal to  $229\mu s$  as shown in Fig. 4.

## V. CONCLUSION

Although the reduced model (5eq model) is simple to imply the compressible multiphase flow from numerical point of view, it can be concluded that the multiphase model (7eq model) is more accurate than the reduced model as shown previously into Fig. 3(a) and Fig. 3(b). It is obvious that HLLC scheme is more accurate than HLL scheme. On the other hand, HLLC scheme last long time to produce the results than HLL scheme.

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