Monotonicity of Dependence Concepts from Independent Random Vector into Dependent Random Vector

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Abstract—When the failure function is monotone, some monotonic reliability methods are used to gratefully simplify and facilitate the reliability computations. However, these methods often work in a transformed iso-probabilistic space. To this end, a monotonic simulator or transformation is needed in order that the transformed failure function is still monotone. This note proves at first that the output distribution of failure function is invariant under the transformation. And then it presents some conditions under which the transformed function is still monotone in the newly obtained space. These concern the copulas and the dependence concepts. In many engineering applications, the Gaussian copulas are often used to approximate the real word copulas while the available information on the random variables is limited to the set of marginal distributions and the covariances. So this note catches an importance on the conditional monotonicity of the often used transformation from an independent random vector into a dependent random vector with Gaussian copulas.

Keywords—Monotonic / Rosenblatt / Nataf transformation; dependence concepts; completely positive matrices; Gaussian copulas.

I. INTRODUCTION

IN many cases, stochastic models are monotone in the uncertain model inputs [1]. For example, the failure function in many mechanical or physical systems happens to be monotone in some uncertain parameters, at least in the neighbourhood of the failure event [2, 3]. The monotonicity is often used to precisely compute the failure probability or extreme quantile.

In what follows, we will use capital letters to represent random variables, lowercase letters to represent their values and the letters with underline to represent their random variable vectors or value vectors. We assume that the one dimensional margin is continuous and strictly increasing (so is the random variable). Let the failure of a structure in a system be quantified by the failure function $z = G(\underline{x})$, where $\underline{x} = (x_1, x_2, ..., x_p)^T \in R^p$ is a vector of uncertain parameters. We assume that its joint density function $f_{\underline{X}}(\underline{x})$ is known (or its one dimensional margins and covariance matrix are available). For each $\underline{x}$ in D (the domain of $G(\bullet)$, or physical space), we can uniformly compute the function

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value $G(\underline{x})$ via a numerical code. If $\underline{x}_1 = (x_{11}, x_{12}, ..., x_{1p})^T$ and $\underline{x}_2 = (x_{21}, x_{22}, ..., x_{2p})^T$ with $\underline{x}_1, \underline{x}_2 \in D$, we say that $\underline{x}_1$ is less than or equal to $\underline{x}_2$ for a partial order over D (and write $\underline{x}_1 \leq \underline{x}_2$) if $x_{1i} \leq x_{2i}$ for $1 \leq i \leq p$. Specifically, we assume that $G(\bullet)$ is continuous and strictly monotone for the partial order over D. Under this assumption, $G(\bullet)$ is a strictly monotonic function with respect to each parameter x_i while all others are fixed. Since $Z = G(\underline{X})$ is also a continuous random variable depending on $\underline{X}$, the failure probability can be defined as $P_f = P(Z < 0)$ and the τ-quantile of Z can be defined as $F_Z^{-1}(\tau) = \inf\{z : F_Z(z) \geq \tau\}$.

In the presence of the monotonicity of failure function for a partial order over its domain, some methods can gratefully simplify the computation of failure probability or quantile estimation. On the failure probability computation, the article [4] presented an application of Random Set Theory to calculate the upper and lower bounds on the probability of predicted rock mass response. For this method, the failure function domain D is divided into some progressive refines boxes (called local elements A_i). Then the failure function should be evaluated at each vertex and the integral of the joint pdf $f_{\underline{X}}(\underline{x})$ should be calculated over each A_i. The method has its own shortcomings. On one hand, there are a great number of vertexes to be evaluated and a lot of integral to be calculated when the dimension is higher. On the other hand, the integral computation is difficult when the joint density function is irregular, especially in the dependence case. Fortunately, the articles [2, 3, 5] presented some methods to facilitate the failure probability computation, in which the failure probability can be transformed into a multi-dimensional volume delimited by the transformed failure surface. The dichotomic methods can be used to precisely estimate the interesting volume when the transformed failure function is still monotone. As to quantiles estimation, the article [6] shows a simulation-based quantile estimator which can be guaranteed to be 100% with a finite sample size.

These methods must work in an iso-probabilistic space (the random variables are i.i.d). The presented articles only introduced the independent case in which the components of the random vector are independent. In this case, the inverse

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iso-probabilistic transformation is monotone and the transformed failure function is still monotone in the newly obtained space. However, it may not be the case when applying a transformation on a dependent $\underline{X}$ into an iso-probabilistic space (such as Rosenblatt or Nataf transformation [7, 8]). In fact, its inverse transformation is monotone under some conditions. Aim of this paper is therefore to point out and discuss the favourable conditions under which the transformation from an independent (uniform or Gaussian) random vector into a dependent random vector is monotone. It concerns the following monotonicity of dependence concepts. The monotonicity can be considered as a characterization of dependence concepts.

The paper proceeds as follows. In section II of this paper, a general framework for the random variable vectors transformation is presented and then the conditions under which the inverse transformation is monotone are discussed. A brief history of dependence concepts is also presented in this section. The section III presents the Rosenblatt transformation and a sufficient condition under which the inverse Rosenblatt transformation is monotone. A sufficient condition on the correlation matrix is deduced when the joint distribution is a standard normal distribution. The Nataf distribution approximation is introduced in the section IV. The next section concerns the orthogonal transformation. We will present some conditions on the correlated normal distributions which the inverse Nataf transformation is monotone. The 6th section introduces copulas and its primary applications in simulation. In fact, this section gives us the condition on the interesting monotonicity of dependence concepts. Finally, the extension and the conclusion are identified in section VII.

II. RANDOM VARIABLE VECTORS TRANSFORMATION

A p-dimensional continuous random variable vector $\underline{X} = (X_1, X_2, ..., X_p)^T$ holds on:

1. known with joint density function (pdf) of $\underline{X}$ in the physical space: $f_{\underline{X}}(\underline{x})$ or know with joint cdf : $F_{\underline{X}}(\underline{x})$;

2. related to an other q-dimensional random variable vector $\underline{Y}$: $\underline{y} = T(\underline{x})$ and $\underline{Y} = (Y_1, Y_2, ..., Y_q)^T$ with the joint pdf $f_{\underline{Y}}(\underline{y})$ in a transformed space;

3. for $p = q$, having unique inverse: $\underline{x} = T^{-1}(\underline{y})$.

If $p = q$, then the relation between $\underline{X}$ and $\underline{Y}$ can be obtained by the transformation [9]:

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}(\underline{x})|J|, \qquad (1)$$

where J is the Jacobean transformation matrix with $J_{ij} = \partial x_i / \partial y_j$ and $|J| = |\det J| \neq 0$. The Jacobean matrix plays an important role in the transformation. However, the transformation may not be unique and J is not a square

matrix when $p \neq q$. In fact, if $p < q$, let augmented vector $\underline{x}^* = (\underline{x}^T; y_{p+1} \cdots y_q)^T$, we can then obtain an augmented transformation $\underline{y} = T^*(\underline{x}^*)$ having unique inverse $\underline{x} = T^{*-1}(\underline{y})$. The relation between $\underline{X}$ and $\underline{Y}$ can be obtained by:

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}(\underline{x})f_{Y_{p+1} \cdots Y_q | \underline{X}}(y_{p+1} \cdots y_q | \underline{x})|J^*|, \qquad (2)$$

where

$$J^* = \begin{pmatrix} & J \\ 0 & I_{(q-p)\times(q-p)} \end{pmatrix} \qquad (3)$$

is an augmented Jacobean matrix.

We can prove **that $G(\underline{X})$ and $G \circ T^{-1}(\underline{Y})$ have the same distribution**.

Demonstration: Let $G(\underline{x}^*) = G(\underline{x})$. Using the Jacobean variables change for integral, we have:

$$\begin{aligned} P(G \circ T^{-1}(\underline{Y}) \leq z) &= \int_{G \circ T^{-1}(\underline{y}) \leq z} f_{\underline{Y}}(\underline{y}) d\underline{y} \\ &= \int_{G(\underline{x}^*) \leq z} f_{\underline{Y}}(\underline{y}(\underline{x}^*))|(J^*)^{-1}| d\underline{x}^* \\ &= \int_{G(\underline{x}^*) \leq z} f_{\underline{X}}(\underline{x})f_{Y_{p+1} \cdots Y_q | \underline{X}}(y_{p+1} \cdots y_q | \underline{x}) d\underline{x}^* \\ &= \int_{G(\underline{x}) \leq z} f_{\underline{X}}(\underline{x}) \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_{Y_{p+1} \cdots Y_q | \underline{X}}(y_{p+1} \cdots y_q | \underline{x}) dy_{p+1} \cdots dy_q \right\} d\underline{x} \\ &= \int_{G(\underline{x}) \leq z} f_{\underline{X}}(\underline{x}) d\underline{x} = P(G(\underline{X}) \leq z) \end{aligned}$$

$$(4)$$

So the outputs distribution of failure function is invariant via inputs variables transformation.

The monotonicity of transformed failure function is determined by that of the inverse transformation. The monotonic transformation is defined as following:

Definition1: **monotonic transformation** *if* $J_{ij} \geq 0$ for all $1 \leq i, j \leq p$, *then* $\underline{x} = T^{-1}(\underline{y})$ is an increasing function on $\underline{y}$. We can say that the transformation $\underline{x} = T^{-1}(\underline{y})$ is a monotonic transformation, or that the matrix J^{-1} is of monotone kind. Hereafter, $J_{ij} \geq 0$ for all $1 \leq i, j \leq p$ can be denoted by $J \geq 0$. However, if $J \geq 0$, it may not deduce that $\underline{y} = T(\underline{x})$ is a monotonic transformation on $\underline{x}$. Moreover, if $J^{-1} \geq 0$, it can deduce that $\underline{y} = T(\underline{x})$ is monotone on $\underline{x}$. In general, if $J \geq 0$, it can not deduce that $J^{-1} \geq 0$, except J is a diagonal matrix.

This definition means that each x_i is an increasing function on $\underline{y}$. So we have the following proposition.

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Proposition1: If G is monotonic on $\underline{x}$ and $\underline{x} = T^{-1}(\underline{y})$ is monotone on y, then the transformed function $H = G \circ T^{-1}$ is monotone on y. Hence the monotonicity of inverse transformation constrains that of the transformed failure function.

The next subsections present Rosenblatt and Nataf methods for transforming arbitrary random vector into independent standard normal random vector (Gaussian space) or independent uniform[1] random vector (uniform space) and then discuss the conditional monotonicity of the inverse transformation.

Before we begin, it might be useful to give a brief introduction about dependence concepts. In 1966, Lehmann [10] had introduced "Some Concepts of Dependence" involving dependent bivariables. The quadrant dependence and regression dependence had been presented in his paper. For example, when a pair of variables (X, Y) satisfies:

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y) \quad (5)$$

for all x, y, it will be of positively quadrant dependence. Since the following form,

$$E(XY) - E(X)E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty}[F(x,y) - F_X(x)F_Y(y)]dxdy \quad (6)$$

a pair of positively quadrant dependent variables is of nonnegative covariance. The next year, Esary and al. [11] developed the positively quadrant dependence for association of random variables as the following condition: for all strictly increasing function f and g, if:

$$Cov(f(\underline{X}), g(\underline{X})) \geq 0 \quad (7)$$

the random variables $\underline{X}$ will be associated. The association is clearly invariant under all strictly increasing transformation of $\underline{X}$. It is easily proved that the independent random variables are associated. To research the association of correlated normal variables, Ruschendorf [12] introduced in 1981 a monotonic linear transformation from the uncorrelated normal variables into some correlated normal variables. It is proved that the normal variables with completely positive covariance matrix are associated. In 1982, using a different method from Ruschendorf's, Loren D. Pitt [13] proved that all positively correlated normal variables are associated.

III. ROSENBLATT TRANSFORMATION AND ITS CONDITIONAL MONOTONICITY

When the joint distribution function is available, Rosenblatt transformation [7, 8, 14] is often used to transform the dependent random vector into an independent one. Notationally, $\underline{V} = (V_1, V_2, ..., V_p)^T$ will refer to the transformed vector of independent random variables in the uniform space, while $\underline{X} = (X_1, X_2, ..., X_p)^T$ is the original

[1] The components independently follow the uniform distribution.

vector of statistically dependent random variables. Let the joint density function and the joint probability distribution function of $\underline{X}$ be known as $f_{\underline{X}}(\underline{x})$ and $F_{\underline{X}}(\underline{x})$ respectively; and let the marginal density function and the marginal distribution function of X_i be known as $f_i(x_i)$ and $F_i(x_i)$ respectively. The conditional density functions and the conditional probability distributions are available. Let $f_1^c = f_1(x_1)$ and $F_1^c = F_1(x_1)$. For all $i = 2, \cdots, p$, let the following conditional distribution sequence be denoted by

$$f_i^c = f_i^c(x_i) = f_i(x_i \,|\, X_1 = x_1, X_2 = x_2, \,..., \, X_{i-1} = x_{i-1}),$$
$$(8)$$
$$F_i^c = F_i^c(x_i) = F_i(x_i \,|\, X_1 = x_1, X_2 = x_2, \,..., \, X_{i-1} = x_{i-1}).$$
$$(9)$$

Then the independent, random vector $\underline{V}$ in the uniform space can be obtained by the following Rosenblatt transformation:

$$\underline{v} = (v_1, v_2, ..., v_p)^T = (F_1^c(x_1), F_2^c(x_2), \cdots, F_p^c(x_p))^T \quad (11)$$

where $V_1, V_2, ..., V_p$ are uniformly and independently distributed on [0, 1].

Its inverse form is implicit and the Jacobean matrix is difficult to be obtained. For simplicity, the inverse of the Jacobean matrix of this transformation is obtained first. Let $\partial_{ij}^c = \partial F_i^c / \partial x_j$, we obtain:

$$(J^{-1})_{ij} = \frac{\partial v_i}{\partial x_j} = \begin{cases} 0 & if \quad i < j \\ f_i^c & if \quad i = j \\ \partial_{ij}^c = \dfrac{\partial F_i^c}{\partial x_j} & if \quad i > j \end{cases}.$$

$$\text{Or } J^{-1} = \begin{bmatrix} f_1^c & & & & \\ \partial_{21}^c & f_2^c & & O & \\ \partial_{31}^c & \partial_{32}^c & f_3^c & & \\ \vdots & \vdots & \vdots & \ddots & \\ \partial_{p1}^c & \partial_{p2}^c & \partial_{p3}^c & \cdots & f_p^c \end{bmatrix}. \quad (12)$$

For researching the monotonicity of the inverse Rosenblatt transformation, J should be obtained. The computation of J is presented in appendix A. It is complex but it can show the necessary condition to get the monotonic transformation (or to get $J \geq 0$). Hereas a sufficient condition is given to get $J \geq 0$ which concerns the M-Matrices introduced by Ostrowski. The matrix whose all off-diagonal entries are nonpositive and all principal minors are positive is called M-matrices. An important result on M-

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matrix is that each M-matrix is on monotone kind (an inverse-positive matrix). We assume that $f_{\underline{X}}(\underline{x}) > 0$ in the physical space. Then we can obtain $f_i^c > 0$ and all principal minors of J^{-1} are positive. So, if $\partial F_i^c / \partial x_j \leq 0$ for all $i > j$, J^{-1} will be an M-matrix [15, 16] such that $J \geq 0$.

This sufficient condition ($\partial F_i^c / \partial x_j \leq 0$ for all $i > j$) concerns the conditionally increasing in sequence (CIS) of the random vector [17]. A random vector $\underline{X} = (X_1, X_2, ..., X_p)^T$ is said to be CIS if the conditional probabilities

$$P(X_i > x_i \mid X_1 = x_1, X_2 = x_2, ..., X_{i-1} = x_{i-1}), \quad (13)$$

are increasing in $(x_1, x_2, ..., x_{i-1})$ for every $i \in \{2, \cdots, p\}$. It is a beautiful decision whether the inverse Rosenblatt transformation could be used when the joint distribution function is known or available. However, the necessary conditional joint density functions are difficult to obtain except in rare situation. The joint distribution approximation is often used in practice. For example the Nataf models provide a well behaving in some cases [18].

We shall write down the transformation when $F_{\underline{X}}(\underline{x})$ is a p-dimensional standard normal pdf $\varphi_p(\underline{x}, R')$ with correlation matrix $R' = (\rho_{ij}')$ $1 \leq i, j \leq p$. In fact, if $F_{\underline{X}}(\underline{x})$ is not a p-dimensional standard normal distribution, under the assumption of Gaussian copulas, it can be transformed into a p-dimensional standard normal distribution by a monotonic transformation. It concerns the first step of the following Nataf transformation. Since the Pearson correlation coefficient is not invariant but available under this transformation, ρ_{ij}' is used to distinguish the original ρ_{ij}. Let $\Lambda^k = (\rho_{ij}')$ $1 \leq i, j \leq k$, $\Lambda_{ij}^{(k)}$ be the cofactor of ρ_{ij}' in Λ^k and denote $\Lambda^{(k)} = \det(\Lambda^k)$. The sufficient condition ($\partial F_i^c / \partial x_j \leq 0$ for all $i > j$) is equivalent to $\Lambda_{ij}^{(i)} \leq 0$ for all $i > j$. And then we have the following proposition.

Proposition2: If $R' \geq 0$ and if $\Lambda_{ij}^{(i)} \leq 0$ for all $i > j$, the inverse Rosenblatt transformation is monotone.

The proof for the above proposition is presented in Appendix B. The correlation matrix is in practice available and whether $\Lambda_{ij}^{(i)} \leq 0$ *for $i > j$* is easily to be verified.

IV. NATAF TRANSFORMATION

Nataf transformation [7, 8, 19] may be applied to transform correlated random variables into uncorrelated standard normal variables. When the marginal probability distributions and correlation data are available, the non-Gaussian random variables can be approximated by multivariate Gaussian distribution, called Nataf distribution approximation. That is in fact the first step of Nataf transformation. This approximation may exhibit undesirable when the variables are highly non-Gaussian [18]. The second step concerns orthogonal transformation which may transform the correlated standard normal variables into an uncorrelated (or independent) standard normal vector.

Let i-th marginal cdf and pdf of $\underline{X}$ be noted by F_i and f_i respectively. By virtue of Nataf transformation, the original random vector $\underline{X}$ can be transformed into a random variable vector $\underline{Y}$ with p-dimensional standard normal pdf $\varphi_p(\underline{y}, R')$. This transformation is denoted by T_1 which has the unique inverse as the following forms:

$$\underline{y} = T_1(\underline{x}) = \left(\Phi^{-1}[F_1(x_1)], \ \Phi^{-1}[F_2(x_2)], \ ..., \ \Phi^{-1}[F_p(x_p)] \right)^T \quad (14)$$

$$\underline{x} = T_1^{-1}(\underline{y}) = \left(F_1^{-1}[\Phi(y_1)], \ F_2^{-1}[\Phi(y_2)], \ ..., \ F_p^{-1}[\Phi(y_p)] \right)^T \quad (15)$$

The joint pdf of $\underline{X}$ is given by:

$$f_X(\underline{x}) = \frac{1}{|J|} \varphi_p(\underline{y}, \rho), \quad (16)$$

where the Jacobean matrix J is obtained by:

$$J_{ij} = \frac{\partial x_i}{\partial y_j} = \begin{cases} 0 & if \quad i \neq j \\ \dfrac{\varphi(y_j)}{f_i(x_i)} & if \quad i = j \end{cases}, \quad (17)$$

and where the Jacobean $|J|$ is obtained by:

$$|J| = \frac{\varphi(y_1)\varphi(y_2)...\varphi(y_p)}{f_1(x_1)f_2(x_2)...f_p(x_p)}. \quad (18)$$

Clearly the Jacobean matrix of T_1 is a positive diagonal matrix and both T_1 and T_1^{-1} are monotonic transformation.

The random vector $\underline{Y}$, obtained by T_1, has jointly standard Gaussian probability density function $\varphi_p(\underline{y}, R')$, zero means, unit standard deviations and correlation matrix $R' = (\rho_{ij}')$. It should be noted that the correlations of newly obtained variables are different form that of the original. The new correlation matrix is in fact obtained from an advanced method. Let the known $m_i = E(X_i)$, $\sigma_i = \sqrt{Var(X_i)}$ and the known ρ_{ij} denote the original correlation coefficient of bivariables X_i and X_j, the unknown ρ_{ij}' can be obtained by:

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$$\rho_{ij} = E[(\frac{X_i - m_i}{\sigma_i})(\frac{X_j - m_j}{\sigma_j})]$$

$$= \iint (\frac{F_i^{-1}(\Phi(y_i) - m_i)}{\sigma_i})(\frac{F_j^{-1}(\Phi(y_j) - m_j)}{\sigma_j}) \quad (19)$$
$$\cdot \varphi_2(y_i, y_j, \rho'_{ij}) dy_i dy_j$$

Since ρ_{ij} is a strictly increasing function of ρ'_{ij}, this equation can be iteratively solved for ρ'_{ij} [18].

The beauty here is not only its inverse is a monotonic transformation, but also the transformed random vector has Gaussian copula. For a random vector with Gaussian copula, if its components are pairwise independent or uncorrelated, it will be independent. It means that a Gaussian distribution vector is independent if and only if its correlation matrix is a unit matrix. The rest what we will do is the orthogonalization of the newly obtained random vector $\underline{Y}$. It can be achieved by the second step T_2 of Nataf distribution. By T_2, the joint standard normal random vector $\underline{Y}$ can be transformed into a resulting random vector $\underline{U}$ whose correlation matrix is a unit matrix I. It means that $\underline{U}$ has an independent standardised density function $\varphi_p(\underline{u}, I)$. This orthogonalization concerns the decompositions of the covariance matrix in which the non-negative decomposition is interesting in our research.

V. ORTHOGONAL TRANSFORMATION OF JOINT STANDARD NORMAL RANDOM VARIABLES

Considering the presented random vector $\underline{Y}$ with p-dimensional standard normal pdf $\varphi_p(y, R')$, we know that it has zero means and covariance matrix $C_Y = R'$. Now for the uncorrelated standard normal vector $\underline{U}$ with pdf $\varphi_p(\underline{u}, I)$, a linear transformation with a matrix A can be found, such that:

$$\underline{U} = A\underline{Y}. \quad (20)$$

Under this linear transformation with matrix A, the covariance matrix C_Y is also transformed into the uncorrelated vector $\underline{U}$'s covariance matrix $C_U = I$.

$$C_U = E(\underline{U}\underline{U}^T) = E[(A\underline{Y})(A\underline{Y})^T]$$
$$= A(E[\underline{Y}\underline{Y}^T])A^T = AC_Y A^T \quad (21)$$

So we can obtain:

$$C_Y = A^{-1}A^{-T}. \quad (22)$$

The covariance matrix is a positive definite matrix and it keeps the Cholesky decomposition $C_Y = LL^T$. One of the matrixes A may be equal to L^{-1}. We can then get $\underline{U} = L^{-1}\underline{Y}$ or $\underline{Y} = L\underline{U}$. L is also a linear transformation matrix. Considering the relation between $\underline{y}$ and $\underline{u}$, we can obtain the Jacobean matrix of the transformation by:

$$J_{ij} = \partial y_j / \partial u_i = (L)_{ij}. \quad (23)$$

So we have:

$$\varphi_p(\underline{u}, I) = \varphi_p(y, R', \theta)|L|. \quad (24)$$

Clearly, the inverse orthogonal transformation can be monotone while $L \geq 0$. The result depends only on the correlation matrix R'. Let M_{ij}^k denote the minor of R' with rows $1, \cdots, k, i$ and columns $1, \cdots, k, j$ for $k \leq i, j \leq p$. We have the following proposition [20]:

Proposition3: If $R' \geq 0$, then R' has a Cholesky factorization $R' = LL^T$ with $L \geq 0$ if and only if $M_{ij}^k \geq 0$ for all $k \leq i, j \leq p$; and then the inverse Nataf transformation under Cholesky factorization is monotone.

There is another alternative manner to get an orthogonal transformation. Note that a random variable vector $\underline{W}$ with an independent standardised distribution pdf $\varphi_q(\underline{w}, I)$. There is an $p \times q$ matrix B such that:

$$\underline{Y} = B\underline{W}. \quad (25)$$

Then we have:

$$C_Y = E[\underline{Y}\underline{Y}^T] = B(E[\underline{W}\underline{W}^T])B^T = BB^T. \quad (26)$$

We need the following definition concerning completely positive matrix [13]. A real $p \times p$ positive definite matrix C is called completely positive if C can be factored as BB^T for some $p \times q$ non-negative real matrix B for $q < \infty$. The minimum value of q is the CP rank of C, denoted by *cprank(C)*. An obvious fact is that $cprank(C) \geq rank(C)$ for any completely positive C. Let CP_p denote the set of all completely positive $p \times p$ matrices. We have the following result.

Proposition4: If $R' \in CP_p$, then $R' = BB^T$ with a $p \times q$ non-negative real matrix B; and then the inverse Nataf transformation under non-negative factorization is monotone.

Some interesting results concerning the completely positive matrices are worth presenting hereafter.

Proposition5: If $R' \in CP_p$ and $p < 5$, then $R' = BB^T$ with a $p \times p$ non-negative real matrix B, *see* [21]. The monotonicity of dependence concepts is available for the small dimensional (< 5) positive correlated variables.

Proposition6: If $R' = (\rho'_{ij}) \geq 0$ and $2I - R'$ is positive semi-definite, then $R' \in CP_p$, *see* [16]. $2I - R'$ is called comparison matrix of R'.

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Proposition7: If $R' = (\rho'_{ij}) \geq 0$ and $\sum_{j \neq i} \rho'_{ij} \leq 1$ for all j (or i), R' is said to be diagonally dominant, and then $R' \in CP_p$, *see* [22]. The lower correlated variables are favourable.

Conjecture1: If $R' \in CP_p$ and $p \geq 4$, then $cprank(R') \leq int[p^2/4]$, *see* [23].

This conjecture can help us to estimate the maximum dimension of the newly obtained iso-probabilistic space. Unlike Rosenblatt transformation, Nataf transformation may product an augmentation of dimension.

VI. Monotonicity of Random Variables Transformation via Copulas

Copulas are functions that join multivariate distribution functions to their one dimensional marginal distribution functions. They are of interest to us for the reason as a way of studying scale-free measures of dependence [24]. A p-copula (p-dimensional copula) is a function C: $[0, 1]^p \rightarrow [0, 1]$ which satisfies:

1) for every $\underline{s} = (s_1, s_2, ..., s_p)^T$ in $[0,1]^p$, $C(\underline{s}) = 0$ if at least one coordinate of $\underline{s}$ is 0, and $C(\underline{s}) = s_i$ whenever all coordinates of $\underline{s}$ are 1 excepte s_i;

2) C is a positive probability measure.

Sklar's theorem: the joint distribution function F of the continuous random vector $\underline{X} = (X_1, X_2, ..., X_p)^T$ with univariate marginals $F_1, F_2, ..., F_p$ can be expressed, for every $\underline{x} \in IR^p$, by

$$F(\underline{x}) = C(F_1(x_1), F_2(x_2), ..., F_p(x_p)). \tag{27}$$

where the p-copula C is uniquely determined on $RangeF_1 \times RangeF_2 \times ... \times RangeF_p$.

The copula is a class of joint distribution functions of $\underline{X}$ having the same dependence. When a copula function is fixed, the different joint distribution functions of same class can be obtained by different one-dimensional margins. Under assumption that all random variables are continuous, the above result provides a method of constructing copulas from joint distribution functions.

$$C(\underline{s}) = F(F_1^{-1}(s_1), F_2^{-1}(s_2), ..., F_p^{-1}(s_p)). \tag{28}$$

Since the copula is also a distribution function, from standard textbooks, we know that the density function is its p-derivative, if it exists. In particular, a p-copula is said to be absolutely continuous if:

$$C(\underline{s}) = \int_0^{s^1} \cdots \int_0^{s^p} \frac{\partial^p C(\underline{t})}{\partial t_1 \cdots \partial t_p} d\underline{t}. \tag{29}$$

If copula is absolutely continuous, the expression of the copula density c can be defined as:

$$c(\underline{s}) = \frac{\partial^p C(\underline{s})}{\partial s_1 \cdots \partial s_p}. \tag{30}$$

Like the conditional distribution, we can obtain the conditional copula:

$$\begin{aligned}
&C(s_1, \cdots s_i | T_1 = t_1, \cdots, T_j = t_j) \\
&= \frac{\partial^j C(s_1, \cdots s_i, t_1, \cdots t_j)}{\partial t_1 \cdots \partial t_j} / \frac{\partial^j C(t_1, \cdots t_j)}{\partial t^1 \cdots \partial t^j}.
\end{aligned} \tag{31}$$

It is also conditional distribution and can be noted by $C(s_1, \cdots s_i | t_1, \cdots, t_j)$.

One of the primary applications of copulas is in simulation. By virtue of Sklar's theorem, we need only generate a vector $\underline{s}$ of observations of uniform random vector $\underline{S}$ whose joint distribution function is C. This can be realised by generating the independent uniform random variables $v_1, v_2, ..., v_p$ (can be noted by $\underline{v} = (v_1, v_2, ..., v_p)^T$) since the copula C is known, like the Rosenblatt transformation. Let $s_1 = v_1$ and for $i = 2, ..., p$,

$$\begin{aligned}
v_i = C_i^c &= C(s_i | s_1, \cdots s_{i-1}) \\
&= \frac{\partial^{i-1} C(s_1, \cdots s_i)}{\partial s_1 \cdots \partial s_{i-1}} / \frac{\partial^{i-1} C(s_1, \cdots s_{i-1})}{\partial s_1 \cdots \partial s_{i-1}}.
\end{aligned} \tag{32}$$

For every $i = 2, ..., p$, evaluate the inverse of the conditional copula function C_i^{-c} at v_i to generate:

$$s_i = C_i^{-c} = C^{-1}(v_i | s_1, \cdots, s_{i-1}). \tag{33}$$

The inverse can be derived either analytically or numerically. In fact, this simulation process is a type of dependence concepts. Its monotonicity depends on convexity of the copulas function. This can be stated by the monotonic condition of the inverse Rosenblatt transformation.

Because the copula is a joint distribution function, the monotonicity of dependence concept depends on that of the inverse Rosenblatt transformation on the copula function. Then one sufficient condition is that $\partial C_i^c / \partial s_j \leq 0$ for all $i > j$. In this case, we can say that the random vector $\underline{S}$ with copula $C(\underline{s})$ can be monotonically constructed by a random vector $\underline{V}$ with independent copula Π. For example, a 2D random vector (S_1, S_2) with copula $C(s_1, s_2)$ can be obtained by the following transformation:

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$$\begin{cases} s_1 = v_1 \\ \dfrac{\partial C(s_1, s_2)}{\partial s_1}\Big|_{(v_1, s_2)} = v_2 . \end{cases} \qquad (34)$$

Clearly, s_1 is monotone on v_1, v_2 and s_2 is monotone on v_2. s_2 will be monotone on v_1 when $\partial^2 C(s_1, s_2)/\partial s_1^2 \le 0$.

To postulate that a random vector $\underline{X} = (X_1, X_2, ..., X_p)^T$ is of Gaussian copula is equivalent to assume that:

a) the univariate random variables $X_1, X_2, ..., X_p$ have one dimensional margins $s_i = F_i(x)$;

b) these margins are linked by a unique copula function C called Gaussian copula such that:

$$C_{R'}(\underline{s}) = \int_{-\infty}^{\Phi^{-1}(s_1)} \cdots \int_{-\infty}^{\Phi^{-1}(s_p)} \varphi_p(\underline{z}, R') d\underline{z} . \qquad (35)$$

where $\varphi_p(\underline{z}, R')$ is the p-dimensional standard normal pdf of zero means, unit standard deviations and correlation matrix R'. The Gaussian copula is absolutely continuous. Given any correlation matrix, we can generate a Gaussian copula with the presented simulation process. According to its monotonicity, we have the following sufficient condition.

Proposition8: If $R' \ge 0$ and if $\Lambda_{ij}^{(i)} \le 0$ for all $i > j$, the inverse transformation via Gaussian copula is monotone.
The proof for the above proposition is presented in Appendix C.

VII. EXTENSION AND CONCLUSION

This paper proved at first that the output distribution of failure function is invariant via inputs variables transformation. Then the theoretical conditions presented in this paper provide substantial evidence how the monotonicity of dependence concept can be achieved. Since the Gaussian copulas are often used to approximate the real world copulas in the industrial application, we gave the important correlation conditions under which the positively correlated normal random vector can be monotonically constructed by an independent normal random vector or uniform random vector. The presented Rosenblatt and Nataf transformation on the distributions with Gaussian copulas are different for the monotonicity of dependence concept. Nataf transformation is in fact an orthogonal transformation whose monotonicity concerns completely positive matrices. For this transformation, the dimension of the newly obtained iso-probabilistic space is equal to the cp-rank of the completely positive correlation matrices. Since, for every completely matrices, its cp-rank is always greater than or equal to its rank (the physical space dimension). It may product an augmentation of dimension while monotonic transformation, especially for the higher physical space dimension (> 4). In comparison with Nataf transformation, the Rosenblatt transformation always holds on the same dimension.

The presented correlation conditions are essential for getting a monotonic transformation. In fact, the monotonicity of dependence concept is independent on the random variables' order; the permuted random variables may give us satisfactory correlation matrices. Even though the permutation can not transform a non-completely positive matrix into a completely matrix, it may achieve a positive Cholescky factorization. For example, with a suitable permutation, all the 3-dimensional positively correlated normal random variable can be construction by a monotonic transformation (both inverse Nataf and Rosenblatt).

As for the normal distribution with some negative correlated variables, for example $Cov(U_1, U_2) \le 0$, we may use $U_1' = -U_1$ to change the correlation. Clearly, if U_1 follows normal distribution, $U_1 = -U_1$ on distribution. The increasing function will be changed into a decreasing function by this special transformation. But we can rechange it into an increasing one while the simulation in the iso-probabilistic space [3].

APPENDIX

A: The computation of Rosenblatt transformation matrices is as follows:

Let $|\bullet|$ denote the determinant of matrix. From the equation (12), we have $J_{ij} = 0$ for all $i < j \le p$. Since $\det(J^{-1}) = \prod_{i=1}^{p} f_i^c$, we can easily find $J_{ii} = 1/f_i^c$ for all $i \le p$; $J_{i(i-1)} = -\dfrac{\partial_{i(i-1)}^c}{f_{i-1}^c f_i^c}$ for all $2 \le i \le p$;

$$J_{i(i-2)} = \frac{1}{f_{i-2}^c f_{i-1}^c f_i^c} \begin{vmatrix} \partial_{(i-1)(i-2)}^c & f_{i-1}^c \\ \partial_{i(i-2)}^c & \partial_{i(i-1)}^c \end{vmatrix} \text{ for all } 3 \le i \le p .$$

Now let $1 \le d \le p - 1$, $\partial_{ii}^c = f_i^c$ and $\partial_{ij}^c = 0$ for $i < j \le p$, we can obtain $J_{i(i-d)} = \dfrac{\Delta_i^d}{f_{i-d}^c f_{i-d+1}^c \cdots f_i^c}$ for all $d + 1 \le i \le p$, where

$$\Delta_i^d = (-1)^d \begin{vmatrix} \partial_{(i-d+1)(i-d)}^c & \partial_{(i-d+1)(i-d+1)}^c & \cdots & \partial_{(i-d+1)(i-1)}^c \\ \partial_{(i-d+2)(i-d)}^c & \partial_{(i-d+2)(i-d+1)}^c & \cdots & \partial_{(i-d+2)(i-1)}^c \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{i(i-2)}^c & \partial_{i(i-d+1)}^c & \cdots & \partial_{i(i-1)}^c \end{vmatrix} . \qquad (36)$$

Finally, $\Delta_i^d \ge 0$ is the sufficient and necessary condition under which the inverse Rosenblatt transformation is monotone.

B: The proof of proposition 2 is as follows:
Let $\Lambda^k = (\rho_{ij}')$ $1 \le i, j \le k$, $\Lambda_{ij}^{(k)}$ be the cofactor of ρ_{ij}' in Λ^k, denote $\Lambda^{(k)} = \det(\Lambda^k)$ and $u_i = \Phi^{-1}(v_i)$. The resulting Rosenblatt transformation is then given by:

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$$\underline{y}=\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{pmatrix}=\begin{pmatrix} \Phi(x_1) \\ \Phi\left(\dfrac{x_2+\left(\Lambda_{21}^{(2)}/\Lambda_{22}^{(2)}\right)x_1}{\sqrt{\Lambda^{(2)}/\Lambda_{22}^{(2)}}}\right) \\ \vdots \\ \Phi\left(\dfrac{x_p+\sum_{j=1}^{p-1}\left(\Lambda_{pj}^{(p)}/\Lambda_{pp}^{(p)}\right)x_j}{\sqrt{\Lambda^{(p)}/\Lambda_{pp}^{(p)}}}\right) \end{pmatrix};$$

$$\underline{u}=\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix}=\begin{pmatrix} x_1 \\ \dfrac{\Lambda_{22}^{(2)}x_2+\Lambda_{21}^{(2)}x_1}{\sqrt{\Lambda^{(2)}\Lambda_{22}^{(2)}}} \\ \vdots \\ \dfrac{\Lambda_{pp}^{(p)}x_p+\sum_{j=1}^{p-1}\Lambda_{pj}^{(p)}x_j}{\sqrt{\Lambda^{(p)}\Lambda_{pp}^{(p)}}} \end{pmatrix}. \qquad (37)$$

Let

$$\underline{w}=(u^1,u^2\sqrt{\Lambda^{(2)}\Lambda_{22}^{(2)}},\cdots,u^i\sqrt{\Lambda^{(i)}\Lambda_{ii}^{(i)}}),\cdots,u^p\sqrt{\Lambda^{(p)}\Lambda_{pp}^{(p)}})^T$$

and $\Omega^{-1}=(\Omega^{-1})_{ij}=\begin{cases} 0 & if \quad i<j \\ \Lambda_{ij}^{(i)} & if \quad i\geq j \end{cases}$,

we can then obtain:

$$\underline{w}=\Omega^{-1}\underline{x}\ or\ \underline{x}=\Omega\underline{w}. \qquad (38)$$

The Jacobean transformation J is equivalent to Ω for researching the monotonicity of the inverse Rosenblatt transformation and the sufficient condition ($\partial F_i^c/\partial x_j\leq 0$ for all $i>j$) is equivalent to $\Lambda_{ij}^{(i)}\leq 0$ for all $i>j$.

As to the sufficient and necessary condition, assume that $\partial_{ij}^c=\Lambda_{ij}^{(i)}$ for all $i\geq j$ and $\partial_{ij}^c=0$ for all $i<j$ in the formulation (36), we can obtain Δ_i^d.

C: The proof of proposition 8 is as follows:
Let Let $\Lambda^i=(\rho_{jk}')\ 1\leq j,k\leq i$ denote the correlation matrix of the random vector $\underline{X}=(X_1,X_2,...,X_i)^T$. The i-dimentional Gaussian copulas can be denoted by:

$$C_i=C(s_1,\cdots s_i)=\int_{-\infty}^{\Phi^{-1}(s_1)}\cdots\int_{-\infty}^{\Phi^{-1}(s_i)}\varphi_i(\underline{z},\Lambda^i)d\underline{z}. \qquad (39)$$

Since the correlation matrix is symmetric, we can set $A=(a_{jk})=(\Lambda^i)^{-1}$ for all $j,k\leq i$, and $B=(b_{jk})=(\Lambda^{i-1})^{-1}$ for all $j,k\leq i-1$. Both A and B are symmetric matrix.

Let:

$$\Psi_A=(\Phi^{-1}(s_1),...,\Phi^{-1}(s_{i-1}),z_i)^T$$

and $\Psi_B=(\Phi^{-1}(s_1),...,\Phi^{-1}(s_{i-1}))^T$.

We can obtain:

$$\frac{\partial^{i-1}C_i}{\partial s_1\cdots\partial s_{i-1}}$$
$$=\sqrt{|A|/(2\pi)^{i/2}}\cdot\{\varphi[\Phi^{-1}(s_1)]\cdots\varphi[\Phi^{-1}(s_{i-1})]\}^{-1}; \qquad (40\text{-}1)$$
$$\cdot\int_{-\infty}^{\Phi^{-1}(s_i)}\exp\left\{-\frac{1}{2}\Psi_A^T A\Psi_A\right\}dz_i$$

$$\frac{\partial^{i-1}C_{i-1}}{\partial s_1\cdots\partial s_{i-1}}$$
$$=\sqrt{|B|/(2\pi)^{(i-1)/2}}\cdot\{\varphi[\Phi^{-1}(s_1)]\cdots\varphi[\Phi^{-1}(s_{i-1})]\}^{-1}. \qquad (40\text{-}2)$$
$$\cdot\exp\left\{-\frac{1}{2}\Psi_B^T B\Psi_B\right\}$$

The conditional Gaussian copula C_i^c can be denoted by:

$$C_i^c=\frac{\partial^{i-1}C_i}{\partial s_1\cdots\partial s_{i-1}}\Big/\frac{\partial^{i-1}C_{i-1}}{\partial s_1\cdots\partial s_{i-1}}$$
$$=\sqrt{|A|/2\pi|B|}\cdot\exp\left\{\frac{1}{2}\Psi_B^T B\Psi_B\right\}. \qquad (41)$$
$$\cdot\int_{-\infty}^{\Phi^{-1}(s_i)}\exp\left\{-\frac{1}{2}\Psi_A^T A\Psi_A\right\}dz_i$$

Let $\quad\alpha=(a_{1i},a_{2i},\cdots,a_{i-1i})^T\quad$ and $\quad A_0=(a_{jk})\quad$ for all $j,k\leq i-1$. Since $A=\begin{pmatrix} A_0 & \alpha \\ \alpha^T & a_{ii} \end{pmatrix}$, we can obtain:

$$-\frac{1}{2}\Psi_A^T A\Psi_A$$
$$=-\frac{1}{2}\Psi_B^T A_0\Psi_B+\frac{1}{2}\frac{(\alpha^T\Psi_B)^2}{a_{ii}}-\frac{1}{2}\left(\frac{a_{ii}z_i+\alpha^T\Psi_B}{\sqrt{a_{ii}}}\right)^2. \qquad (42)$$

So,

$$C_i^c=\sqrt{|A|/a_{ii}|B|}$$
$$\cdot\exp\left\{\frac{1}{2}\Psi_B^T(B-A_0)\Psi_B+\frac{1}{2}\frac{(\alpha^T\Psi_B)^2}{a_{ii}}\right\}. \qquad (43)$$
$$\cdot\Phi\left(\frac{a_{ii}\Phi^{-1}(s_i)+\alpha^T\Psi_B}{\sqrt{a_{ii}}}\right)$$

Let

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$$E = \exp\left\{\frac{1}{2}\Psi_B^T(B - A_0)\Psi_B + \frac{1}{2}\frac{(\alpha^T\Psi_B)^2}{a_{ii}}\right\},$$

$$F = \Phi\left(\frac{a_{ii}\Phi^{-1}(s_i) + \alpha^T\Psi_B}{\sqrt{a_{ii}}}\right),$$

$$f = \varphi\left(\frac{a_{ii}\Phi^{-1}(s_i) + \alpha^T\Psi_B}{\sqrt{a_{ii}}}\right)$$

$$\beta_t = (b_{1t} - a_{1t},\ b_{2t} - a_{2t},..., \ b_{i-1t} - a_{i-1t})^T.$$

Differentiating C_i^c with respect to s_i, we have the conditional copula density:

$$c_i^c = \frac{\partial C_i^c}{\partial s_i} = \sqrt{|A|/|B|} \cdot \frac{E \cdot f}{\varphi[\Phi^{-1}(s_i)]}. \tag{44-1}$$

Differentiating C_i^c with respect to s_t for all $t = 1, \cdots i - 1$, we can obtain:

$$\frac{\partial C_i^c}{\partial s_t} = \sqrt{|A|/a_{ii}|B|} \cdot \frac{E \cdot F}{\varphi[\Phi^{-1}(s_t)]}$$
$$\cdot \left\{(\beta_t^T + \frac{a_{ii}}{a_{ii}}\alpha^T)\Psi_B + \frac{a_{ii}f}{\sqrt{a_{ii}}F}\right\}. \tag{44-2}$$

Since $(\beta_t^T + \frac{a_{ii}}{a_{ii}}\alpha^T) = 0$ (the proof is presented hereafter) and $a_{ii} = \det(A)\Lambda_{it}^{(i)}$, finally, we have:

$$\frac{\partial C_i^c}{\partial s_t} = |A|\sqrt{|A|/|B|} \cdot \frac{E \cdot f}{a_{ii}\varphi[\Phi^{-1}(s_t)]} \cdot \Lambda_{it}^{(i)}. \tag{45}$$

Hereas the first factor is always positive (theoretically non-negative) and the second is the the cofactor of $\rho_{it}^{'}$ in Λ^i. Thus we have proved that if $R' \geq 0$ and if $\Lambda_{ij}^{(i)} \leq 0$ for all $i > j$, the inverse transformation via Gaussian copula is monotone.

The proof of $(\beta_t^T + \frac{a_{ii}}{a_{ii}}\alpha^T) = 0$ is as follows:

Let $\lambda = (\rho_{1i}^{'}, \rho_{2i}^{'}, \cdots, \rho_{i-1i}^{'})^T$, we have $\Lambda^i = \begin{pmatrix} \Lambda^{i-1} & \lambda \\ \lambda^T & 1 \end{pmatrix}$. Since Λ^{i-1} is a positive definite matrices, there exist an invertible matrix G such that $G^T\Lambda^{i-1}G = I$. From $(G^T\Lambda^{i-1}G)^{-1} = I$, we can then obtain $B = (\Lambda^{i-1})^{-1} = GG^T$.

Let $H_1 = \begin{pmatrix} G & 0 \\ 0 & 1 \end{pmatrix}$, we have $H_1^T\Lambda^i H_1 = \begin{pmatrix} I & G^T\lambda \\ \lambda^T G & 1 \end{pmatrix}$. Let

$$H_2 = \begin{pmatrix} I & -G^T\lambda \\ 0 & 1 \end{pmatrix},$$

$$H = H_1 H_2 = \begin{pmatrix} G & -B\lambda \\ 0 & 1 \end{pmatrix}.$$

We have then $H^T\Lambda^i H = \begin{pmatrix} I & 0 \\ 0 & 1 - \lambda^T B\lambda \end{pmatrix}$.

From the same manner, we obtain:

$$A = (\Lambda^i)^{-1} = H\begin{pmatrix} I & 0 \\ 0 & \frac{1}{1 - \lambda^T B\lambda} \end{pmatrix}H^T$$
$$= \begin{pmatrix} B + \frac{B\lambda(B\lambda)^T}{1 - \lambda^T B\lambda} & \frac{-B\lambda}{1 - \lambda^T B\lambda} \\ \frac{-(B\lambda)^T}{1 - \lambda^T B\lambda} & \frac{1}{1 - \lambda^T B\lambda} \end{pmatrix}. \tag{46}$$

Because $A = (a_{jk})$ for all $j, k \leq i$ and $\alpha = (a_{1i}, a_{2i}, \cdots, a_{i-1i})^T$, we can then obtain $\frac{1}{1 - \lambda^T B\lambda} = a_{ii}$ and $\frac{-B\lambda}{1 - \lambda^T B\lambda} = \alpha$. Finally we have $A = \begin{pmatrix} B + \alpha\alpha^T / a_{ii} & \alpha \\ \alpha^T & a_{ii} \end{pmatrix}$ and then $a_{jt} = b_{jt} + a_{ii}a_{ji} / a_{ii}$ for all $j, t = 1, \cdots i - 1$. This is the end of proof.

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